General relativistic magnetic monopole

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ABSTRACT. The general relativistic version of the 't Hooft-Polyakov magnetic monopole is studied. We confirm a previously derived result indicating that for a sufficiently large symmetry breaking mass, the monopole is a black hole. The critical mass is evaluated and the existence of an event horizon examined with some detail.

RESUMEN. Presentamos un estudio del monopolo de t'Hooft-Polyakov incluyendo el campo gravitacional. El estudio confirma un resultado obtenido previamente que indica la conversión del monopolo en un hoyo negro para un valor de la masa asociada a la ruptura de simetría suficientemente grande. Calculamos también el valor crítico de dicha masa y examinamos con cierto detalle la existencia del horizonte de eventos

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1. INTRODUCTION

't Hooft [1] and Polyakov [2] have shown that magnetic monopoles exist in all gauge theories with spontaneously broken symmetry if the symmetry group admits a compact covering. Such monopoles are regular stable solutions of the field equations with finite energy. Later on, Bogomol'nyi and Marinov [3] obtained numerical solutions of the field equations which describe such a monopole.

The inclusion of gravitational effects in gauge field theories has been considered by Van Nieuwenhuizen, Wilkinson and Perry [4], and other authors [5-9]. An interesting problem is the existence of regular solutions of the Einstein-Yang-Mills-Higgs equation, and the possible existence of black holes with 'hairs', that is, parameters other than mass, angular

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momentum and electric charge. It has been found recently that such regular solutions corresponding to a magnetic monopole exist in general relativity, provided the mass for spontaneous symmetry breaking is small; if, on the other hand, this mass is comparable to the Planck mass, the spacetime corresponds to a Reissner-Nordstrom solution with a black hole horizon [10-11].

Given the complexity of the equations and the many parameters involved, we think it is worth to repeat the above mentioned calculations (particularly those of Refs. [10-11]) with a different integration method and a wider range of parameters. In particular, we calculate the norm of the timelike Killing vector. This norm is a particularly important quantity because it defines the horizon of a black hole (it vanishes at the horizon), and has not been evaluated in previous works.

Our model has a SU(2) gauge field with a Higgs field. As pointed out in Ref. [4], gravity does not alter drastically the form of the Yang-Mills and Higgs fields if the mass v related to the vacuum expectation value of the Higgs field is much smaller than the Planck mass; however, black hole solutions exist if the mass v exceeds a critical value [10-11]. Indeed, the numerical solutions for small v are quite similar to those found by Bogomol'nyi and Marinov [3], and are presented in the figures.

The interested readers are referred to Refs. [1-3] for the concept of a magnetic monopole in flat space, and to Refs. [4-11] for the extension of gauge field theories to general relativity.

2. Field equations

The basic equations were obtained by van Nieuwenhuizen *et al.* [4]. For the sake of completeness, we outline a slightly different derivation of these equations. The starting point is the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\langle D_\mu\phi, D_\nu\phi\rangle - V(\|\phi\|) - \tfrac{1}{4}\langle G^{\mu\nu}, G_{\mu\nu}\rangle\right], \tag{1}$$

where

$$D_\mu\phi = \partial_\mu\phi - ie[A_\mu, \phi] \tag{2}$$

and

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]. \tag{3}$$

Here $\langle A, B\rangle \equiv \mathrm{Trace}(A^\dagger B)$, $\|\phi\|^2 \equiv \langle\phi,\phi\rangle$ and V is the scalar field potential (we set $\hbar = c = 1$).

Now, let the gauge group be SU(2) and set

$$\phi = iQ(r)\frac{x^{(n)}\sigma_{(n)}}{r}, \tag{4}$$

$$A^\mu = W(r)\frac{K^\mu_{(n)}\sigma_{(n)}}{r^2}, \tag{5}$$

where $\sigma_{(n)}$ are the Pauli matrices, $x^{(n)}$ are Cartesian coordinates and $K^{\mu}_{(n)}$ are the three Killing vectors of the rotation group. Thus, A^{μ} is a pure gauge except at $r = 0$.

In general, a static and spherically symmetric metric has the form

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\Omega^2. \tag{6}$$

Using this metric, one finds the field equations:

$$e^{-\frac{1}{2}(\nu+\lambda)} \frac{d}{dr}\left[e^{\frac{1}{2}(\nu-\lambda)} r^2 \frac{dQ}{dr}\right] - 2(1+2eW)^2 Q - \frac{1}{2} r^2 \frac{dV}{dQ} = 0, \tag{7}$$

$$e^{-\frac{1}{2}(\nu+\lambda)} \frac{d}{dr}\left[e^{\frac{1}{2}(\nu-\lambda)} \frac{dW}{dr}\right] - 2(1+2eW)\left[\frac{W(1+eW)}{r^2} + eQ^2\right] = 0 \tag{8}$$

$$\begin{aligned} T^t_{\ t} + T^r_{\ r} &= \frac{4}{r^2}\left[(1+2eW)^2 Q^2 + \frac{2W^2(1+eW)^2}{r^2}\right] + 2V(Q) \\ &= -(8\pi G)^{-1}\left[\frac{2(e^{-\lambda}-1)}{r^2} + \frac{e^{-\lambda}}{r}\frac{d}{dr}(\nu-\lambda)\right], \end{aligned} \tag{9}$$

$$\begin{aligned} T^t_{\ t} + T^r_{\ r} &= 2e^{-\lambda}\left[\left(\frac{dQ}{dr}\right)^2 + \frac{2}{r^2}\left(\frac{dW}{dr}\right)^2\right] \\ &= (8\pi G)^{-1} \frac{e^{-\lambda}}{r}\frac{d}{dr}(\nu+\lambda). \end{aligned} \tag{10}$$

Far from the monopole core, the scalar field has the value $\|\phi\| = \sqrt{2}v$, which co-rresponds to the true vacuum, that is $V(v) = 0 = dV/dQ(v)$, and the gauge field is $W = -(2e)^{-1} \equiv g$. In that region, the spacetime is described by the Reissner-Nordstrom metric of a magnetic monopole with charge g.

The norm of the timelike Killing vector is e^{ν}. By definition, this norm vanishes if there is a black hole horizon. The other gravitational potential λ must vanish at the origin in order to fulfill the condition of elementary flatness.

It is worth noticing that, according to Eq. (10), the derivative of $\lambda + \nu$ is always positive. Since λ and ν vanish at infinity, it follows that $\lambda + \nu < 0$. For a black hole solution, $\nu \to -\infty$ before $\lambda \to \infty$, a fact which has been overlooked in Refs. [11] and [12], where only the potential λ is evaluated. Notice also that, for a regular solution, $\lambda(0) = 0$ and $\nu(0) < 0$.

Near the center of the core, the fields have the approximate forms $Q = ar$ and $W = br^2$, which imply an equation of state

$$\epsilon \equiv T^t_{\ t} = (8\pi G)^{-1}\big[3(a^2 + 4b^2) + V(0)\big], \tag{11}$$

$$p \equiv -T^r_{\ r} = (8\pi G)^{-1}\big[a^2 - 4b^2) + V(0)\big], \tag{12}$$

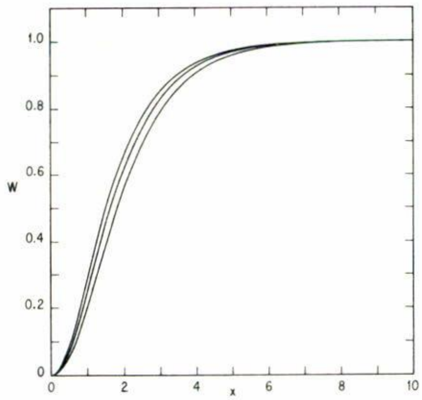

FIGURE 1. Dimensionless scalar field q as a function of the dimensionless radial distance x, for three values of the parameter α', and for $\eta = 10^{-6}$.

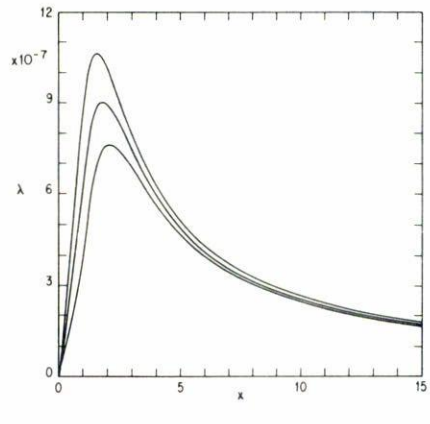

FIGURE 2. Same as Fig. 1 for the dimensionless gauge field w.

Thus, the spacetime in the center of the monopole is not exactly DeSitter, as it is sometimes claimed.

Our integration method closely follows the one used in Ref. [3]. First, we assume that the potential V has the form

$$V(Q) = \tfrac{1}{2}\alpha(Q^2 - v^2)^2. \tag{13}$$

Next, we set $Q = W = W' = \lambda = 0$ at $r = 0$, in order to guarantee that the solutions are regular (the potential ν can be eliminated from the equations). Finally, we choose particular values for Q', W'' and λ' at $r = 0$ and integrate from the origin to infinity, and look for solutions which tend asymptotically to $Q = v$, $W = -g$ and $e^{-\lambda} = 1 - 2GM/r + g^2/r^2$, where M is the mass of the monopole which depends on the initial values for Q', W'' and λ'.

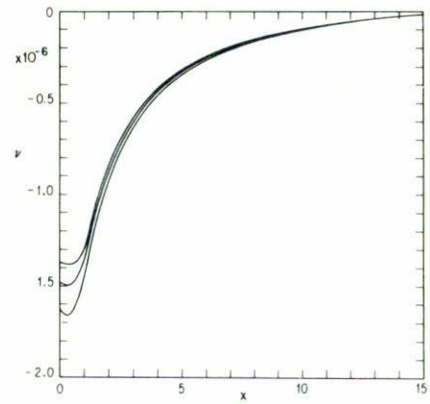

FIGURE 3. Same as Fig. 1 for the gravitational potential λ.

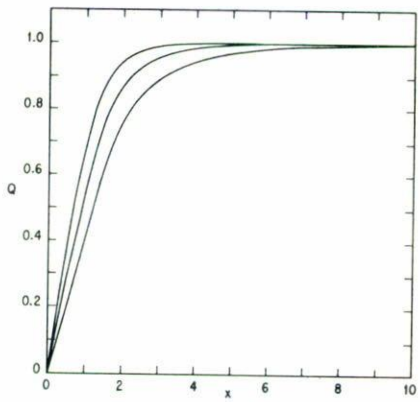

FIGURE 4. Same as Fig. 1 for the gravitational potential ν.

3. RESULTS AND DISCUSSIONS

For the purpose of numerical calculations, it is convenient to define dimensionless functions q and w as $Q = vq, W = -w/2e$, and set $r = x/ev$. With this rescaling, the field equations contain only two parameters: $\alpha' \equiv \alpha/e^2$ and $\eta \equiv 8\pi G v^2$. The parameter η is essentially the squared ratio of the GUT and Planck masses.

Values of η in the interval $[10^{-8}, 10^{-1}]$ were chosen as representative. Within this range, all values of this parameter give essentially the same results up to a scaling factor of order η for the potentials λ and ν. For larger values, black hole solutions exist according to the value of the parameter α', which is a measure of the strength of the scalar field potential; different values from 0 to 10 were taken for α'.

In Figs. 1 and 2, we have plotted some typical solutions for the dimensionless functions

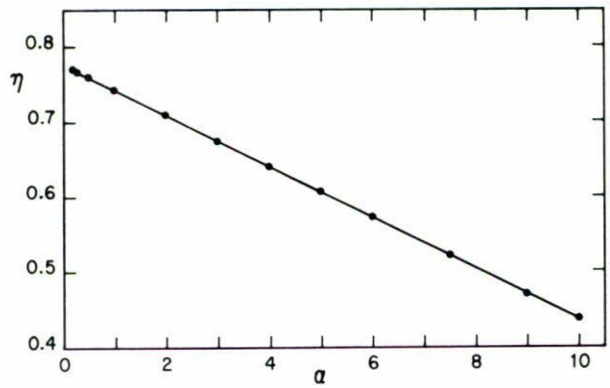

FIGURE 5. Critical value of $\eta(\eta^*)$ as a function of α'.

q and w, respectively. Due to the smallness of the gravitational effects, neither the scalar field q nor the gauge field w differ appreciably from the solutions found by Bogomol'nyi and Marinov [3].

In Figs. 3 and 4, we show the gravitational potentials which correspond to the functions in Figs. 1 and 2. Both λ and ν remain finite and tend to the Reissner-Nordstrom solution when the dimensionless radial distance x tends to infinity.

The typical values of the monopole mass turn out to be 1.234, 1.290 and 1.356 (in units of v) for $\alpha' = 0.3$, 1 and 3. These values are very close to those found by Bogomol'nyi and Marinov (1.160, 1.238 and 1.326 respectively) for the case without a gravitational field.

For η above a critical value η^*, the function $\nu \to -\infty$, which implies that a black hole is formed. This critical value depends on the parameter α'; the larger this last parameter, the smaller is the critical value of η for the existence of a black hole solution. This result is plotted in Fig. 5, where we show the threshold value of η for the disappearance of regular solutions and the existence of black hole horizons.

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