Fuzzy sets and physics

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ABSTRACT. A brief introduction to the mathematical theory of fuzzy sets and its applications is presented. Also the relation of this relatively new branch of mathematics with classical and quantum physics is discussed.

RESUMEN. Se presenta una breve introducción a la teoría matemática de los conjuntos borrosos y sus aplicaciones. También se discute la relación de esta rama relativamente nueva de las matemáticas con la física clásica y cuántica.

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1. INTRODUCTION

According to Kaufmann and Gupta [1], since the inception of the theory of "fuzzy" or "diffuse" sets in 1965, over 7000 research papers, reports, monographs and books have been published in both the "soft" social sciences and in the "hard" natural and engineering sciences. There even exist journals [2] solely devoted to this relatively new branch of mathematics.

Why fuzzy mathematical concepts and techniques are applied in so many other fields of knowledge and remain practically unknown in the realm of physics? Are these concepts or techniques useful to physicists? To try to answer these questions we will proceed first to present some basic concepts and applications of fuzzy mathematics in Sect. 2. Sect. 3 is devoted to discuss its relation with physics.

2. THEORY

In what follows we have preferred to use the term "fuzzy mathematics" (FM), instead of the more common "fuzzy sets" as a generic term to designate the methods for dealing

*On leave from Instituto de Física, UNAM, Apartado postal 20-364, 01000 México, D.F., México with phenomena that are vague, imprecise or too complex to be susceptible of study by conventional mathematical means. This choice of nomenclature is based on pedagogical reasons since FM can be roughly divided in three parts: fuzzy events (fuzzy sets), fuzzy syllogisms (multivalued and fuzzy logic) and possibility theory.

a) Fuzzy sets

The first technical paper on fuzzy sets was published by Zadeh [3] in 1965 offering a mathematical formalism to group elements into classes which are ill-defined or not sharply bounded. To generalize the classical concept of a set, where an element *is* contained or *is not* contained in it, a fuzzy set is defined as a set where its elements have different degrees of membership other than total or null membership. A good example of a fuzzy set could be a cloud since the interior points closer to the center belong "more" to the cloud than those in its diffuse boundary. In contrast to a classical or "sharp" set (where we can assign to each element a value 1 if it is contained in the set or a value 0 if it is not contained in the set), we can define for a fuzzy set A a membership function $\mu_{\mathbf{A}}(x)$ that can take intermediate values in the closed interval [0,1] qualifying the degree of membership of an element x.

For example, let us consider the set A of all real numbers that are much greater than the real number r, that is,

$$A = \{x | x \in R, x \gg r\}. \tag{1}$$

This fuzzy set may be defined by a membership function such as

$$\mu_{\mathbf{A}}(x) = \begin{cases} 0, & \text{for } x \leq r, \\ (x-r)/x, & \text{for } x \geq r, \end{cases} \tag{2}$$

which assigns to x values closer to 1 as x grows beyond $x > r$.

A useful concept is the α-cut of a fuzzy set (which is shown schematically in the Fig. 1 for a set defined on R^2) since α-cuts help to generalize the notion of subsets (a larger value of α correspond to a larger degree of membership). Analogously to sharp sets we can also define for fuzzy sets the concepts of complement A^c, union $A \cup B$ and intersection $A \cap B$. One of the possible generalizations is

$$\mu_{\mathbf{A}^c}(x) = 1 - \mu_{\mathbf{A}}(x), \tag{3}$$

$$\mu_{\mathbf{A}\cup\mathbf{B}}(x) = \max\, [\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x)], \tag{4}$$

$$\mu_{\mathbf{A}\cap\mathbf{B}}(x) = \min\, [\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x)], \tag{5}$$

which corresponds to the usual operations for sharp sets when the membership grades are restricted to the set $\{0,1\}$. However, it can be observed that operations for sets such as union and intersection yield better descriptions for fuzzy sets than for sharp sets. For example, in the case of sets defined by opposite characteristics (say $A = \{x | x$ is a short

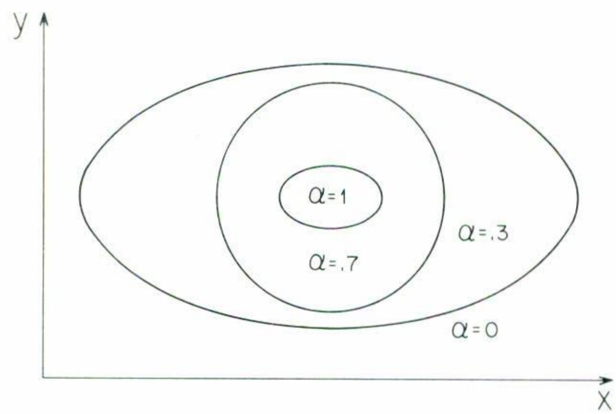

FIGURE 1. α-cuts of an arbitrary fuzzy set defined in R^2.

person} and $B = \{x|x \text{ is a tall person}\}$), their union and intersection for sharp sets yields the empty and total sets, respectively, whereas for fuzzy sets Eqs. (4) and (5) yields more detailed descriptions (all people may belong to the union and intersection sets with different membership grades; to middle height persons correspond the smallest values in $A \cup B$ and the largest values in $A \cap B$).

Many classical concepts in mathematics such as relations, algebraic structures, graph theory, differential calculus, geometry, analysis, topology, theory of catastrophes, etc., have been generalized to have their fuzzy counterpart exhibiting new or restricted properties. For the reader interested in learning about the general aspects of FM topics and their applications there is a number of books that include theory and selected references [1,4-10].

b) Fuzzy syllogisms

Classical or two-valued logic deals only with propositions that are either true or false, an assumption that has been questioned by researchers in different epochs, including Aristotle [11] and B. Russel. The latter author wrote [12]: "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence."

The two-valued logic has been extended axiomatically in different ways to include a third truth value that may be called indeterminate. Furthermore, three-valued logic has been also extended to $n \geq 3$ and infinite-valued logic whose truth values are represented by all real numbers in the interval [0,1]. It can be shown [13,14] that these kinds of multivalued logics are closely related to fuzzy set theory (like two-valued logic is related to Boolean algebra and classical set theory [15]).

c) Possibility theory vs. probability theory

Since fuzzy sets and fuzzy logic deal with certain forms of uncertainty, ambiguity or vagueness, one can ask how these theories are related to the successful theory of probability [16]. In order to elucidate some of their differences let us first introduce in the next example the FM concept of "possibility".

Consider the i-dependent statement "Prof. Z teaches i courses this semester". A probability distribution p_i may be associated to i after observing Prof. Z for 40 semesters while the possibility distribution π_i can be interpreted as the degree of ease with which Prof. Z can teach i courses. Acceptable values for p_i and π_i are $p = (.4, .3, .2, .075, .025, 0, 0, 0, 0, 0, ...)$ and $\pi = (1, 1, 1, .2, .2, .1, .05, .03, .03, .01, \ ...)$, where for convenience the distributions are written in vectorial form and only the corresponding values for the first 10 values of i are shown.

Although axioms of the different versions of possibility theory are more flexible than those of probability theory (for example $\sum_i \pi_i$ may not be necessarily equal to 1), there should be a certain degree of consistency or compatibility between their results. For example, for events $\{x_i\}$ with corresponding possibilities $\{\pi_i\}$ and probabilities $\{p_i\}$, a large value of $\sum_i \pi_i p_i$ indicates that there is a good degree of compatibility between them. Furthermore, common sense would require that knowledge of possibilities conveys some information about the probabilities, but not vice versa; if an event x is impossible ($\pi_j \cong 0$) then it is also improbable ($p_j \cong 0$). However, it is not true that an event that is possible is also probable.

To illustrate situations where inexact, or fuzzy concepts can be discussed rigorously withing the confines of an extension of probability theory, let us consider a one-dimensional normalized probability density $f(x)$, which could describe very different systems:

i) An age distribution of a large group of people (being x = time t).
ii) The resistance distribution of a large number of resistors (being x = resistance R).

If we qualify each x with a membership function $\mu(x) \in [0, 1]$, then the value of

$$H = \int_{\infty}^{\infty} \mu(x) f(x) dx \tag{6}$$

will allow us to characterize the whole distribution with the fuzzy characteristic that μ qualifies. For example, in the first case, if we assign to $\mu(t)$ larger values for older people, H will yield the "total age" of the distribution. In the second case, if $\mu(R)$ is chosen to peak around a desired or ideal value of the resistance R_0, then the value of H qualifies how "ideal" is the set of given resistors. We can notice that, in contrast to the well-defined function f, the choice of the function μ is arbitrary since it qualifies subjective concepts such as "total age" and "ideal".

d) General applications

FM has provided methods and algorithms where uncertain, vague or ambiguous description or reasoning are required. To give a glance of the wide range of its applicability we can mention contributions in the fields of artificial intelligence, economics, pattern recognition (fuzzy clustering), medicine (in the process of diagnosis of disease), ecology, psychology and cognitive science, ethology (to classify and qualify principal postures of animals under observation), theory of information, neural networks and natural language. In many cases the appropriate algorithms are performed by expert systems.

However, the main applications of FM seem to be centered around systems science and decision making, since for these fields FM has probed to be a useful tool to implement key concepts such as control, regulation, adaptation, communication and organization.

In the consumer product market, aircraft control devices, cameras, washing machines and dozens of other consumer goods have appeared. The Japanese government and Japanese companies, for example, have poured millions of dollars into research of fuzzy systems trough grants and centers like the Laboratory for International Fuzzy Engineering in Yokohama.

3. Discussion

a) Development of physics

We claim that the uncertainty, vagueness and ambiguity which are inherent to FM can be attributed to either: *i)* presence of subjectivity in the variables chosen or to, *ii)* lack of complete information of the system in consideration or an incomplete definition of the system.

To exhibit the component of subjectivity mentioned above let us recall that the choices of membership function in Eq. (1) (to describe the fuzzy set of real number x "much larger" than r) and in Eq. (6) (to describe fuzzy concepts such as the "total age" of a distribution or how "good" it is) are arbitrary, as in the general case of assigning values to the variables in fuzzy logic and possibility theory. They are arbitrary since there is not a unique procedure to quantify the fuzzy concepts used above, such as very large, old, ideal, not very true, possible, etc. (notice that the concept of fuzzy is itself fuzzy!).

Nevertheless, the presence of subjective variables in a theory, once they are defined, does not diminish necessarily its usefulness. A similar situation has occurred in physics since in the first phases of its development, subjective notions such as "quantity of movement" or "heat" of a macroscopic body were subjective but useful. The former concept could be quantified in different ways, but in the process of becoming part of the physical sciences (or better, to give birth to them), mechanics further developed by noticing that certain quantities are, in general, more useful (like $p = mv$ or $K = mv^2/2$) than others. In other words, the process of quantifying certain subjective notions was one of the first steps to create mechanics and thermodynamics.

From the above reasoning a scientist could be tempted to infer that only a certain amount of "mathematization", could help to generate further scientific advances, a statement which is not always true, since it neglects the importance of qualitative discoveries (for example, Darwin's theory of evolution had an enormous impact and did not involve mathematics at all).

To illustrate how a fuzzy variable can account for a partial description of a system let us consider the problem of the evaluation of a color. From the physicists point of view, that particular color is uniquely represented by the frequency of an electromagnetic wave. For experimental psychologists [17], however, the meaning of that color is totally different; it might well be a fuzzy concept since it involves aspects of perception and semantics, as well.

In general, a scientist in the process of research may conscious or unconsciously define fuzzy concepts in the form of trial variables, functions, models, etc., to approach some *ideal* conditions (diminish experimental errors, increase efficiency, minimize energy, etc.). Those new concepts that later on are proven to be successful become less fuzzy or more sharp.

To explain why FM is practically unknown in the realm of physics, it can be argued that the relative simplicity and regularity of the systems under its study have led to a considerable advance based on concepts for which classical mathematics has provided an excellent framework. To further discuss the origin of regularities or uniformities in nature, in the next section we will present aspects of the theory of quantum mechanics that lead both to regular and fuzzy behavior in nature.

b) Conection of fuzzy mathematics with quantum physics

In regard to the regularity of physical systems mentioned in the last section, there is an important aspect of quantum physics that rejects fuzziness; the atomic quantum states have specific shapes and frequencies which are uniquely predetermined by the wave nature of the electrons. The fact that atoms (for example, two silver atoms in the ground state) are completely identical imposes a regularity which is in contradistinction to the situation in classical physics, where the allowed energy forms a continuum for bound states. In other words, if atoms could obey classical mechanics, then it would be rather difficult to assemble a large number of them with similar orbits.

On the other hand, there are ambiguous and vague aspects inherent to the actual theory of quantum mechanics that could lead to descriptions involving FM.

For example, in order to apply FM to the wave-particle duality problem, let us recall the well known two-slit particle interference experiment [18] (see Fig. 2). A beam that penetrates a screen through two slits shows the characteristic intensity patterns, which are quite different from the simple sum of intensities expected of two separate beams emerging from the slits on the basis of the classical picture of particles. The pattern of intensity is in fact the same as if obtained from a wave passing through two slits.

In this particular case we can construct in term of the physical parameters of the experiment a membership function μ_p of the fuzzy set of particles P to characterize the ambiguity wave-particle. For example, if large values of μ_p describe particle behavior and small values of μ_p describe wave behavior, then $\mu_p(\lambda)$ could be chosen proportional to Δ/λ, where Δ is the fixed spatial resolution of the detector and $\lambda = h/p$ is the Broglie wavelength associated to the beam particle with momentum p. For small values of Δ/λ the wave behavior of the particles is clearly shown in part (b) of Fig. 2, whereas for large values of Δ/λ, any physical detector covers or straddles several wiggles of the probability

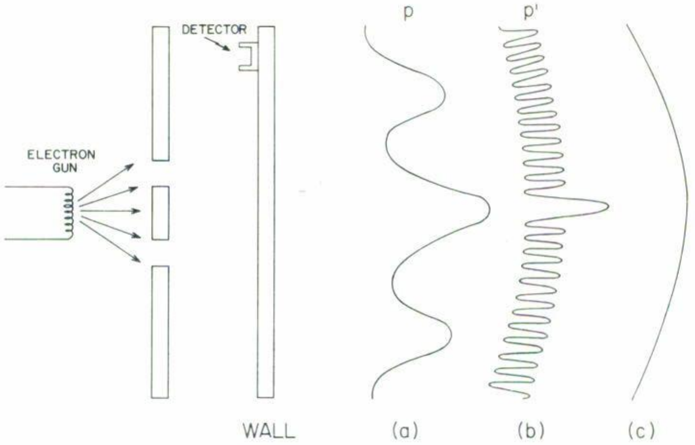

FIGURE 2. Interference experiment with electrons. If the wire is at a negative voltage with respect to the box, electrons emitted by the wire (electron gun) be accelerated toward the walls and some will pass through the holes. Panel (a) shows schematically the interference patern for electrons with momentum p clearly exhibiting wave behavior and panel (b) shows their interference for electrons with momentum $p'(p' > p)$. Panel (c) shows the observed interference pattern of panel (b) due to the finite resolution Δ of the detector. The smooth resulting curve corresponds to "classical" particle behavior.

curve, so that the measurements show the smooth curve drawn in part (c) of the Fig. 2, as expected for particles in classical physics. Furthermore, in order for μ_p to be a membership function of the fuzzy set P, the values of μ_p can be fitted to lie between zero and one by normalizing Δ/λ by a multiplicative factor.

On the other hand, there are uncertainties that naturally arise in the theory of quantum mechanics. Whenever two operators do not commute, the states of their corresponding quantum mechanical observables can not be measured precisely, that is, those observables are said to be incompatible (like the position x and the momentum component p_x). That is, the nature of quantum mechanics implies the existence of physical variables which measurement is necessarily fuzzy! This situation has been formalized in the language of fuzzy sets in a number of articles [19-21] and, at least in a book [22].

To avoid confusions due to the nomenclature it is important to mention that in Refs. [21] and [22] the term "stochastic" is employed to describe quantum mechanical measurements as fuzzy events, which is not a goal of the better known *stochastical theories of quantum mechanics*. To distinguish withing this context the meaning of the terms fuzzy and stochastic, it should be mentioned that the truly stochastic theories of quantum mechanics are classified [23], according to their goals, in two main groups. In the first group there are those theories which aim to provide a stochastical interpretation of quantum mechanics and in the second group there are more ambitious theories which aim, through introduction of new physical ideas, to englobe quantum mechanics. The more evolved model of the second group is *stochastic electrodynamics*, defined as the theory of the electron in the presence of a vacuum stochastic electromagnetic field (zero point-field) [24]. In *quantum electrodynamics* this zero point-field is virtual while in *stochastic electrodynamics* is real.

c) Final remarks

The main concepts of the so called theory of fuzzy sets were presented as a generic term of fuzzy mathematics (FM) to include fuzzy events, fuzzy syllogisms and possibility theory. The summary of FM presented in Sect. 2 do not belong to standard physics courses, although for a college physics student it is very easy to fully appreciate these concepts since it is only required to know basis concepts of algebra, calculus and probability. Furthermore, students and researchers in interdisciplinary fields may find the concepts and formalism of FM helpful.

From the perspective of fuzzy mathematics, we mention in Sect. 3 the subjectivity in the development of physics, which is an interesting topic within the broader field of philosophy of science. We also discussed the importance of quantum mechanics in the regular behavior observed in the microscopic world. This regularity helps to understand the success of classical (non-fuzzy) mathematics in the study of physics. Important concepts in quantum mechanics such as ambiguity and vague measurements can be formalized in the language of FM.

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