Symmetries, Noether's theorem and inequivalent Lagrangians applied to nonconservative systems

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ABSTRACT. Symmetry properties, Noether's theorem and inequivalent Lagrangians are discussed and applied to simple one-dimensional nonconservative systems. In the most elementary instances, the standard constants of the motion combined with other conserved quantities associated with previously overlooked spacetime symmetries of the action lead to a complete algebraic solution of the equations of motion. In the more difficult case of the damped harmonic oscillator, two inequivalent Lagrangians are employed. Noether invariances of the corresponding actions are identified by inspection, allowing the determination of two independent constants of the motion from which the general solution to the equation of motion is algebraically found.

RESUMEN. En este trabajo se discuten las propiedades de simetría, el teorema de Noether y los lagrangianos no equivalentes, así como su aplicación a sistemas unidimensionales no conservativos simples. En las instancias más elementales, las constantes de movimiento estándar, combinadas con otras cantidades conservadas asociadas a simetrías espaciotemporales de la acción previamente no percibidas, conducen a solución algebraica completa de las ecuaciones de movimiento. En el caso más difícil de un oscilador armónico amortiguado, se emplean dos lagrangianos no equivalentes. Las invariancias de Noether de las acciones correspondientes son identificadas por inspección, permitiendo la determinación de dos constantes de movimiento independientes a partir de las cuales se encuentra algebraicamente la solución general de la ecuación de movimiento.

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1. INTRODUCTION: NOETHER'S THEOREM

In Lagrangian dynamics the general connection between symmetry (invariance) properties and conserved quantities is provided by Noether's theorem [1]. For discrete systems, this theorem asserts that if, given the infinitesimal transformation (ϵ is an infinitesimal parameter),

$$t \to t' = t + \epsilon X(q(t), t), \tag{1a}$$

$$q_i(t) \to q_i'(t') = q_i(t) + \epsilon\psi_i(q(t), t), \tag{1b}$$

the action integral remains invariant, that is,

$$\delta S = \int_{t_1'}^{t_2'} \mathcal{L}\left(q'(t'), \frac{dq'(t')}{dt'}, t'\right) dt' - \int_{t_1}^{t_2} \mathcal{L}\left(q(t), \frac{dq(t)}{dt}, t\right) dt = 0, \tag{2}$$

then the quantity

$$C = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\dot{q}_i X - \psi_i) - \mathcal{L} X \tag{3}$$

is a constant of the motion, where $\mathcal{L}(q, \dot{q}, t)$ is the Lagrangian of the system. A straightforward proof of this result runs as follows, where in all forthcoming computations only terms up to the first order in ϵ are retained. First of all one has

$$\frac{dt'}{dt} = 1 + \epsilon \dot{X}, \qquad \frac{dt}{dt'} = (1 + \epsilon \dot{X})^{-1} = 1 - \epsilon \dot{X}, \tag{4}$$

and

$$\frac{dq_i'(t')}{dt'} = \frac{dt}{dt'}\frac{dq_i'(t')}{dt} = (1 - \epsilon \dot{X})(\dot{q}_i + \epsilon \dot{\psi}_i) = \dot{q}_i + \epsilon \xi_i, \tag{5}$$

where

$$\xi_i = \dot{\psi}_i - \dot{q}_i \dot{X}. \tag{6}$$

Thus,

$$\delta S = \int_{t_1}^{t_2} \mathcal{L}(q + \epsilon \psi, \dot{q} + \epsilon \xi, t + \epsilon X)(1 + \epsilon \dot{X})\, dt - \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt = 0 \tag{7}$$

leads at once (so long as ϵ and the interval of integration are arbitrary) to the Noether condition

$$\sum_i \left\{ \psi_i \frac{\partial \mathcal{L}}{\partial q_i} + (\dot{\psi}_i - \dot{q}_i \dot{X}) \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\} + \mathcal{L}\dot{X} + \frac{\partial \mathcal{L}}{\partial t} X = 0. \tag{8}$$

Upon making use of Lagrange's equations and the well-known result

$$\frac{d}{dt}\left\{ \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \right\} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{9}$$

one easily finds that Noether's condition (8) reduces to

$$\frac{d}{dt}\left\{ \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\dot{q}_i X - \psi_i) - \mathcal{L} X \right\} = 0, \tag{10}$$

which proves Eq. (3).

This result can be further generalized in the following fashion. Suppose the action is quasi-invariant under the transformation (1), that is, assume there exists a function $G(q,t)$ such that

$$\delta S = \int_{t_1'}^{t_2'} \mathcal{L}\left(q'(t'), \frac{dq'(t')}{dt'}, t'\right) dt' - \int_{t_1}^{t_2} \left\{ \mathcal{L}\left(q(t), \frac{dq(t)}{dt}, t\right) + \epsilon \frac{d}{dt} G(q(t),t) \right\} dt = 0. \tag{11}$$

Then Noether's condition becomes

$$\sum_i \left\{ \psi_i \frac{\partial \mathcal{L}}{\partial q_i} + (\dot{\psi}_i - \dot{q}_i \dot{X}) \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right\} + \mathcal{L}\dot{X} + \frac{\partial \mathcal{L}}{\partial t} X = \dot{G}, \tag{12}$$

and the conserved quantity associated with the symmetry is

$$\bar{C} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} (\dot{q}_i X - \psi_i) - \mathcal{L} X + G. \tag{13}$$

Notice that Eq. (12) provides, in terms of the Lagrangian alone, the necessary and sufficient condition for the action to be quasi-invariant under the transformation (1). Noether's condition (12), once solved for X, ψ_i and G, yields the full group of continuous symmetries of the action associated with the Lagrangian $\mathcal{L}$ [2]. This systematic and powerful approach has often been used to find constants of the motion (also called "invariants") for both conservative [2,3] and nonconservative [3,4] systems. Frequently the number of invariants thus found is sufficient for one to solve completely the equations of motion by algebraic means. However, this is hardly a practical procedure to solve equations of motion because, in the process of seeking solutions to Eq. (12), one stumbles against the original equations of motion one wanted to solve or even more difficult partial differential equations.

Here we wish to consider Noether's theorem as a method for finding invariants and sometimes solving equations of motion through identification of continuous symmetries of the action by nothing more than inspection. From the latter point of view the standard applications [1] of Noether's theorem in particle mechanics are the following: (i) The Jacobi integral is derived as a consequence of invariance under time translations; (ii) invariance with respect to translations along a cyclic variable q_k is used to prove conservation of the conjugate canonical momentum p_k; (iii) conservation of linear or angular momentum is deduced as a consequence of translational or rotational symmetry of the usual Lagrangian $\mathcal{L} = T - V$. Regrettably, joint space and time symmetries are seldom discussed in classical mechanics textbooks in the context of discrete mechanical systems. Typically, invariance under simultaneous transformations of space and time and the corresponding conserved quantities are a subject dealt with only in the relativistic theory of fields [5].

In this paper we try to fill this gap by supplying illustrations of spacetime symmetries displayed by simple one-dimensional systems. With this purpose we apply Noether's theorem to discrete nonconservative systems, to wit, a particle submitted to two kinds of nonconservative force: (i) A dissipative force linear in the velocity; (ii) a nonconservative force quadratic in the particle's velocity. In the absence of additional conservative forces nontrivial spacetime invariances of the action —that apparently had been overlooked so far— are explored to derive constants of the motion which lead to a complete solution of the equations of motion by algebraic means. In the less elementary case of the damped harmonic oscillator, use is made of two Lagrangians that do not differ by a total time derivative. Then, by inspection, Noether symmetries of the corresponding actions are identified which imply two independent constants of the motion, so that, again, the general solution to the equation of motion can be algebraically found.

2. DAMPED "FREE" PARTICLE

As a first and very simple example, consider a linearly damped "free" particle of mass m, whose equation of motion is

$$\ddot{x} + \lambda \dot{x} = 0, \tag{14}$$

where $\lambda > 0$ is the friction coefficient. A Lagrangian that yields Eq. (14) is [6]

$$\mathcal{L} = e^{\lambda t} \frac{m\dot{x}^2}{2}. \tag{15}$$

By just looking at the action integral associated with the above Lagrangian we are led to consider the *finite* transformation (α is a constant)

$$t' = t + \alpha, \tag{16a}$$

$$x'(t') = e^{-\lambda\alpha/2} x(t), \tag{16b}$$

that represents a time translation accompanied with a space dilatation. We find for the varied action

$$S' = \int_{t_1'}^{t_2'} e^{\lambda t'} \frac{m}{2} \left(\frac{dx'}{dt'}\right)^2 dt' = \int_{t_1+\alpha}^{t_2+\alpha} e^{\lambda t} e^{\lambda\alpha} \frac{m}{2} e^{-\lambda\alpha} \dot{x}(t)^2 \, dt'$$

$$= \int_{t_1}^{t_2} e^{\lambda t} \frac{m}{2} \dot{x}(t)^2 \, dt = S, \tag{17}$$

upon making the change of integration variable $t' = t + \alpha$ in the second integral in Eq. (17). Therefore, the action is invariant and Noether's theorem applies. The infinitesimal version of Eqs. (16) is obtained by taking $\alpha = \epsilon$, that is,

$$t' = t + \epsilon, \qquad x' = x - \tfrac{1}{2}\epsilon\lambda x, \tag{18}$$

whence, by comparison with Eqs. (1),

$$X = 1, \qquad \psi = -\tfrac{1}{2}\lambda x. \tag{19}$$

Alternatively, one may check the invariance of the action by inserting Eqs. (19) and (15) into Eq. (8) and verifying that it is identically satisfied. One way or another, we find from (3) the energylike integral of the motion

$$e^{\lambda t}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\lambda x\dot{x}\right) = C, \tag{20}$$

where C is a constant.

Since the Lagrangian (15) does not depend on x, the action is obviously invariant under infinitesimal space translations, so that with $X = 0$ and $\psi = 1$ Eq. (3) establishes that the canonical momentum conjugate to x is conserved, that is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = e^{\lambda t} m\dot{x} = D. \tag{21}$$

From Eq. (20) we get

$$x = \frac{2}{m\lambda\dot{x}}\left(Ce^{-\lambda t} - \tfrac{1}{2}m\dot{x}^2\right), \tag{22}$$

which, with the help of Eq. (21), takes the final form

$$x(t) = A + Be^{-\lambda t}, \tag{23}$$

where A and B are arbitrary constants. The function $x(t)$ given by Eq. (23) is the general solution to the equation of motion (14), that thus has been solved without any integration by just exploring the symmetry properties of the Lagrangian (15) and its corresponding action integral.

3. QUADRATIC FRICTION

Consider now a particle of mass m submitted to a nonconservative force proportional to the square of the particle's velocity, the corresponding equation of motion being

$$\ddot{x} + \gamma\dot{x}^2 = 0. \tag{24}$$

A well-known Lagrangian that yields Eq. (24) is [7,8]

$$\mathcal{L} = \tfrac{1}{2}m\dot{x}^2 e^{2\gamma x}. \tag{25}$$

The form of the above Lagrangian suggests that we investigate the effect on the action of the finite transformation

$$x'(t') = x(t) + \alpha, \tag{26a}$$

$$t' = e^{2\gamma\alpha}t. \tag{26b}$$

Notice that this transformation corresponds to a space translation together with a time dilatation, that is, here time and space play a role exactly opposite to that in Eqs. (16). The varied action becomes

$$S' = \int_{t_1'}^{t_2'} \frac{m}{2}\left(\frac{dx'}{dt'}\right)^2 e^{2\gamma x'}\,dt' = \int_{t_1}^{t_2} \frac{m}{2} e^{-4\gamma\alpha}\dot{x}(t)^2 e^{2\gamma x} e^{2\gamma\alpha} e^{2\gamma\alpha}\,dt = S, \tag{27}$$

so that the action is invariant and Noether's theorem is applicable. It is worth remarking that in the present case the Lagrangian is not invariant under the transformation (26), only the action is.

The infinitesimal version of Eqs. (26) with $\alpha = \epsilon$ furnishes $\psi = 1$ and $X = 2\gamma t$, so that Eq. (3) becomes

$$(\gamma t\dot{x} - 1)\dot{x}e^{2\gamma x} = C, \tag{28}$$

where C is a constant. Since the Lagrangian (25) does not depend explicitly on time, the action is invariant under time translations, and with $X = 1$ and $\psi = 0$ Eq. (3) yields the Jacobi integral as another conserved quantity:

$$\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L} = \tfrac{1}{2}m\dot{x}^2 e^{2\gamma x} = \tfrac{1}{2}mD^2, \tag{29}$$

with D a constant. This last equality is equivalent to

$$\dot{x}e^{\gamma x} = D. \tag{30}$$

Elimination of $\dot{x}$ from Eq. (30) followed by its insertion into Eq. (28) gives immediately

$$x(t) = A + \frac{1}{\gamma}\ln(B + \gamma t), \tag{31}$$

with A and B arbitrary constants. Again, the symmetry properties of the action engendered by the Lagrangian (25) have enabled us to find the general solution (31) to the equation of motion (24) by purely algebraic means.

4. Damped harmonic oscillator and inequivalent Lagrangians

Perhaps the previous cases may be regarded as too simple since the equations of motion are actually first-order differential equations for the velocity, the position as a function of time being given by two immediate integrations. As a nontrivial example, let us now discuss the damped oscillator. A well-known Lagrangian for the damped harmonic oscillator is [6]

$$\mathcal{L} = e^{\lambda t}\left(\tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}m\omega^2 x^2\right), \tag{32}$$

that generates the equation of motion

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0. \tag{33}$$

One can easily check that the action associated with the Lagrangian (32) is still invariant under the transformation (16), so that Noether's theorem now yields the conserved energylike quantity

$$e^{\lambda t}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\omega^2 x^2 + \tfrac{1}{2}m\lambda x\dot{x}\right) = C. \tag{34}$$

If one rewrites this equation in terms of the canonical momentum $p = e^{\lambda t}m\dot{x}$ one finds

$$e^{-\lambda t}\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 e^{\lambda t} + \frac{\lambda}{2}xp = C, \tag{35}$$

which is exactly the same constant of the motion obtained previously [9] by means of a time-dependent canonical transformation.

Independent constants of the motion for the damped harmonic oscillator have been found before by several different methods [7,10]. As to the use of Noether's theorem, it depends crucially on the form of the Lagrangian chosen. In the case of the damped "free" particle, for instance, the complicated Lagrangian [11]

$$\bar{\mathcal{L}} = (1 + e^{\lambda t})\frac{m\dot{x}^2}{4} + \frac{m\lambda}{2}x\dot{x}\ln\dot{x} - \frac{m\lambda^2}{4}x^2 \tag{36}$$

also gives rise to the equation of motion (14), but its associated action has none of the symmetries discussed in Sect. 2. Moreover, it is a known fact that the symmetries of a given action constitute a proper subgroup of the group of continuous symmetries of the equations of motion. In the case of the harmonic oscillator, for example, the standard action possesses a five-parameter invariance group that is a subgroup of the eight-parameter Lie group of the equation of motion [2]. As other elementary illustrations, notice that Eq. (14) is invariant under temporal translations, but the action induced by the Lagrangian (15) is not. Also, Eq. (24) is invariant with respect to a pure time dilatation $t' = \beta t$, whereas the action generated by the Lagrangian (25) does not exhibit such a symmetry.

The conclusion is that in order to realize all continuous symmetries of the equations of motion as Noether symmetries one must take into account more than one action. In other words, one has to deal with classes of inequivalent Lagrangians, that is, Lagrangians that do not differ by a total time derivative but give rise to equations of motion whose solutions are the same [12]. If some of these Lagrangians are known in advance, the Noether symmetries of their corresponding actions may be explored to produce independent constants of the motion in number sufficient to allow an algebraic solution of the equations of motion.

As an illustration of this point, let us resume the discussion of the damped harmonic oscillator. By mere inspection we have been unable to find a further symmetry of the action induced by the Lagrangian (32) that might lead to a second independent constant of the motion, which would then permit us to algebraically solve Eq. (33). But an inequivalent Lagrangian for Eq. (33) is known in the form [13]

$$\mathcal{L}_1 = \frac{2\dot{x} + \lambda x}{2\Omega x} \arctan\left(\frac{2\dot{x} + \lambda x}{2\Omega x}\right) - \tfrac{1}{2}\ln(\dot{x}^2 + \lambda x\dot{x} + \omega^2 x^2), \tag{37}$$

with

$$\Omega = (\omega^2 - \lambda^2/4)^{1/2}. \tag{38}$$

It is plain to see that the equation of motion (33) has an additional symmetry, namely, it is invariant under a pure space dilatation $x' = \beta x$, whose infinitesimal version reads

$$x'(t') = (1 + \epsilon)x(t), \tag{39a}$$

$$t' = t. \tag{39b}$$

With the help of $\ln(1 + \epsilon) = \epsilon$ one easily checks that the above transformation leaves the action associated with $\mathcal{L}_1$ quasi-invariant, more precisely, that Eq. (11) holds for the Lagrangian $\mathcal{L}_1$ with

$$G = -t. \tag{40}$$

By comparing Eqs. (39) with Eqs. (1) one identifies $X = 0$ and $\psi = x$, so that Eq. (13) with $\mathcal{L} = \mathcal{L}_1$ becomes

$$-x\frac{\partial \mathcal{L}_1}{\partial \dot{x}} + G = \frac{\delta}{\Omega}, \tag{41}$$

where the value of the constant of the motion was written as δ/Ω for convenience. An explicit calculation reduces the last equation to the form

$$-\arctan\left(\frac{2\dot{x} + \lambda x}{2\Omega x}\right) - \Omega t = \delta, \tag{42}$$

that is,

$$\dot{x} = -x\left[\frac{\lambda}{2} + \Omega\tan(\Omega t + \delta)\right]. \tag{43}$$

We have thus achieved our aim, for Eq. (42) is an independent constant of the motion with respect to Eq. (34). By inserting Eq. (43) into Eq. (34) and making use of the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$, one finds

$$mx^2\Omega^2\sec^2(\Omega t + \delta) = 2Ce^{-\lambda t}, \tag{44}$$

or, equivalently,

$$x(t) = Ae^{-\lambda t/2}\cos(\Omega t + \delta), \tag{45}$$

which is the general solution to the equation of motion of the damped harmonic oscillator.

5. CONCLUDING REMARKS

As a method for unveiling the whole group of continuous symmetries of a given set of equations of motion, Noether's theorem is of limited value inasmuch as a single action in general does not reflect all invariance properties of the equations of motion. In order to realize the latter as Noether symmetries it is necessary to deal with several inequivalent Lagrangians. In favorable circumstances, however, the Noether symmetry group of a single action may be large enough to furnish as many independent constants of the motion as are sufficient to completely solve the equations of motion by algebraic means. When this is not so but inequivalent Lagrangians are available, the sufficient number of invariants may sometimes be obtained by exploring easily detectable Noether symmetries of the corresponding inequivalent actions. Anyway, the approach based on Noether's theorem should be contrast with the one that relies on the invariance properties of the equations of motion themselves [3,7,14,15].

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