On the radiation by a charge in a material medium

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Abstract. In this work we obtain the exact non linear equation of motion, including radiation reaction, for a charge moving in an isotropic, homogeneous and linear medium. From this equation we recover the known results for the dynamics of charges in vacuum and discuss the involved approximations. Using the hypothesis of uniform motion in the exact non linear equation we obtain the usual results of the Cherenkov effect. Also we analyze several conceptual points in order to get a better understanding of the Cherenkov effect. We study the energy balance and show explicitly the existence of an external force to avoid a contradiction with the electromagnetic theory; we show that, in this case, the charge behaves as a transducer transforming completely mechanical work into radiation. We also discuss the physical origin of the Cherenkov radiation, giving an explanation why there is radiation even when the charge is in uniform motion.

Resumen. Se obtiene la ecuación de movimiento no lineal, incluyendo reacción de radiación, para una carga moviéndose en un medio lineal, homogéneo e isótropo. De esta ecuación se obtienen los resultados conocidos para la dinámica de cargas en el vacío y se discuten las aproximaciones que se utilizaron para ello. Utilizando la hipótesis de movimiento uniforme en la ecuación no lineal se obtienen los resultados conocidos para el efecto Cherenkov. Se analizan algunos aspectos conceptuales que ayudarán a un mejor entendimiento del efecto Cherenkov. Se estudia el balance de energía y se muestra que en este caso debe existir una fuerza externa para evitar una contradicción con la teoría electromagnética, es decir, en este caso la partícula se comporta como un transductor que convierte completamente energía mecánica en radiación.

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1. Introduction

One of the most important and well known results in classical electrodynamics is that, in vacuum, a charge radiates only when it is accelerated. On the other hand, it is also a well known result that when a charge travels with constant velocity through a material medium, it radiates if its velocity is greater than the velocity of light in that medium; that is $v > c/\sqrt{\epsilon}$, where ϵ is the dielectric function of the medium (we shall assume

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throughout this work $\mu = 1$, with μ the magnetic permeability). This phenomenon, known as the Cherenkov effect, is of great importance in high energy physics where it is used to determine the velocity of elementary particles. Owing to its usefulness this effect is discussed in several texts of electromagnetism [1-7]. In the early fifties Motz and Schiff [8] studied briefly the Cherenkov effect in a dispersive medium; ten years later Volkoff [9] clarified the apparent contradiction between the Cherenkov effect and Gauss's Law by taking into account the discontinuous nature of the electric field on the Cherenkov cone. In this excellent work it is evident the care needed to handle the electromagnetic fields on the surface of the Cherenkov cone, in absence of dispersion, because of the extremely singular behavior of the fields.

Some years ago two works appeared [10, 11] where the Cherenkov effect was discussed completing the analysis available in most common texts. Nevertheless, in the methods presented in the literature there are several points which deserve a careful discussion. As an example we can point out that in the method followed by Jackson [3], Panofsky and Phillips [5], and Good and Nelson [6] it is not clear how radiation can be predicted for a charge moving with constant velocity using Lienard-Wièchert potentials for an accelerated charge. This is a common source of confusion for the student. Furthermore, there is not an explicit mention of an external force needed to restore the energy lost by radiation, making difficult the understanding of the energy balance.

There are at least two methods to determine the energy radiated per unit time in the Cherenkov effect. One is due to Landau [2], who equals the power radiated by the charge with the work done in unit time by a stopping force $e\mathbf{E}$ self-exerted on the particle by the self-field. The other method [4], uses Poynting's vector to determine the energy radiated per unit time.

One of the most interesting and challenging problems that arises when the dynamics of a charge is studied is the analysis of the effects produced by the radiated fields on its source, that is, the radiation reaction. Hence it is interesting to discuss the dynamical effects on a charge due to the emission of Cherenkov radiation. In the present work we make explicit and thoroughly develop an implicit idea in the work of Landau [2] on the ionization losses by fast particles in matter. This idea suggests us to face the Cherenkov effect as a radiation reaction problem. This approach leads to a deep analysis of radiation reaction beyond the usual treatments in the texts of electromagnetism [4, 5, 12-16] and journals devoted to teaching. Then, we first discuss several physical aspects of the derivation of the equation of motion for a radiating charge, and exhibit the usual approximations that must be avoided to obtain a useful expression for the radiation reaction force for a charge moving in a medium. Our approach shows clearly how a charge in uniform motion can radiate and, as a particular case, we obtain the usual results of the Cherenkov effect. Also we clarify the energy balance for a charge moving in a medium and some physical aspects of the origin of Cherenkov radiation showing that if we assume a constant velocity for the radiating particle, there must exist an external force acting on it, so in this case we can think the particle as a transducer transforming completely mechanical work into radiation.

It is interesting to remark that the results obtained with the hypothesis of constant velocity in the discussion of the Cherenkov effect agree with usual experiments [17] because the change in the velocity of the particle, traveling through a medium, is so small that can be neglected for all practical purposes.

2. THE RADIATION REACTION IN A MATERIAL MEDIUM

The starting point in the analysis of the dynamics of a charge in vacuum is the postulate of the conservation of linear momentum of the system constituted by the charge, the fields produced by it, and the external fields [4]; this leads to the equation

$$\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}} = m_0\mathbf{a}. \tag{1}$$

Using the same reasoning it can be demonstrated that this equation is satisfied even when the particle is moving through a dielectric medium. In this case the effects of the medium are represented through the explicit form of each term in the preceding equation; for example, the electromagnetic fields in $\mathbf{F}_{\text{ext}}$ and $\mathbf{F}_{\text{self}}$ are different from their respective expressions in vacuum owing to the perturbation on the fields caused by the dielectric.

In order to determine the self-interaction force on a charge distribution ρ moving through a dielectric we assume that it is spherically symmetric, non rotating, and that its center follows the trajectory $\mathbf{r}(t)$, so the self-force can be expressed as

$$\mathbf{F}_{\text{self}}(\mathbf{r},t) = \int d^3x\rho(|\mathbf{x}-\mathbf{r}(t)|)\Big(\mathbf{E}_{\text{self}}(\mathbf{x},t) + \frac{\mathbf{v}(t)}{c}\times\mathbf{B}_{\text{self}}(\mathbf{x},t)\Big), \tag{2}$$

where $\mathbf{E}_{\text{self}}(\mathbf{x},t), \mathbf{B}_{\text{self}}(\mathbf{x},t)$ are the fields produced by the charge. It must be emphasized that, unlike the self-fields in vacuum, these fields are not produced directly and exclusively by the charge; they are the inhomogeneous solutions to the macroscopic Maxwell equations with the moving charge distribution as source. These fields contain then the response of matter to the perturbation produced by the charge. In this sense it is clearly a self-interaction phenomenon.

Since in the Cherenkov effect the velocity is assumed to be constant, we can follow two possible ways to determine $\mathbf{F}_{\text{self}}$: we can use the frame in which $\mathbf{v}(t)\neq 0$, or we can use the rest frame of the particle, where the medium moves with velocity $-\mathbf{v}(t)$, finding the self-fields with the Minkowsky equations for moving mediums. For simplicity we follow the first method.

In order to obtain the self-fields we work in the Coulomb gauge. In this gauge the fields are determined by the transverse component of the vectorial potential $\mathbf{A}(\mathbf{x},t)$ by means of the equations

$$\mathbf{E}_{\text{self}} = -\frac{1}{c}\frac{\partial\mathbf{A}_\perp}{\partial t}, \qquad \mathbf{B}_{\text{self}} = \nabla\times\mathbf{A}_\perp. \tag{3}$$

We omit the contribution $\mathbf{E}' = -\nabla\phi_c$ because, as is well known, in Coulomb gauge ϕ_c is an electrostatic potential and for a spherically symmetric charge distributions its contribution to the self-force vanishes [3, 4, 13]. At the same time the Fourier transform of $\mathbf{A}_\perp, \tilde{\mathbf{A}}_\perp(\mathbf{k},\omega)$, satisfies the equation

$$\left(k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}\right)\tilde{\mathbf{A}}_\perp(\mathbf{k},\omega) = \frac{4\pi}{c}\tilde{\mathbf{J}}_\perp(\mathbf{k},\omega), \tag{4}$$

where

$$\tilde{\mathbf{J}}_\perp(\mathbf{k},\omega) = \hat{\mathbf{k}} \times (\tilde{\mathbf{J}} \times \hat{\mathbf{k}}). \tag{5}$$

To obtain $\mathbf{A}(\mathbf{x},t)$ it is usually enough to take the inverse Fourier transform of $\tilde{\mathbf{A}}(\mathbf{k},\omega)$; this is possible since $\mathbf{J}(\mathbf{x},t)$, and consequently $\tilde{\mathbf{J}}(\mathbf{k},\omega)$, are known functions. Nevertheless, when the radiation reaction problem is studied, as in this case, the trajectory of the particle is unknown and then $\mathbf{J}(\mathbf{x},t)$ and $\tilde{\mathbf{J}}(\mathbf{k},\omega)$ are not specified beforehand. Moreover, it is risky to assume that the system of Maxwell's equations is complete even when $\mathbf{J}(\mathbf{x},t)$ is unknown, but it is the usual and unique starting point with which we can count [18, 19]. It is evident that this fact deserves a careful and deep study, but now we use this hypothesis in the same way as in the study of radiation reaction in vacuum. That is, we consider $\mathbf{J}$ as unknown and at the end, in order to recover the usual results, we assume a particular $\mathbf{J} = \rho\mathbf{v}$, with constant $\mathbf{v}$. The merits and deficiencies of this hypothesis will be seen in the obtained results.

As a point of departure we have the general solution to the Maxwell equations that can be obtained from Eq. (4), that is

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}_{\text{hom}} + \frac{2}{(2\pi)^{3/2}c}\int dt' \int d^3k \int d\omega \tilde{\rho}(k) \frac{\hat{\mathbf{k}} \times (\mathbf{v}(t') \times \hat{\mathbf{k}})}{k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}} \cdot e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{r}(t'))-\omega(t-t')]}. \tag{6}$$

From this expression it is easy to get the fields which contain the contributions of the charge and the medium,

$$\mathbf{E}_{\text{self}}(\mathbf{x},t) = -\frac{2}{(2\pi)^{3/2}c^2}\int dt' \int d^3k \int d\omega \tilde{\rho}(\mathbf{k}) \left(\frac{-i\omega\hat{\mathbf{k}} \times (\mathbf{v}(t') \times \hat{\mathbf{k}})}{k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}}\right) \cdot e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{r}(t'))-\omega(t-t')]}, \tag{7}$$

and

$$\mathbf{B}_{\text{self}}(\mathbf{x},t) = \frac{2}{(2\pi)^{3/2}c}\int dt' \int d^3k \int d\omega \tilde{\rho}(\mathbf{k}) \left(\frac{i\mathbf{k} \times \mathbf{v}(t')}{k^2 - \frac{\omega^2\epsilon(\omega)}{c^2}}\right) \cdot e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{r}(t'))-\omega(t-t')]}. \tag{8}$$

The self-force can be obtained directly from these equations. The electric and magnetic contributions to it are, respectively,

$$\mathbf{F}_{\substack{\text{self}\\ \text{elec}}}(\mathbf{r},t) = -\frac{2}{c^2}\int dt' \int d^3k \int d\omega |\tilde{\rho}(\mathbf{k})|^2 \left(\frac{-i\omega\hat{\mathbf{k}}\times(\mathbf{v}(t')\times\hat{\mathbf{k}})}{k^2-\frac{\omega^2\epsilon(\omega)}{c^2}}\right)\cdot e^{i[\mathbf{k}\cdot(\mathbf{r}(t)-\mathbf{r}(t'))-\omega(t-t')]} \tag{9a}$$

and

$$\mathbf{F}_{\substack{\text{self}\\ \text{mag}}}(\mathbf{r},t) = -\frac{2}{c^2}\int dt' \int d^3k \int d\omega |\tilde{\rho}(\mathbf{k})|^2 k \left(\frac{i\mathbf{v}(t)\times(\hat{\mathbf{k}}\times\mathbf{v}(t'))}{k^2-\frac{\omega^2\epsilon(\omega)}{c^2}}\right)\cdot e^{i[\mathbf{k}\cdot(\mathbf{r}(t)-\mathbf{r}(t'))-\omega(t-t')]}. \tag{9b}$$

The preceding expressions are as general as the solutions to the macroscopic Maxwell equations and the Lorentz force. From them it is possible to get all the necessary information to discuss some special points of interest. Now, it is impossible to continue our analysis because $\epsilon(\omega)$ is unspecified, and the general characteristics of this function are not enough to trace the most relevant aspects of the self-force. But, using a particular model for $\epsilon(\omega)$, such as that of Drude-Lorentz, we can integrate it, in an approximated way, over ω. The results are useless for our purposes. For a thoroughly discussion of the Cherenkov effect taking dispersion into account see Allison [17].

In order to reproduce the usual results for radiation reaction in vacuum [4, 5, 20-23], and get a better idea of the approximations involved, it is enough to assume that $\epsilon(\omega)$ is constant. Although this supposition is unphysical, it permits us to integrate over ω and get some interesting results. After integrating we obtain

$$\begin{aligned}\mathbf{F}_{\text{self}} = {} & 4\pi \int dt' \int d^3k \frac{|\tilde{\rho}(\mathbf{k})|^2}{\epsilon}\hat{\mathbf{k}}\times(\mathbf{v}(t')\times\hat{\mathbf{k}})\cos\left(\frac{kc(t-t')}{\sqrt{\epsilon}}\right)\Theta(t-t')\cdot \\ & \cdot e^{i\mathbf{k}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]} \\ & - 4\pi i \int dt' \int d^3k \frac{|\tilde{\rho}(\mathbf{k})|^2}{c\sqrt{\epsilon}}\mathbf{v}(t)\times(\hat{\mathbf{k}}\times\mathbf{v}(t'))\sin\left(\frac{kc(t-t')}{\sqrt{\epsilon}}\right)\Theta(t-t')\cdot \\ & \cdot e^{i\mathbf{k}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]},\end{aligned} \tag{10}$$

where Θ is the Heaviside step function. It must be noted that in the limit $\epsilon \to 1$ this equation does not reduced to the usual equation for a charge moving in vacuum [20-23], because in that case the analysis of the radiation reaction is done in the rest frame of the particle, that is $\mathbf{v}(t) = 0$, consequently there is not a magnetic contribution. On the other hand, Eq. (8) is a non-linear one; this can be seen easily expanding the product

$$v(t')e^{i\mathbf{k}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]}$$

in a Taylor's series around time t. In this expansion terms appear involving products of $\mathbf{v}(t)$ with its higher order derivatives or products of them, so to obtain a linear equation, as in the usual analysis of radiation reaction in vacuum, it will be necessary to approximate the exponential to unity. However, this approximation is not enough since in the magnetic contribution the product $\mathbf{v}(t)\cdot\mathbf{v}(t')$ appears. Then it is necessary to impose the restriction $\mathbf{v}(t) = 0$, which means that the description is made in the rest frame of the charge. Both approximations and the limit $\epsilon \to 1$ lead us to the usual results.

Carrying out all angular integrations in Eq. (10) and approximating the exponential to unity lead to

$$\mathbf{F}_{\text{self}} = -\frac{32\pi^2}{3c}\int dt'\mathbf{a}(t')\int dkk\frac{|\tilde{\rho}(\mathbf{k})|^2}{\sqrt{\epsilon}}\sin\left(\frac{kc(t-t')}{\sqrt{\epsilon}}\right)\Theta(t-t'), \tag{11}$$

and defining

$$g(t-t') = \int dkk\frac{|\tilde{\rho}(\mathbf{k})|^2}{\sqrt{\epsilon}}\sin\left(\frac{kc(t-t')}{\sqrt{\epsilon}}\right) \tag{12}$$

results in

$$\mathbf{F}_{\text{self}} = -\frac{32\pi^2}{3c}\int dt' g(t-t')\mathbf{a}(t')\Theta(t-t'), \tag{13}$$

which, in the limit $\epsilon = 1$, is exactly the linear integrodifferential equation for the radiation reaction discussed in several papers [20-23]. In the case of a point particle Eq. (13) reduces to the famous Abraham-Lorentz expression

$$\mathbf{F}_{\text{self AL}} = \frac{2e^2}{3c^3}\,\dot{\mathbf{a}} - \delta m_e\mathbf{a}, \tag{14}$$

where δm_e is the electromagnetic mass, which is 4/3 of the electrostatic energy divided by c^2, a result usually regarded as a contradiction with special relativity. The respective expression in a non dispersive dielectric medium is

$$\mathbf{F}_{\text{self AL}} = \frac{2e^2}{3c^3}\sqrt{\epsilon}\dot{\mathbf{a}} - \delta m'_e\mathbf{a}, \tag{15}$$

which does not give us more information that the equation in vacuum. In this case the electromagnetic mass $\epsilon\delta m_e$ represents not only the contribution of the vacuum self-field to the inertia, but also the contribution due to the medium.

3. The Cherenkov effect

Although Eq. (13) is important because it led us to obtain a linear equation of motion with the well known virtues of the equation of motion for extended charges in vacuum, it is only an approximated expression for the problem of self-interaction for a particle moving through matter. If we want to obtain the usual results of the Cherenkov effect from this equation we have to face a problem: it has been used the approximation $\mathbf{v}(t) = 0$ which hinders us to analyze the case of our interest, when $v > c/\sqrt{\epsilon}$. However, although our non-linear Eq. (10), which is exact and valid even for dispersive mediums, is unhandy, it takes a simple form in the case of the Cherenkov effect where $\mathbf{v}$ is constant.

As we mentioned before, to arrive to Eq. (10) we assumed the consistency of Maxwell equations even when the current density is unknown. Now, if we assume the particle moving with constant velocity its trajectory becomes known and we will be able to determine $\mathbf{F}_{\text{self}}$ and the radiation emitted; as we shall see, with this result it is possible to understand the energy balance.

One of the most famous derivations of the Abraham-Lorentz equation is that due to Planck [24]. This derivation assumes that the power radiated is equal to the work done by the radiation reaction force per unit time, so if we use the known self-force expression with constant velocity in order to get the unknown power radiated, which in this case is proportional to the radiated energy per unit length, we must obtain the usual expression for the Cherenkov effect. As we shall see, this really happens. Although this idea is implicit in Landau's work [2] it has not been developed. In this work we make it.

Taking into account the idea of Planck, our next step is to introduce in Eq. (10), which contains the information of the dispersive medium and the non-linear terms, our hypothesis of constant velocity for the particle. So

$$e^{i[\mathbf{k}\cdot(\mathbf{r}(t)-\mathbf{r}(t'))]} = e^{i\mathbf{k}\cdot\mathbf{v}(t-t')}. \tag{16}$$

The integration over t' can be carried out and it leads us to $\delta(\omega + \mathbf{k}\cdot\mathbf{v})$; without loss of generality we can choose the particle moving in the x direction, then performing the integrations we get

$$\mathbf{F}_{\substack{\text{self}\\ \text{elec}}} = \frac{4\pi i}{vc^2}\int d\omega \int d^2k |\tilde{\rho}(\mathbf{k})|^2 \frac{\omega\mathbf{v}\left(1 - \frac{\omega^2}{k^2v^2}\right)}{q^2 + \frac{\omega^2}{v^2}\left(1 - \frac{v^2\epsilon(\omega)}{c^2}\right)}, \tag{17a}$$

where

$$q^2 = k_y^2 + k_z^2$$

and

$$\mathbf{F}_{\substack{\text{self}\\ \text{mag}}} = 0. \tag{17b}$$

Because of the symmetry of the problem the transversal components to the velocity are zero, and using $d^2k = q\,dq\,d\varphi$ and the dispersion relation $\omega^2/k^2 = c^2/\epsilon(\omega)$, Eq. (17a) transforms into

$$\mathbf{F}_{\substack{\text{self}\\\text{elec}}} = \frac{8\pi^2 i}{vc^2}\mathbf{v}\int d\omega\int dq\; q|\tilde{\rho}(\mathbf{k})|^2\frac{\omega\left(1-\frac{c^2}{v^2\epsilon(\omega)}\right)}{q^2+\frac{\omega^2}{v^2}\left(1-\frac{v^2\epsilon(\omega)}{c^2}\right)}. \tag{18}$$

In the limit of a point particle it reduces to

$$\mathbf{F}_{\substack{\text{self}\\\text{elec}}} = \frac{ie^2}{\pi c^2}\frac{\mathbf{v}}{v}\int d\omega\int dqq\omega\frac{\left(1-\frac{c^2}{v^2\epsilon(\omega)}\right)}{q^2+\frac{\omega^2}{v^2}\left(1-\frac{v^2\epsilon(\omega)}{c^2}\right)}. \tag{19}$$

This expression coincides with that obtained by Landau [2].

It must be emphasized that if a non dispersive medium is considered [$\epsilon(\omega)$ is constant], the integrand of the preceding equations depends in a crucial way on the magnitude of the velocity of the particle with respect to velocity of light. If the speed of the particle is less than $c/\sqrt{\epsilon}$, the integrand is an odd function with no poles, and the integration limits are symmetric so $\mathbf{F}_{\text{self}} = 0$; on the other hand, if the speed of the particle is greater than $c/\sqrt{\epsilon}$, the integrand have poles and the integral can be different from zero. We have then as a consequence that if the particle travels with constant velocity in vacuum or through a medium with velocity $v < c/\sqrt{\epsilon}$ it does not radiate, but when its velocity is $v > c/\sqrt{\epsilon}$ in the medium, or if it could travel in vacuum faster than light, then there would be radiation producing a self-force different from zero, $\mathbf{F}_{\text{self}} \neq 0$.

In the dispersive case the self-force depends on the detailed behaviour of $\epsilon(\omega)$, so without a model for it we can not obtain general conclusions. In fact we can see from Eq. (19) that the specific behaviour of $v^2\epsilon(\omega)/c^2$ determine if there are or not poles in that equation.

If the velocity of the charge is constant it implies that the power radiated is

$$\mathcal{P} = \frac{d\mathcal{E}}{dt} = v\frac{d\mathcal{E}}{dx} = -\mathbf{F}_{\text{self}}\cdot\mathbf{v}, \tag{20}$$

then

$$\frac{d\mathcal{E}}{dx} = -\frac{ie^2}{\pi c^2}\int d\omega\omega\left(1-\frac{c^2}{v^2\epsilon(\omega)}\right)\int dq\frac{q}{q^2+\frac{\omega^2}{v^2}\left(1-\frac{v^2\epsilon(\omega)}{c^2}\right)}. \tag{21}$$

Performing the integration over q we get

$$\frac{d\mathcal{E}}{dx} = \frac{e^2}{c^2}\int d\omega\omega\left(1-\frac{c^2}{v^2\epsilon(\omega)}\right), \tag{22}$$

which is precisely the usual result [3, 25] for the Cherenkov effect.

On the other hand, when we integrate Eq. (10) over t' we get a Dirac's delta function which impose the condition $\omega^2 = (\mathbf{k}\cdot\mathbf{v})^2 = k^2 v^2 \cos\theta$. This condition and the dispersion relation imply that

$$\frac{c^2}{v^2\epsilon(\omega)} = \cos^2\theta_c, \tag{23}$$

or

$$\cos\theta_c = \pm\frac{c}{v\sqrt{\epsilon}}. \tag{24}$$

So we have obtained the Cherenkov angle, θ_c, which defines the cone within which there can be radiation. We must remark that this result is independent of the charge distribution. However, the spectral distribution depends on the charge distribution [25].

Another point that deserves special attention is the following: Planck's idea about radiation reaction, according to which the power radiated is equal to the work done by the self force per unit time, plays a fundamental role in the determination of the energy lost per unit length in Cherenkov radiation. As it has been shown in another paper [26] this idea, which seems so reasonable and suggestive, is in general wrong in vacuum. However, in this case it has given excellent results; it is so, as we shall see, due to the presence of a medium and the uniform motion of the charge.

As we have seen, the existence of radiation when the velocity of the particle is greater than $c/\sqrt{\epsilon}$ has as a consequence the existence of a radiation reaction force that disappears when $v < c/\sqrt{\epsilon}$. In this way it is clear that if the Cherenkov effect can be faced as a radiation reaction problem, it is also clear that the relation with the radiation reaction in vacuum is not immediate; the most relevant differences reside in the linear character of the usual equation of motion in vacuum. Besides, in vacuum the trajectory and the velocity of the particle are unknown even when the external force is known, while in the Cherenkov effect the velocity is known. This is precisely the point that we shall discuss in the following section from the perspective of the energy balance. As we see, the Cherenkov effect considered as a radiation reaction problem leads to the problem of radiation reaction itself, a yet properly unsolved classical problem.

4. Energy balance

As we can see from the radiation reaction problem in material mediums, even when it is assumed a constant velocity for the particle there is radiation if the speed of the particle is greater than $c/\sqrt{\epsilon}$. However, it is also important to remark that without the presence of an external force acting on the particle there will be an obscure source of energy that restores the radiated energy. This fact is the basic idea in Planck's theory of the radiation reaction. We can show easily that in the Cherenkov effect it is strictly necessary the presence of an external force to restore the energy lost by the charge. To see this we will use Poynting's expression for the energy balance [3],

$$-\frac{dU_{\text{mec}}}{dt} = -\int d^3x\,\mathbf{J}\cdot\mathbf{E} = \oint \mathbf{S}\cdot dSur + \frac{dU_{\text{elm}}}{dt}, \tag{25}$$

where, as usual, the first term on the right side is related to the power radiated,

$$\mathcal{P} = \oint \mathbf{S} \cdot dSur. \tag{26}$$

Let us first analyze the left hand side of Eq. (25). It is clear that $\mathbf{J}$ is the current density, which in this case is the convected current produced by the charge itself in uniform motion, that is $\mathbf{J} = \rho\mathbf{v}$. $\mathbf{E}$ is the electric field and it is necessary to clarify what kind of field we are working with. In some modern texts [4] where Poynting's theorem is derived, $\mathbf{E}$ is an external electric field, that is a field not produced by $\mathbf{J}$ itself. However, this point of view has some problems [27]. In other deductions Eq. (25) is obtained through a formal manipulation of the Maxwell equations, so $\mathbf{E}$ represents the general solution to the inhomogeneous Maxwell equations and then includes not only the homogeneous solution $\mathbf{E}_h$, that can be associated with external fields, but the particular solution $\mathbf{E}_i = \mathbf{E}_{\text{self}}$, which is associated with the fields produced by $\mathbf{J}$. So we have that

$$\int d^3x\ \mathbf{J} \cdot \mathbf{E} = \int d^3x\ \rho\mathbf{v} \cdot (\mathbf{E}_h + \mathbf{E}_{\text{self}}) = \mathbf{v} \cdot (\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}}), \tag{27}$$

but as we have assumed $\mathbf{v} =$ constant

$$\frac{dU_{\text{mec}}}{dt} = 0$$

and then

$$\mathbf{F}_{\text{ext}} = -\mathbf{F}_{\text{self}}$$

where the external force can have in general an electromagnetic and a non electromagnetic contribution. Using this result in Eq. (25) gives us

$$\mathcal{P} = -\frac{dU_{\text{elm}}}{dt}, \tag{28}$$

and from Eq. (20)

$$-\mathbf{F}_{\text{self}} \cdot \mathbf{v} = \mathbf{F}_{\text{ext}} \cdot \mathbf{v} = -\frac{dU_{\text{elm}}}{dt} = \mathcal{P}. \tag{29}$$

It is interesting to remark that although a charge in uniform motion in a medium produces static fields, these fields induce a time dependent polarization in the medium, due to the change with time of the distance from the charge to every element of the medium. So U_{elm} varies with time.

Now we have three interesting points: In the first place we can see how Jackson's approach to the Cherenkov effect is also a radiation damping problem, hence the full equivalence of Landau's and Jackson's approaches, as we could expect; in the second place, from Eq. (29) we can conclude that if $\mathcal{P} \neq 0$ then there must necessarily exist an external force which restores to the charge the energy lost by radiation. This external force can be mechanical or electromagnetic, but if it does not exist it is impossible to maintain a constant velocity for the particle; finally, we have shown Eq. (29) to be a consequence of Poynting's theorem, so in this case we show Planck's conjecture is correct and the charge behaves as a transducer which transform completely the energy obtained from the external force into radiation. This situation does not appear in vacuum, being Schott's term a prove of it. This term is in part responsible for the confusions with the energy balance in the radiation reaction problem in vacuum. We must emphasize that this conspicuous fact is not discussed in most texts on electromagnetism [4, 5, 12-16, 28]. It must be also remarked that our Eq. (10) remains valid even when the velocity of the charge changes; in real experiments [17] there is not an external force to restore the energy lost by radiation, but the change in the velocity of the charge is so small that the hypothesis of constant velocity is quite well.

5. Velocity and radiation fields

Although it is apparent the transformation of mechanical energy into radiation even with a constant velocity of the charge, it is not clear the relation between Cherenkov radiation and the usual assumptions used to analyze the motion of a charge in vacuum. When the radiation emitted by a charge moving in vacuum is studied it is strongly remarked that the radiation field varies as $|\mathbf{x}-\mathbf{r}'|^{-1}$ and depends on the acceleration, meanwhile the velocity field varies as $|\mathbf{x}-\mathbf{r}'|^{-2}$ and it is considered as a field bounded to the particle.

The root of many of the difficulties in the understanding of the Cherenkov effect lies precisely in this separation between the velocity and radiation fields, since in the Cherenkov effect the radiated field depends on the velocity. According to this fact, it is necessary to clarify this apparent contradiction.

In order to clarify the problem it will be useful to analyze the behaviour of the electromagnetic potential in terms of the ratio of the velocity of the charge to the velocity of light in the dielectric. In order to do it, it will be enough to consider a non dispersive medium. In the space $(\mathbf{x},t)$ the electromagnetic potential $\mathbf{A}_\perp(\mathbf{x},t)$ satisfies the equation

$$\left(\nabla^2 - \frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_\perp(\mathbf{x},t) = -\frac{4\pi}{c}\mathbf{J}_\perp(\mathbf{x},t); \tag{30}$$

for a point particle traveling with constant velocity the current density $\mathbf{J}_\perp$ is proportional to $\delta(\mathbf{x}-\mathbf{v}t)$. If, as we did before, we take the motion in the x direction, then

$$\mathbf{J}_\perp(\mathbf{x},t) = \mathbf{J}_\perp(x-vt,y,z,); \tag{31}$$

we can expect the same functional dependence for the potential $\mathbf{A}_\perp$, namely,

$$\mathbf{A}_\perp(\mathbf{x},t) = \mathbf{A}_\perp(x-vt,y,z,). \tag{32}$$

This means that at time $t + t_0$ the field has the same functional form that it had in the previous position $-vt_0$, so the potential must satisfy

$$\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} = v^2 \frac{\partial^2 \mathbf{A}_\perp}{\partial x^2}. \tag{33}$$

This relation permit us to write the expression (30) in the form

$$\nabla_R^2 \mathbf{A}_\perp + \frac{1}{v^2}\left(1 - \frac{v^2}{c^2}\epsilon\right)\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}_\perp, \tag{34}$$

where

$$\nabla_R^2 = \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

As we can see immediately, the electromagnetic potential $\mathbf{A}$ behaves in a completely different way if the particle's velocity is less or greater than $c/\sqrt{\epsilon}$, since if the coefficient of the second time-derivative is positive the preceding differential equation is an elliptical one, of Poisson's type. This fact is characteristic of the three dimensional electrostatic behaviour. It is convenient to define the variable

$$\tilde{x} = \mathrm{x}(1 - \epsilon\beta^2)^{-1/2}, \tag{35}$$

in terms of which the Eq. (34) can be written as

$$\left(\nabla_R^2 + \frac{\partial^2}{\partial \tilde{x}^2}\right)\mathbf{A}_\perp = \tilde{\nabla}^2 \mathbf{A}_\perp = -\frac{4\pi}{c}\mathbf{J}_\perp. \tag{36}$$

On the other hand, if $v > c/\sqrt{\epsilon}$, the equation will be an hyperbolic one, of D'Alembert's type, but in two spatial and one time dimensions. In this way the behaviour of the wave motion depends notably on the number of spatial dimensions; indeed, we have seen that in the Cherenkov effect the propagation of radiation is quite different from that in three spatial dimensions.

By means of the analysis of the wave equation, in two or three dimensions [6], we can give a better idea of the preceding discussion. The Cauchy solution of the three dimensional wave equation shows that a perturbation produced at time t' and at position $\mathbf{r}'$ travels in space like an spherical wave front centered in $\mathbf{r}'$, without changing its form, but with its size diminished in such way that behind the wave front there is not any perturbation at all. Of course the wave front emerges from $\mathbf{r}'$. In the bidimensional case, if a perturbation is produced it will travel without changing its form, but there will exist a remanent wake behind the wave front. This kind of behavior is also found in the inhomogeneous solution of the wave equation. These differences in the wave behaviour can be easily appreciated from their respective Green functions: in the three dimensional case the Green function is $|\mathbf{x} - \mathbf{r}'|^{-1}\delta(t' + [|\mathbf{x} - \mathbf{r}'|/c] - t)$, which shows the change in the amplitude and how the pulse is concentrated in a sphere, defined by the argument of the δ function, that grows up with time; furthermore the perturbation at $(\mathbf{x},t)$ depends only on what happened in just one point in the past $(\mathbf{r}',t')$. In the bidimensional case the Green function is $2c[c^2(t-t')^2-|\mathbf{x}-\mathbf{r}'|^2]^{-1/2}\cdot\Theta[c(t-t')-|\mathbf{x}-\mathbf{r}'|]$, which, like in the three dimensional case, shows a singularity in $|\mathbf{x}-\mathbf{r}'|=c(t-t')$, but for $|\mathbf{x}-\mathbf{r}'|<c(t-t')$ there is a wake, a remanent perturbation. In the case of Cherenkov radiation the field at the time t and position $\mathbf{x}$ is a consequence of what happened in the interior region defined by the Heaviside function, whose boundary defines an edge called the Cherenkov cone that at the same time represents indeed the radiation.

According to the preceding discussion, when the velocity of the charge is $v<c/\sqrt{\epsilon}$ we have the usual electrostatic behaviour that varies as $|\mathbf{x}-\mathbf{r}'|^{-2}$, but when $v>c/\sqrt{\epsilon}$ the behaviour is ruled by a bidimensional wave equation whose dependence on $|\mathbf{x}-\mathbf{r}'|$ changes drastically.

The Green function reflects the physical fact that all the elements of the medium inside the cone contribute to the radiation: the envelope of all the wave fronts produced by every radiating dipole forms the Cherenkov radiation front. In vacuum, the Green function in three dimensions shows how the charge in motion is the unique source of radiation.

As we have shown, in the Cherenkov effect there are not contradictions. If $v<c/\sqrt{\epsilon}$ we have an usual static behaviour, but if $v>c/\sqrt{\epsilon}$ the behaviour of the electromagnetic potential is ruled by the bidimensional wave equation whose characteristics are remarkably different from those of the three dimensional wave behaviour. From the mathematical point of view, the change in the field behaviour when the particle goes from the case of "no propagation" (no radiation) to the case of "propagation" (radiation), previously related to the presence or lack of poles in Eq. (19), now appears in the electromagnetic potentials as a change in the behaviour of the functions describing it. In the first case they are described in terms of modified Hankel functions K, with an asymptotic behaviour that falls off rapidly with $|\mathbf{x}-\mathbf{r}'|-1$, while in the second case the potentials are described in terms of Hankel functions H which asymptotically vary as $|\mathbf{x}-\mathbf{r}'|^{-1/2}$, but since we are working with a bidimensional problem, then in this case we have a typical radiative behaviour.

6. Conclusions

The Cherenkov effect permits us to analyze several physical aspects of classical electrodynamics, in particular the challenging problem that arises from considering the effect produced on the dynamics of a charge when we take into account the radiation produced by it. It must be emphasized that the usual approaches to radiation reaction are inadequate to face the Cherenkov effect since the common linear approximation is too coarse to make the job. Our analysis shows several interesting and conceptually difficult points, like those in the derivation of the radiation damping force itself, and the remarkable compatibility of the energy balance for a radiating charge in a medium and Planck's approach in vacuum. Also our approach makes apparent that the Cherenkov effect is a good example where the distinction between self-field and external fields in Poynting's theorem is significant [27], and clarifies the energy balance in the Cherenkov effect. Hence this effect becomes a problem of great conceptual and didactic importance, as we have seen throughout this work. We have clarified in an elementary way most of the difficulties the students face in studying the Cherenkov effect: the apparent contradiction between Cherenkov radiation and the usual radiation criterion used in the analysis of radiation in vacuum, within the framework of classical electrodynamics.

On the other hand, the energy balance used in electromagnetism has some difficulties, mainly when we include the radiation reaction force in the analysis of the motion of charges.

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