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Susy quantum mechanics and cochains*

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Abstract. We consider susy transformations in quantum mechanics. We show they give rise to a 1 cochain. We relate the breaking of susy to the non triviality of this cochain.

Resumen. Considerando transformaciones supersimétricas en mecánica cuántica, se muestra que ellas dan lugar a una co-cadena. El rompimiento de supersimetría se relaciona con el carácter no trivial de esta co-cadena.

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It has been pointed out that the existence of non trivial cochains (cocycles) may be a signal of anomalies in some quantum field theories [1]. This fact motivated us to reconsider susy quantum mechanics so as to try to relate the breaking of susy with a non trivial 1 cochain.

1 cochains appear whenever we consider transformations which are symmetries of the action but not of the Lagrangian. At the quantum level this is reflected as a phase (ω_1) in the equation describing the wave function transformation [2]:

$$q \xrightarrow{g} q^g \qquad \mathcal{U}(g)\psi(q) = e^{i2\pi\omega_1(q,g)}\psi(q^g), \tag{1}$$

ω_1 is called trivial if there exists an $\alpha(q)$ such that

$$2\pi\omega_1(q,g) = \alpha(q^g) - \alpha(q).$$

A well known example [2] of cocycles is provided by Galileo's transformations for a free particle. In that case

$$q^v = q - vt, \qquad \mathcal{L}(q,\dot{q},t) = \frac{1}{2m}\dot{q}^2,$$

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$$\delta\mathcal{L} = \frac{d}{dt} m \left(\tfrac{1}{2} v^2 t - qv\right), \tag{2}$$

the generator of the symmetry, *i.e.* the conserved charge is

$$Q = mqv - m\dot{q}vt. \tag{3}$$

Owing to the canonical commutation relations, the wave function transforms under Galileo's transformations according to

$$e^{iQ}\psi(q) = e^{i2\pi\omega_1(q,v)}\psi(q^v = q - vt),$$

where

$$\omega_1(q,v) = mqv - \tfrac{1}{2}mv^2 t.$$

In this case ω_1 is trivial and

$$2\pi\alpha(q) = -\frac{mq^2}{2t}.$$

It is convenient to introduce

$$\phi = e^{+i2\pi\alpha(q)}\psi(q).$$

In terms of ϕ the Schrödinger equation for ψ is

$$i\frac{\partial}{\partial t}\exp\left(\frac{imq^2}{2t}\right)\phi = \frac{p^2}{2m}\exp\left(\frac{imq^2}{2t}\right)\phi.$$

Simplifying this equation we obtain

$$i\frac{\partial\phi}{\partial t} = H'\phi,$$

where

$$H' = \frac{1}{2m}\left(p + \frac{mq}{t}\right)^2 - \frac{mq^2}{2t^2}.$$

This Hamiltonian leads to the Lagrangian

$$\begin{aligned}\mathcal{L}' &= m\left(\frac{\dot{q}^2}{2} - \frac{q\dot{q}}{t} + \frac{q^2}{2t^2}\right)\\ &= \mathcal{L} + \frac{d}{dt}\big(2\pi\alpha(q)\big).\end{aligned} \tag{4}$$

This new Lagrangian is invariant under Galileo transformations and therefore at the quantum level it is free of cocycles. At the classical level $\mathcal{L}'$ and $\mathcal{L}$ (Eq. 4) are equivalent, at the quantum level they differ in the appearance of a 1 cochain.

Let us now consider the theory described by the Lagrangian [3]

$$\mathcal{L} = \tfrac{1}{2}\dot{x}^2 + \tfrac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \tfrac{1}{2}\big(u(x)\big)^2 - \tfrac{1}{2}[\psi^*, \psi]u'(x), \tag{5}$$

where the prime over u denotes a derivative with respect to the x and ψ and ψ^* are grassmanian variables which satisfy

$$\{\psi^*, \psi\} = \{\psi, \psi\} = \{\psi^*, \psi^*\} = 0.$$

This Lagrangian describes a supersymmetric theory as under the susy transformations:

$$\delta x = i(\psi\epsilon - \epsilon^*\psi^*),$$

$$\delta\psi = \epsilon^*(\dot{x} - iu),$$

$$\delta\psi^* = \epsilon(\dot{x} + iu),$$

with ϵ, ϵ^* constant grassmanian quantities, the variation of the Lagrangian is given by

$$\delta\mathcal{L} = \frac{1}{2}\frac{d}{dt}\left(\tilde{Q}_1\epsilon + \epsilon^*\tilde{Q}_2\right), \tag{6}$$

where

$$\tilde{Q}_1 = (i\dot{x} + u)\psi, \qquad \tilde{Q}_2 = (i\dot{x} - u)\psi^*. \tag{7}$$

Since the Lagrangian is not invariant [see Eq. 6] we expect the appearance of a 1 cochain.

Taking into account the constraints

$$\phi^{(1)} = p_\psi - \tfrac{i}{2}\psi^*, \qquad \phi^{(2)} = p_\psi^* - \tfrac{i}{2}\psi,$$

the theory can be quantized by imposing

$$[x, p] = i, \qquad \{\psi^*, \psi\} = 1.$$

It is convenient to choose the representation [3]

$$\psi^* = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \psi = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In this representation we recover susy quantum mechanics in the form originally given by Witten [4]

$$H = \tfrac{1}{2}\big(p^2 + u^2(x)\big) + \tfrac{1}{2}\sigma_3 u'(x). \tag{8}$$

This Hamiltonian can be expressed in terms of the supercharges Q_i, the generators of the supersymmetry algebra,

$$H = \tfrac{1}{2}Q_1^2 = \tfrac{1}{2}Q_2^2,$$

where

$$Q_1 = p\sigma_1 + u\sigma_2,$$

$$Q_2 = p\sigma_2 + u\sigma_1,$$

with σ_i the Pauli matrices. Clearly

$$[Q_1, H] = [Q_2, H] = 0.$$

Finite susy transformations are generated by

$$G_i = e^{i\epsilon Q_i}.$$

For an arbitrary operator the infinitesimal transformation reduces to

$$\delta_i \mathcal{O} = i[\epsilon Q_i, \mathcal{O}].$$

In particular notice that (in the following we will restraint to Q_1, although a similar analysis can be applied to Q_2)

$$\delta_1 x = i[p\epsilon\sigma_1 + u\epsilon\sigma_2, x] = \epsilon\sigma_1. \tag{9}$$

Let $\psi(x)$ be a solution of the Schrödinger equation:

$$\left\{\tfrac{1}{2}\left(p^2 + u(x)\right)^2 + \tfrac{1}{2}\sigma_3 u'(x)\right\}\psi(x) = E\psi(x). \tag{10}$$

Consider a susy transformation on $\psi(x)$:

$$\begin{aligned} e^{i\epsilon Q_1}\psi(x) &= e^{i\epsilon u(x)\sigma_2} e^{i\epsilon p\sigma_1}\psi(x) \\ &= e^{i\epsilon u(x)\sigma_2}\psi(x + \epsilon\sigma_1) = e^{i2\pi\omega_1}\psi(x'). \end{aligned} \tag{11}$$

This result shows that susy transformations give rise to a 1 cochain. (In fact we are dealing with a ray proyective representation). ω_1 is trivial if it can be written as

$$2\pi\omega_1 = \alpha_0(x') - \alpha_0(x).$$

Substituing ω_1, $x' = x + \epsilon\sigma_1$ and expanding we obtain

$$u(x)\sigma_2 = \sigma_1 \frac{d\alpha_0}{dx}, \tag{12}$$

which can be solved to yield

$$\alpha_0(x) = \pm i\sigma_3 \int_{x_0}^{x} u(y)\, dy. \tag{13}$$

We can redefine $\psi(x)$ so as to eliminate ω_1:

$$\Phi(x) = e^{i\alpha_0(x)}\psi(x) = e^{-\sigma_3 \int_{x_0}^{x} u(y)\, dy}\psi(x).$$

At this point a few comments are in order:

Although we have formally solved Eq. (12), we still must require that we get a sensible result, *i.e.* that we obtain a new wave function $\Phi(x)$ which is physically acceptable. If this is not so, we will consider that the cochain is non-trivial.

Up to a constant, the cochain $e^{i\alpha_0(x)}$ is the wave function of the ground state of susy quantum mechanics. It is a well known fact [3, 4] that the requirement of normalisability of this states leads to a criterium for the breaking of susy.

In solving Eq. (12) to show the triviality of the cochain, we do not get any constraint on $u(x)$. However, the new wave function $\Phi(x)$ must be well defined over the whole space, in particular for $x \to \pm\infty$. This requirement is fulfilled if the leading term of $u(x)$ is an odd function of x. On the other hand if the leading term of $u(x)$ is an even function of x, $\int u(x)\, dx$ is odd and $\Phi(x)$ is not finite either in $x = +\infty$ or $x = -\infty$.

Summarizing, we have shown that susy transformations in quantum mechanics give rise to a 1 cochain. Although we can formally show the triviality of the cochain, demanding a physicaly acceptable wave function requires $u(x)$ to be an odd function of x, in accordance with a well known criterion for susy breaking. We can therefore conclude that, at least for the example under consideration, the non-triviality of the cochain is an indication of the quantum mechanical breaking of the symmetry.

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