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Self-dual spin–3 and 4 theories

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ABSTRACT. We present self-dual spin–3 and 4 actions using relevant Dreibein fields. Since these actions start with a Chern-Simons like kinetic term (and therefore cannot be obtained through dimensional reduction) one might wonder whether they need the presence of auxiliary ghost-killing fields. It turns out that these actions must contain, even in this three dimensional case, auxiliary fields. Auxiliary scalars do not break self-duality since their free actions do not contain kinetic terms.

RESUMEN. En este artículo presentamos acciones autoduales para campos de espín–3 y 4 usando la representación triádica. Como estas acciones contienen como término cinético un término generalizado de Chern-Simons, es natural preguntarse si, también en ese caso, se necesitan campos auxiliares para matar a los fantasmas. Resulta que estas acciones deben contener, aun en este caso tridimensional, campos auxiliares. Los escalares auxiliares no rompen la autodualidad de la acción complexiva puesto que sus acciones libres no contienen términos cinéticos.

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Self-dual theories for odd dimensions were discovered time ago by Townsend, Pilch and van Nieuwenhuizen [1]. For Abelian vector theories, they can be shown to be classically and quantum mechanically equivalent [2] to the Maxwell-Chern-Simons (MCS) [3] model, if one allows a non minimal coupling in the self-dual model while keeping the minimal one for the gauge invariant second order MCS theory.

Otherwise, although both models propagates one massive spin–1 mode, these theories will not be equivalent if they are minimally coupled to the same sources.

Spin–2 presents a new feature: there are three topological spin–2 theories: linearized topological massive gravity [4], a second order Einstein-CS action [5] and the first order self-dual one [6]. In the vector case the topological massive action is second order, whereas the self-dual one is first order. Spin-two fields present a new feature: exact topological massive gravity [4] is a third oder action while self-dual gravity [5] is, by definition, a

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first order action. Self-dual gravity is a good example of the relevance of the Dreibein representation [7] for higher spin gauge fields: its more compact form is obtained when the spin two fields is represented by the (linearized) unsymmetrized second rank tensor w_{pa}, where p is the gauge index and a is the flat remnant of a Lorentz index. Its gauge variation is given by $\delta w_{pa} = \partial_p \xi_a$.

When dealing with higher spin particle ($s \geq 3$) one is always concerned with whether they can have consistent interactions with other basic elementary systems or (at least) with themselves. Along this direction, the existence of higher-spin self interacting bosonic theories has recently been shown [14]. These theories are third-order in the basic fields, and their structure is very similar to metric topological Chern-Simons gravity [4].

In $d = 4$, bosons obey second order field equation. Due to this fact, coupling them to Abelian vectors (when charged) or to gravity (which is always mandatory because of the universality of gravity) leads to considerations of a wide variety of different types of non minimal coupling, once it is shown that the canonical ones do not work, as it is generally the case. The natural solution to this problem comes from charged-string theory models which consistently contain all spins in their spectrum [15].

In dimension 3 we have the peculiarity of the existence of these first order, Dirac-like, bosonic self-dual theories for spin 1 and 2. It seems to us worthwhile to construct flat models for spin–3 and spin–4 in order to investigate whether they can be consistently coupled either to Abelian vectors or to gravity.

Here we report about the precise, Dirac-like, self-dual actions we found for spin 3 and 4. We want to mention an additional (more technical) problem.

Massive spin-3 fields in dimensions $d \geq 4$ cannot avoid the presence of auxiliary fields as it is clearly shown by dimensional reduction from their massless, gauge invariant $d+1$ dimensional spin–3 ascendant action [8]. In $d = 3$ it is hard to imagine what might be the 4–dimensional ascendant of a three dimensional self-dual action (whose kinetic term is essentially given by $\sim w_{(3)}\epsilon\partial w_{(3)}$). Therefore, one might ask again whether a self-dual pure spin–3 (or higher) field needs the presence of auxiliary fields. Even if self-dual spin–3 needed no auxiliary fields, one should ask what is the fate of spin–4 since the real higher-spin field has spin–4. This is due to the fact that if one works in the symmetric representation where $w_{(4)}$ is the basic 4–index symmetric tensor which carries the physical massless excitation, $w_{(4)}$ has to be doubly traceless [9], *i.e.*, $w \equiv w_{pprr} = 0$. This condition is uniformly satisfied by any spins-s greater than, *i.e.*, $w_{pprr\ell_1\ldots\ell_4} = 0$.

In the following we will show that both self-dual spin–3 and spin–4 actions require the presence of self-dual auxiliary fields of spin–1 and 0 for the former and spin–2 and 1 for the latter.

The symmetric formulation of massless spin–3 in $d \geq 3$ was given in [9]. The first order Vierbein formulation was presented by Vasiliev [7] and a second order action was introduced in [10]. The associated massive spin–3 models are discussed in [8].

In three dimensions there exist three additional possibilities, (taking into account the analysis performed in [5] for the spin–2 case): the topological massive third-order formulation discovered by Damour and Deser [11], the first-order self-dual action which is presented here and the intermediate second-order action equivalent to these two, similar to the spin–2 intermediate action [12]. Since spin–3 is simpler, we treat it first.

Self-dual spin–3 action is the addition of three layers:

$$S = S_3 + S_{31} + S_{10}, \tag{1}$$

where

$$S_3 \equiv \tfrac{1}{2}\mu\langle w_{p\bar{a}_1\bar{a}_2}\varepsilon^{pmn}\partial_m w_{n\bar{a}_1\bar{a}_2}\rangle - \tfrac{1}{6}\mu^2\langle \varepsilon^{pmn}\varepsilon^{abc}\eta_{pa}w_{m\bar{b}\bar{d}}w_{n\bar{c}\bar{d}}\rangle, \tag{2}$$

$$S_{31} \equiv \mu^2\langle w_p u_p\rangle + \tfrac{1}{2}\alpha\mu\langle u_p\varepsilon^{pmn}\partial_m w_n\rangle + \tfrac{1}{2}\beta\mu^2\langle u_p u_p\rangle, \tag{3}$$

$$S_{10} \equiv \mu\langle \phi\partial_p u_p\rangle + \tfrac{1}{2}\gamma\langle\phi\Box\phi\rangle + \tfrac{1}{2}\delta\mu^2\langle\phi^2\rangle. \tag{4}$$

In three dimensions $[\phi] = m^{1/2} = [w] = [u]$. The basic field $w_{p\bar{a}_1\bar{a}_2}$ is symmetric and traceless in its Dreibein Lorentz indices $w_{p\bar{a}_1\bar{a}_2} = w_{p\bar{a}_2\bar{a}_1}$, $w_{p\bar{a}\bar{a}} = 0$ while p is a world index, unrelated to them. (In the following, a set of barred indices will indicate that the associated tensor is symmetric and traceless in this set.) The algebraically irreducible decomposition of $w_{p\bar{a}_1\bar{a}_2}$ is

$$w_{p\bar{a}_1\bar{a}_2} = w_{\bar{p}\bar{a}_2\bar{a}_1} + \varepsilon_{pa_1b}h_{\bar{b}\bar{a}_2} + \varepsilon_{pa_2b}h_{\bar{b}\bar{a}_1} + b(\eta_{pa_1}w_{a2}\eta_{pa_2}w_{a_1} - \tfrac{2}{3}\eta_{a_1a_2}w_p). \tag{5a}$$

The 15 independent components of $w_{p\bar{a}_1\bar{a}_2}$ are represented by the 7 components of $w_{p\bar{a}_1\bar{a}_2}$, plus the 5 needed to describe $h_{\bar{b}\bar{c}}$, plus the last 3 which determine $w_p \equiv w_{r\bar{r}\bar{p}}$, the unique nonvanishing trace of $w_{p\bar{a}_1\bar{a}_2}$. Taking the trace in Eq. (5a) one obtains $b = 3/10$ and calculating the symmetric part of $\epsilon_b{}^{pa}w_{p\bar{a}\bar{a}}$ one is led to determine $h_{\bar{b}\bar{c}}$:

$$h = h_{\bar{b}\bar{c}} = -\tfrac{1}{6}(\varepsilon_b{}^{pa}w_{p\bar{a}\bar{c}} + \varepsilon_c{}^{pa}w_{p\bar{a}\bar{b}}). \tag{5b}$$

The first interesting fact is that S_3 has the good spin–3 and spin–2 behaviour. The associated field equations $E^{p\bar{a}_1\bar{a}_2} \equiv \delta S^3/\delta w_{p\bar{a}_1\bar{a}_2} = 0$ propagate one parity sensitive spin–3 excitation, do not propagate neither the other possible spin–3 variable nor any spin–2 degree of freedom (those contained in $h^{\mathrm{T}}_{\bar{a}\bar{b}}$, the transverse part of $h_{\bar{a}\bar{b}}$: $\partial_{\bar{a}}h^{\mathrm{T}}_{\bar{a}\bar{b}} = 0$). However, S_3 has spin–1 ghosts and this is the reason why one has to add a second layer which will fix this situation. S_{31} is a pure self-dual vector action for the auxiliary vector u_p plus the simplest contact term $\sim \langle wu_p\rangle$. In general one might also consider terms $\sim \mu\langle w_p\epsilon^{pmn}\partial_m u_n\rangle$ but we have been lucky and there is no need to include them. Addition of these two layers leads to $S_3 + S_{31}$ whose field equations are

$$E^{p\bar{a}_1\bar{a}_2} \equiv \varepsilon^{pmn}\partial_m w_{n\bar{a}_1\bar{a}_2} + \tfrac{1}{6}\mu(\eta_{pa_1}w_{a_2} + \eta_{pa_2}w_{a_1} - w_{a_1\bar{p}\bar{a}_2}w_{a_2\bar{p}\bar{a}_1})$$
$$+ \tfrac{1}{2}\mu\left(\eta_{pa_1}u_{a_2} + \eta_{pa_2}u_{a_1} - \tfrac{2}{3}\eta_{a_1a_2}u_p\right) = 0, \tag{6}$$

$$F^p \equiv \alpha\varepsilon^{pmn}\partial_m u_n + \beta\mu u_p + \mu w_p = 0. \tag{7}$$

These two equations can be analyzed by further breaking of the algebraic decomposition (5a) in terms of its SL(2,ℝ) irreducible representations. We introduce the three dimensional covariant (and non local) T-projectors which, in the vector case, are

$$\begin{aligned} u_p &= u_p^{\mathrm{T}} + \hat{\partial}_p u^{\mathrm{L}}, & \hat{\partial}_p &\equiv \Box^{-1/2}\,\partial_p, \\ \hat{\partial}_p u_p^{\mathrm{T}} &= 0, & \hat{\partial}_p \cdot \hat{\partial}_p &= 1. \end{aligned} \tag{8a}$$

For spin–2 and 3, similar decompositions for symmetric traceless second and third rank tensors have the form

$$h_{\bar{p}\bar{a}} = h^{\mathrm{T}}_{\bar{p}\bar{a}} + \hat{\partial}_{(\bar{p}} h^{\mathrm{L}}_{\bar{a})}, \qquad \hat{\partial}_p h^{\mathrm{T}}_{\bar{p}\bar{a}} = 0 = h^{\mathrm{T}}_{\bar{p}\bar{p}}, \tag{8b}$$

$$w_{\bar{p}\bar{a}\bar{b}} = w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}} + \hat{\partial}_{(p} w^{\mathrm{T}}_{\bar{a}\bar{b})}, \qquad \hat{\partial}_p w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}} = 0 = w^{\mathrm{T}}_{\bar{p}\bar{p}\bar{b}}. \tag{8c}$$

Symmetric traceless transverse $3d$ tensors $(u_p^{\mathrm{T}}, h^{\mathrm{T}}_{\bar{p}\bar{a}}, w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}}, w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}\bar{c}})$ have two independent components corresponding to the two P-sensitive pseudospin–j $(j = 1, 2, 3, 4)$ excitation they can propagate. A final covariant splitting of these set (symmetric, traceless, transverse) tensors is obtained by means of the pure pseudospin–j projectors $p_j^{\pm} w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}\cdots\bar{c}}$ [6]:

$$p_j^{\pm} w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}\cdots c} \equiv w^{\mathrm{T}\pm}_{\bar{p}\bar{a}\bar{b}\cdots\bar{c}} = \tfrac{1}{2} w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}\cdots\bar{c}} \pm \frac{1}{2j}\varepsilon_{(p}{}^{mn}\hat{\partial}_m w_{\bar{n}\bar{a}\bar{b}\cdots\bar{c})}, \tag{9}$$

where the indicated symmetrization is the minimal one and does not carry a normalization coefficient. It is straightforward to check that

$$p_j^{+} + p_j^{-} = 1, \quad p_j^{+} - p_j^{-} = \frac{1}{j}\varepsilon(\cdot\cdot\hat{\partial}\cdots). \tag{10}$$

Armed with these projectors one can analyse the behaviour of $E^{\bar{p}\bar{a}\bar{b}\,\mathrm{T}}$, the spin–3 sector of Eq. (6). It turns out that $E^{\bar{p}\bar{a}\bar{b}\,\mathrm{T}}$ propagates the spin–3^+ part of $w^{\mathrm{T}}_{\bar{p}\bar{a}\bar{b}}$ and annihilates $w^{\mathrm{T}-}_{\bar{p}\bar{a}\bar{b}}$. Then ones goes to the spin–2 sector and it is immediate to verify that $\partial_p E^{p\bar{a}b}$, $\check{E}^{\bar{b}\bar{c}} \equiv \varepsilon_{(bpa} E^{p\bar{a}}{}_{\bar{c})}$ do not allow the propagation of $h^{\mathrm{T}\pm}_{\bar{a}\bar{b}}$. The spin–1 dynamical behaviour is determined by $\partial_{pa}E^{p\bar{a}\bar{b}}$, $\partial_b \check{E}^{\bar{b}\bar{a}}$, $E^b \equiv E^{p\bar{p}\bar{b}}$ and F^p. In order not to have any spin–1 excitation alive we must choose

$$\alpha = \beta = -18. \tag{11}$$

Unfortunately this is not the last step in order to get a pure pseudospin–3^+ propagation. $S_3 + S_{31}$ has scalar ghosts and therefore they have to be destroyed by an auxiliary scalar ϕ. This is the reason for adding the last layer S_{10} defined in Eq. (4) to the first two layers $S_3 + S_{31}$. In principle one should consider the possibility of kinetic terms like $\sim \phi\Box\phi$

which are of second order and would therefore break the full system self-duality. The fields equations derived from S are

$$\delta_w S \sim E^{p\bar{a}_1\bar{a}_2} = 0, \tag{12}$$

$$\delta_u S \sim {}^{\backprime}F^p \equiv F^p - \partial_p \phi = 0, \tag{13}$$

$$\delta_\phi S \sim G \equiv \gamma \Box \phi + \delta \mu^2 \phi + \mu \partial_p u_p = 0. \tag{14}$$

There are five scalar excitations which the system might propagate: $\hat{\partial}_{pab} w_{\bar{p}\bar{a}\bar{b}}$, $\hat{\partial}_{ab} h_{\bar{a}\bar{b}}$, $\hat{\partial}_p w_p$, $\hat{\partial}_p u_p$ and ϕ. However, since $\partial_p E^{p\bar{a}\bar{b}}$ and $\check{E}^{\bar{b}\bar{c}}$ tell us that

$$\mu h_{\bar{b}\bar{c}} = -3\big(\partial_b u_c + \partial_c u_b - \tfrac{2}{3}\eta_{ab}(\partial \cdot u)\big), \tag{15a}$$

$$\partial_b w_c + \partial_c w_b - (\partial_p w_{b\bar{p}\bar{c}} + \partial_p w_{c\bar{p}\bar{b}}) + 3\big(\partial_b u_c + \partial_c u_b - \tfrac{2}{3}\eta_{bc}(\partial . u)\big) = 0, \tag{15b}$$

it is immediate that, if neither $\hat{\partial}_p u_p$ nor $\hat{\partial}_p w_p$ propagate (*i.e.*, $\hat{\partial}_p u_p = 0 = \hat{\partial}_p w_p$), $\hat{\partial}_{pab} w_{\bar{p}\bar{a}\bar{b}}$ and $\hat{\partial}_{pa} h_{\bar{p}\bar{a}}$ will not propagate either. The key equations are the vanishing of $\partial_b E^{p\bar{p}\bar{b}}$, $\partial_p {}^{\backprime}F^p$ and G, where in the first one, one makes use of Eqs. (5a) and (15). They can be written, respectively,

$$\big(12\Box + \tfrac{5}{8}\mu^2\big)\partial \cdot u + \tfrac{1}{2}\mu^2 \partial \cdot w = 0, \tag{16a}$$

$$\mu\beta\partial \cdot u + \mu\partial \cdot w - \Box\phi = 0, \tag{16b}$$

$$\mu\partial \cdot u + (\gamma\Box + \delta\mu^2)\phi = 0. \tag{16c}$$

Introducing the dimensionless operator $x \equiv \mu^{-1}\Box^{1/2}$, it is straightforward to see that the inverse propagator of $\hat{\partial} \cdot w$, $\hat{\partial} \cdot u$, ϕ is

$$\Delta(x) \equiv -(\gamma x^2 + \delta)\big(12x^2 + \tfrac{5}{8}\big) + \tfrac{1}{2}x^2 + \tfrac{1}{2}\beta(\gamma x^2 + \delta). \tag{17}$$

These scalar variables (and consequently $\hat{\partial}_{pab} w_{\bar{p}\bar{a}\bar{b}}$, $\hat{\partial}_{pa} h_{\bar{p}\bar{a}}$) do not propagate if the polynomial $\Delta(x)$ becomes of zero order, *i.e.*, $\Delta(x) \equiv \Delta_4 \cdot x^0 = \Delta_4 \cdot 1$. This condition uniquely determines γ, δ

$$\gamma = 0, \quad \delta = \tfrac{1}{24}. \tag{18}$$

Note that the vanishing of γ makes the action S_{10} of first order (scalars of the self-dual type also appear), leading to the final S being fully first order. Observe that we do not claim mathematical uniqueness for a pure spin-3^+ (or 3^+) $3d$ action: in the scalar sector one could consider coupling terms like $\sim \phi(\partial \cdot w)$. However, it seems to us that, if one starts with the right-spin Dreibein seed (in the case S_3), then S_{31} is unique if we demand that it must be the vector self-dual action coupled in the softest possible ways to S_3 (the coupling term must be, at most, first order and if possible algebraic). The construction of the auxiliary scalar action S_{10} again is unique: it contains the free self-dual scalar action ($\sim \mu^2\phi^2$, no Klein-Gordon kinetic term) and it is next-neighbour coupled to the auxiliary spin–1 field, discarding $\phi(\partial \cdot w)$ which is not of the next-neighbour type.

All these results will be useful when dealing with the much complex case of spin–4.

We start this analysis by introducing the spin–4 part of the final action S_{42} with the right physical behaviour up to the spin–2 sector. It reads

$$\begin{aligned} S_{42} \equiv {} & \tfrac{1}{2}\mu\left\langle w_{p\bar{a}\bar{b}\bar{c}}\varepsilon^{pmn}\partial_m w_{n\bar{a}\bar{b}\bar{c}}\right\rangle - \tfrac{1}{2}\mu^2\left\langle \varepsilon^{pmn}\varepsilon^{abc}\eta_{pa}w_{m\bar{b}\bar{d}_1\bar{d}_2}w_{n\bar{c}\bar{d}_1\bar{d}_2}\right\rangle \\ & + \mu^2\left\langle w_{p\bar{p}\bar{a}\bar{b}}u_{ab}\right\rangle + \tfrac{1}{2}\alpha\mu\left\langle u_{pa}\varepsilon^{pmn}\partial_m u_{na}\right\rangle \\ & + \tfrac{1}{2}\beta\mu^2\left\langle \varepsilon^{pmn}\varepsilon^{abc}\eta_{pa}u_{mb}u_{nc}\right\rangle, \end{aligned} \tag{19}$$

where $w_{p\bar{a}\bar{b}\bar{c}}$ is symmetric and traceless (ST) in its three last barred indices and u_{pa} is an auxiliary self-dual second rank tensor, $[w] = [u] = m^{1/2}$. Their algebraically irreducible representations are, respectively,

$$w_{p\bar{a}\bar{b}\bar{c}} = w_{\bar{p}\bar{a}\bar{b}\bar{c}} + \varepsilon_{p(ad}h_{d\bar{b}\bar{c})} + \tfrac{5}{21}\eta_{p(a}w_{\bar{b}\bar{c})} - \tfrac{2}{21}w_{p(\bar{a}}\eta_{bc)}, \tag{20}$$

$$u_{pa} = u_{\bar{p}\bar{a}} + \varepsilon_{pad}h_d + \tfrac{1}{3}\eta_{pa}u, \qquad h_d = -\tfrac{1}{2}\varepsilon_d{}^{pa}u_{pa}, \tag{21a, b}$$

where $w_{p\bar{p}\bar{b}\bar{c}} \equiv w_{\bar{b}\bar{c}}$ and u_{pp} are the unique non-vanishing contractions which can be made out of $w_{p\bar{a}\bar{b}\bar{c}}$ and u_{pa}, respectively. Symmetrizations are minimal with coefficient one in front and sets of barred indices still indicate ST tensors.

Variations with respect the $w_{p\bar{a}\bar{b}\bar{c}}$ and u_{pa} yield the initial set of field equations

$$\begin{aligned} E_{p\bar{a}\bar{b}\bar{c}} \equiv {} & \varepsilon_p^{\ mn}\partial_m w_{n\bar{a}\bar{b}\bar{c}} + \tfrac{1}{3}\mu\left\{\eta_{p(a}w_{\bar{b}\bar{c})} - w_{(a\bar{b}\bar{c})\bar{p}}\right\} \\ & + \tfrac{1}{3}\mu\left\{\eta_{p(a}u_{\bar{b}\bar{c})} - \tfrac{2}{5}\eta_{(ab}u_{\bar{c}\bar{p})}\right\} = 0, \end{aligned} \tag{22}$$

$$F_{pa} \equiv \mu w_{\bar{p}\bar{a}} + \alpha\varepsilon_p{}^{mn}\partial_m u_{na} + \mu\beta\varepsilon_p{}^{mn}\varepsilon_a^{\ bc}\eta_{nc}u_{mb} = 0. \tag{23}$$

The spin–$4^{\pm}$ excitations are carried on the transverse part of $w_{p\bar{a}\bar{b}\bar{c}}: w^{\mathrm{T}}_{p\bar{a}\bar{b}\bar{c}}, \partial_p w^{\mathrm{T}}_{p\bar{a}\bar{b}\bar{c}} = 0$ while there are two sets of spin–3 variables: those contained in $\hat{\partial}_p w_{p\bar{a}\bar{b}\bar{c}}$ and those defined by $h^{\mathrm{T}}_{\bar{a}\bar{b}\bar{c}}$. Use of the spin–$4^{\pm}$ projectors defined in Eqs. (9) and (10) show that $E_{p\bar{a}\bar{b}\bar{c}}$ uniquely propagate spin–4^{+} (make the spin–4^{-} degree of freedom to cancel) and does not propagate neither $(\hat{\partial}_p w_{\bar{p}\bar{a}\bar{b}\bar{c}})^{\mathrm{T}}$ nor $h^{\mathrm{T}}_{\bar{a}\bar{b}\bar{c}}$. In fact, equations $\partial_p E_{p\bar{a}\bar{b}\bar{c}} = 0 = \varepsilon^{pd}_{(a}E_{p\bar{d}\bar{b}\bar{c})}$ are equivalent to

$$4\mu h_{\bar{a}\bar{b}\bar{c}} = \tfrac{2}{5}\eta_{(ab}\partial_p u_{\bar{p}\bar{c})} - \partial_{(a}u_{\bar{b}\bar{c})}, \tag{24}$$

$$\partial_{(a}w_{\bar{b}\bar{c})} - \partial_p w_{(a\bar{p}\bar{b}\bar{c})} = \tfrac{2}{5}\eta_{(ab}\partial_p u_{\bar{p}\bar{c})} - \partial_{(a}u_{\bar{b}\bar{c})}. \tag{25}$$

These equations imply that both $h_{\bar{a}\bar{b}\bar{c}}$ and $\partial_p w_{\bar{p}\bar{a}\bar{b}\bar{c}}$ are curls of spin-2 objects and therefore their pure spin-3 parts have to vanish.

Four variables describe the spin-2 sector of S_{42}: $(\hat{\partial}_{pa} w_{\bar{p}\bar{a}\bar{b}\bar{c}})^{\mathrm{T}}$, $(\hat{\partial}_p h_{\bar{p}\bar{a}\bar{b}})^{\mathrm{T}}$, $w^{\mathrm{T}}_{\bar{p}\bar{a}}$, $u^{\mathrm{T}}_{\bar{p}\bar{a}}$. The equations which determine their dynamical behaviour are $\partial_{pa} E^{\bar{p}\bar{a}\bar{b}\bar{c}} = 0$, $\check{E}^{\bar{a}\bar{b}\bar{c}} = 0$, $E_{\bar{b}\bar{c}} \equiv E_{p\bar{p}\bar{b}\bar{c}} = 0$ and $F_{pa} = 0$. After some algebra one is led to a separated propagation equation for $u^{\mathrm{T}}_{\bar{p}\bar{a}} \equiv \omega$, $p'\omega \equiv \omega^{\pm}$:

$$\left(x^2 + \tfrac{7}{5} - \tfrac{4}{3}\beta\right)(\omega^+ + \omega^-) + \tfrac{2}{3}x(\alpha x + \beta)\omega^+ + \tfrac{2}{3}x(\alpha x - \beta)\omega^- - \tfrac{4}{3}\alpha x(\omega^+ - \omega^-) = 0. \quad (26)$$

Projecting on this spin-2^+ (2^-) subspaces we obtain the two uncoupled equations which determine their evolution:

$$\left\{x^2(1 + \tfrac{2}{3}\alpha) \mp \tfrac{2}{3}(2\alpha - \beta)x + (\tfrac{7}{5} - \tfrac{4}{3}\beta)\right\}\omega^{\pm} = 0, \quad (27)$$

(either all upper indices or all right down). Non-propagations of one of these two variables determines the values of α, β:

$$\alpha = -\tfrac{3}{2}, \qquad \beta = -3, \quad (28)$$

and, due to Eq. (27), entails the non-propagation of the other companion variable. S_{42} (19) has been uniquely determined requesting its good physical behaviour in its highest spin sector ($s = 4, 3, 2$). However, it contains vector and scalar ghosts. This is the reason why we have to add two additional layers. The most difficult of them is the spin-1 fixing action. Its ambiguity stems in the wide range of mathematically consistent terms one might have to consider *ab initio*.

In principle S_{21} may be

$$\begin{aligned} S_{21} \equiv & -2\lambda_1\mu\langle h_a\partial_b u_{\bar{a}\bar{b}}\rangle + 2\lambda_2\mu\langle v_p\partial_r u_{\bar{r}\bar{p}}\rangle \\ & + \gamma_2\mu\langle h_a\varepsilon^{abc}\partial_b h_c\rangle + \tfrac{1}{2}\gamma_1\mu\langle v_p\varepsilon^{pmn}\partial_m v_n\rangle \\ & + \rho\mu^2\langle h_a^2\rangle + \tfrac{1}{2}\delta\mu^2\langle v_a^2\rangle + 2\varepsilon\mu^2\langle h_p v_p\rangle + 2\kappa\mu\langle h_a\partial_b w_{\bar{b}\bar{a}}\rangle \\ & + 2\varphi\mu\langle v_p\partial_r w_{\bar{r}\bar{p}}\rangle + 2\sigma\mu\langle v_p\varepsilon^{pmn}\partial_m h_n\rangle, \end{aligned} \quad (29)$$

which can be regarded as the addition of the self-dual action for the spin-1 variable h_a contained in u_{pa}, plus the auxiliary self-dual action for the auxiliary vector u_p algebraically coupled through $\sim h \cdot v$, plus more bizarre terms like $\sim h_a\partial_b u_{\bar{b}\bar{a}}$, $h_a\partial_b w_{\bar{b}\bar{a}}$, $v_a\partial_b u_{\bar{b}\bar{a}}$, $v_a\varepsilon^{abc}\partial_b h_c$ and the exotic term $\sim v_a\partial_b w_{\bar{b}\bar{a}}$. We will not consider them, the first because we already have chosen a good kinetic term for u_{pa} [$u_{pa}\varepsilon^{pmn}\partial_m u_{na}$ as in Eq. (19)], the last one because it is not of the next-neighbour type (it is spin-4 $\cdot$ spin-1), and the second, third and fourth because we have decided to choose, whenever possible, algebraic couplings and we have already a spin-2 $\cdot$ spin-1 contact term $\sim h.v$. Therefore we rule out the presence of terms $\sim v_a\partial_b u_{\bar{b}\bar{a}} v_a\varepsilon^{abc}\partial_b h_c$ as well as the need for a term $\sim h_a\partial_b w_{\bar{b}\bar{a}}$, a different coupling term linking spin–4 with spin–2 for the same reason. In other words, we take $\lambda_1 = \lambda_2 = \kappa = \sigma = \varphi = 0$ in S_{21}.

Taking into account Eq. (21b) we write down the modified spin–2 field equations which govern this system (note that $E^{\bar{p}\bar{a}\bar{b}\bar{c}} = 0$ remains intact). They have the form

$$`F_{pa} \equiv F_{pa} + \gamma_2(\partial_p h_a - \partial_a h_p) - \rho\varepsilon_{pab}h_b - \varepsilon\varepsilon_{pab}v_b = 0. \tag{30}$$

An additional vector-like field equation appears after varying v_p,

$$G_p \equiv \gamma_1 \varepsilon_p^{mn}\partial_m v_n + \delta\mu v_p + 2\varepsilon\mu h_p = 0. \tag{31}$$

We want to determine $\gamma_1, \gamma_2, \rho, \delta, \varepsilon$ in such a way that none of the six spin–1 variables: $\omega_8 \equiv (\hat{\partial}_{pab}w_{\bar{p}\bar{a}\bar{b}\bar{c}})^{\mathrm{T}}$, $\omega_9 \equiv (\hat{\partial}_{pa}h_{\bar{p}\bar{a}\bar{b}})^{\mathrm{T}}$, $\omega_{10} \equiv (\hat{\partial}_p u_{\bar{p}\bar{a}})^{\mathrm{T}}$, $\omega_{11} \equiv h_p^{\mathrm{T}}$, $\omega_{12} \equiv h_p^{\mathrm{T}}$, $\omega_{13} \equiv v_p^{\mathrm{T}}$ can propagate. Since ω_8 is given by $\partial_{pab}E_{\bar{p}\bar{a}\bar{b}\bar{c}}$ in terms of the five remaining variables $\omega_9, \ldots, \omega_{13}$, we consider the non propagation of them. They are determined by $\partial_{ab}\check{E}_{\bar{b}\bar{a}\bar{c}} = 0$, $\partial_b E_{\bar{b}\bar{c}} = 0$, $\partial_p `F_{pa} = 0$, $`\check{F}^b = 0$ and $G^p = 0$. After minor algebra and some use of Eq. (24) the five equations become

$$4\mu\partial_{ab}h_{\bar{a}\bar{b}\bar{c}} + \tfrac{8}{5}\Box\partial_a u_{\bar{a}\bar{c}} + \tfrac{1}{5}\partial_c(\partial_{ab}u_{\bar{a}\bar{b}}) = 0, \tag{32}$$

$$-4\partial_{ab}h_{\bar{a}\bar{b}\bar{c}} - \tfrac{1}{3}\varepsilon_c^{pr}\partial_p(\partial_b w_{\bar{b}\bar{r}} + \tfrac{4}{3}\mu\partial_p w_{\bar{p}\bar{c}} + \tfrac{7}{5}\mu\partial_p u_{\bar{p}\bar{c}}) = 0, \tag{33}$$

$$\begin{aligned}\mu\partial_p w_{\bar{p}\bar{a}} - 3\mu\partial_p u_{\bar{p}\bar{a}} + (\rho - 3)\mu\varepsilon_a^{pr}\partial_p h_r + 2\mu\partial_a u \\ +\gamma_2\big(\Box h_a - \partial_a(\partial_p h_p)\big) + \varepsilon\mu\varepsilon_a^{pr}\partial_p v_r = 0,\end{aligned} \tag{34}$$

$$\tfrac{3}{2}\partial_p u_{\bar{p}\bar{b}} + 2(\rho - 3)\mu h_b + 2\varepsilon\mu v_b + \big(2\gamma_2 + \tfrac{2}{3}\big)\varepsilon_b^{pr}\partial_p h_r - \partial_b u = 0 \tag{35}$$

and Eq. (31) as it stands.

Working in a way similar to what we did for the spin–3 case, the vanishing of $\omega_9, \ldots, \omega_{13}$ is equivalent to their non propagation and this is achieved if $\Delta(x) = \Delta_0 x^4 + \cdots + \Delta_4 \cdot 1$ becomes $\Delta_4 \cdot 1$. Straightforward calculations give

$$\begin{aligned}\Delta(x) = &-\tfrac{3}{10}\gamma_1(9\gamma_2 + 8)x^4 + \big\{\tfrac{3}{2}\gamma_1\big(1 - \tfrac{9}{5}\rho'\big) - \tfrac{3}{5}\delta\big(\tfrac{9}{2}\gamma_2 + 4\big)\big\}x^3 \\ &+ \big\{-\tfrac{27}{5}\gamma_1\big(2\gamma_2 + \tfrac{3}{2}\big) - \tfrac{27}{10}\delta\rho' + \tfrac{3}{2}\delta + \tfrac{27}{5}\varepsilon^2\big\}x^2 \\ &- \tfrac{27}{5}\big\{\delta\big(2\gamma_2 + \tfrac{3}{2}\big) + 2\gamma_1\rho'\big\}x + \tfrac{54}{5}\{2\varepsilon^2 - \delta\rho'\} \cdot 1,\end{aligned} \tag{36}$$

where for convenience $\rho' \equiv \rho - 3$. Requesting the vanishing of the coefficients $\Delta_{0,1,2,3}$ of the inverse propagator Δ_x one is led to

$$\begin{aligned}&\gamma_2 = \tfrac{8}{9}, \qquad \rho' = \tfrac{5}{9} = \rho - 3, \quad (\rho = -4\gamma_2), \\ &\gamma_1 = -\tfrac{18}{5}\varepsilon^2, \qquad \delta = 4\gamma_1 = -\tfrac{72}{5}\varepsilon^2.\end{aligned} \tag{37}$$

Redefining $2\varepsilon v_p \to v_p$, the final unique form of S_{21} becomes

$$S_{21} = -\tfrac{8}{9}\mu\langle h_a \varepsilon^{abc}\partial_b h_c\rangle - \tfrac{9}{20}\mu\langle v_p \varepsilon^{pmn}\partial_m v_n\rangle + \tfrac{32}{9}\mu^2\langle h_a^2\rangle - \tfrac{9}{5}\mu^2\langle v_p^2\rangle + \langle h_p v_p\rangle\mu^2. \tag{29b}$$

The action $S_{42}+S_{21}$ has the right physical properties up to spin–1. However, its scalar sector contains ghost which we have to exorcize by introducing an auxiliary self-dual scalar ϕ. Its associated action S_{10} constitutes the last layer we need in order to determine the final pure self-dual spin–4^+ action S.

The most general scalar auxiliary action one can add to $S_{42}+S_{21}$ is

$$\begin{aligned} S_{10} \equiv\ & 2a_1\mu\langle\phi\partial_p u_p\rangle + 2a_2\mu\langle\phi\partial_p h_p\rangle + 2a_7\mu\langle u\partial_p h_p\rangle + 2a_8\mu\langle u\partial_p v_p\rangle \\ & + a_5\mu^2\langle\phi u\rangle + \tfrac{1}{2}a_3\mu^2\langle\phi^2\rangle + \tfrac{1}{2}a_4\langle\phi\Box\phi\rangle \\ & + \tfrac{1}{2}a_6\mu^2\langle u^2\rangle + \tfrac{1}{2}a_9\langle u\Box u\rangle + a_{10}\langle u\Box\phi\rangle. \end{aligned} \tag{38}$$

Taking advantage of what we learned from the spin–3 case, we assume that there will be a final scalar auxiliary fully self-dual action, *i.e.*, that there exists a non trivial S_{10} with vanishing a_4, a_9 and a_{10}. We also assume a vanishing a_7, since this term can be seen as an unpleasant kinetic term to add to the self-dual actions $u_{\bar p\bar a}\varepsilon^{pmn}\partial_m u_{\bar n\bar a}$ and $h_p\varepsilon^{pmn}\partial_m h_n$. The final equations are

$$E_{p\bar a\bar b\bar c} = 0, \tag{22}$$

$$\text{``}F_{pa} \equiv \text{`}F_{pa} + \mu a_5\eta_{pa}\phi + a_2\varepsilon^{m}_{pa}\partial_m\phi + a_6\mu\eta_{pa}u + 2a_8\eta_{pa}(\partial\cdot v) = 0, \tag{39}$$

$$\text{`}G_p \equiv G_p - 2\alpha_1\mu\partial_p\phi - 2a_8\partial_p u = 0, \tag{40}$$

$$H \equiv \frac{\delta S_{10}}{\delta\phi} = 2a_1(\partial\cdot v) + 2a_2(\partial\cdot v) + a_4\mu u + \mu a_3\phi = 0. \tag{41}$$

The scalar sector has eight independent variables:

$$\begin{aligned} &\omega_1 \equiv \hat\partial_{pabc}w_{\bar p\bar a\bar b\bar c}, && \omega_2 \equiv \hat\partial_{pab}h_{\bar p\bar a\bar b}, && \omega_3 \equiv \hat\partial_{ab}w_{\bar a\bar b}, \\ &\omega_4 \equiv \hat\partial_{ab}u_{\bar a\bar b}, && \omega_5 \equiv \hat\partial_a h_a, && \omega_6 \equiv \mu u, \\ &\omega_7 \equiv \hat\partial_a v_a, && \omega_8 \equiv \mu\phi, \end{aligned} \tag{42}$$

whose evolution is determined by $\partial_{pabc}E_{p\bar a\bar b\bar c}$, $\partial_{abc}\check E_{\bar a\bar b\bar c}$, $\partial_{bc}E_{\bar b\bar c}$, $\partial_{pa}\text{``}F_{pa}$, $\partial_b\text{``}\check F_b$, $\partial_p\text{`}G_p$ and H.

The first set of 3 equations is derived from Eq. (22) taking into account the algebraic structure of $w_{p\bar{a}\bar{b}\bar{c}}$ as given in Eq. (20). It turns out to be

$$-5\partial_{pabc}w_{\bar{p}\bar{a}\bar{b}\bar{c}} + \tfrac{5}{21}\Box\partial_{pa}w_{\bar{p}\bar{a}} + 3\Box\partial_{\bar{p}\bar{a}}u_{\bar{p}\bar{a}} = 0, \tag{43}$$

$$4\mu\partial_{pab}h_{\bar{p}\bar{a}\bar{b}} + \tfrac{9}{5}\mu\partial_{pa}u_{\bar{p}\bar{a}} = 0, \tag{44}$$

$$-4\partial_{pab}h_{\bar{p}\bar{a}\bar{b}} + \tfrac{4}{3}\mu\partial_{pa}w_{\bar{p}\bar{a}} + \tfrac{7}{5}\mu\partial_{pa}u_{\bar{p}\bar{a}} = 0. \tag{45}$$

The second set comes from Eq. (39). It consists of

$$\begin{aligned}\partial_{pa}\text{``}F_{pa} \equiv{}& \mu\partial_{pa}w_{\bar{p}\bar{a}} - 3\mu\partial_{pa}u_{\bar{p}\bar{a}} + \mu(2+a_6)\Box u\\ &+ \mu a_5\Box\phi + 2a_8\Box\partial_p v_p = 0,\end{aligned} \tag{46}$$

$$\begin{aligned}\partial_b\text{``}\check{F}_b \equiv{}& \tfrac{3}{2}\partial_{pa}w_{\bar{p}\bar{a}} + \tfrac{10}{9}\mu\partial_p h_p + \mu\partial_p v_p\\ &- \Box u - 2a_2\mu\Box\phi = 0,\end{aligned} \tag{47}$$

$$\text{``}F_{pp} \equiv \partial_p h_p + (2+a_6)\mu u + a_5\mu\phi + 2a_8\partial_p v_p = 0. \tag{48}$$

The last two equations are

$$\partial_p\text{`}G_p \equiv \delta_\mu\partial_p v_p + \mu\partial_p h_p - 2a_1\mu\Box\phi - 2a_8\Box u = 0, \tag{49}$$

and Eq. (41), $H = 0$. In terms of the ω–variables (42), Eqs. (43)–(45) allow to obtain ω_1, ω_2, ω_3 as functions of ω_4. In particular

$$\omega_3 = -\tfrac{3}{20}(9x^2+7)\omega_4. \tag{50}$$

Then it is immediate to realize that Eqs. (46)–(49), (41) become a decoupled subset of the full system. They can be written as

$$-\tfrac{27}{20}(x^2+3)\omega_4 + (2+a_6)\omega_6 + 2a_8x\omega_7 + a_5\omega_8 = 0, \tag{51}$$

$$\tfrac{3}{2}x\omega_4 + \tfrac{10}{9}\omega_5 + \omega_7 - x\omega_6 - 2a_2x\omega_8 = 0, \tag{52}$$

$$x\omega_5 + (2+a_6)\omega_6 + 2a_8x\omega_7 + a_5\omega_8 = 0, \tag{53}$$

$$\omega_5 - 2a_8x\omega_6 + \delta\omega_7 - 2a_1x\omega_8 = 0, \tag{54}$$

$$2a_2x\omega_5 + a_5\omega_6 + 2a_1x\omega_7 + a_3\omega_8 = 0. \tag{55}$$

The inverse of this determinant $\Delta_{(a_1,a_2,a_3,a_5,a_6,a_8)}$ is the propagator of the system. We wish to determine the $a_1,\ldots,a_8$ coefficients in such a way that $\Delta(x)$ is a non-vanishing real number. First we investigate the possibility of having a solution with pure next-neighbours coupling terms, *i.e.*, where $a_2 = 0 = a_5$ (they are spin–2–spin–0 couplings). In this case

$$\Delta(a_2 = 0 = a_5) = -\tfrac{27}{20}x^2(x^2+3)\big(4a_1^2x^2 + a_3(\delta - 2a_8)\big)$$
$$- 18x^2\big(a_a^2a_6' + a_3a_8^2\big) - \tfrac{9}{2}\delta a_3 a_6', \tag{56}$$

where $a_6' \equiv 2 + a_6$. Vanishing of its highest power coefficient leads to

$$a_1 = 0, \tag{57a}$$

and subsequent cancellation of quartic and quadratic terms impose

$$a_3 = 0, \tag{57b}$$

which seems an inconsistent possibility, since in this case Δ in Eq. (56) becomes identically zero. However, since we are not having ϕ–dependent action ($a_1 = a_2 = a_3 = a_5 = 0$), we have to consider the appropriate system of field equations which consists of Eqs. (22), (39) and (40) for these values of $a_{1,2,3,5}$ and does no longer contain Eq. (41). Its crucial decoupled part consists of Eqs. (51)-(54) ($a_1 = a_2 = a_3 = a_5 = 0$) and the non propagating character is determined by imposing to its associated (quartic) determinant the condition to be a non-zero real number. This leads us to determine a_6 and a_8:

$$a_6 = \tfrac{5}{44}, \quad a_8 = -\tfrac{9}{10}. \tag{58}$$

S_{10} takes a very simple form

$$S_{10} = -\tfrac{9}{5}\mu\langle u\partial_p v_p\rangle + \tfrac{22}{5}\mu^2\langle u^2\rangle, \tag{59}$$

where there is no auxiliary scalar field present.

This is the minimal solution. If we relax a little bit the assumption of considering only next-neighbours coupling and investigate the consequence of only imposing $a_2 = 0$ (leaving room for an algebraic non-next-neighbour spin–2–spin–0 coupling), we are led to $a_1 = a_3 = 0$, a_6, a_8 arbitrary and a_5 arbitrary non-vanishing.

Similarly, one might constraint a_5 to vanish and try to determine a_2. In this case one obtains (after redefining $\phi \to a_2\phi$)

$$\begin{aligned} &a_1 = \tfrac{1}{2}\delta, \qquad a_2 = 1, \qquad a_3 = \frac{20\delta^2(2+a_6)}{6a_6 + 12 - 5\delta^2}, \\ &a_6 \neq \tfrac{44}{5}, \qquad a_8 = \tfrac{1}{4}\delta, \end{aligned} \tag{60}$$

and the corresponding full action is a pure spin–4^+ action too.

It is worth observing that the simplest, self-dual, next-neighbour coupled pure spin–4^+ is then given by

$$S = S_{42}[\text{Eq. (19)}] + S_{21}[\text{Eq. (29a)}] + S_{10}[\text{Eq. (59)}], \tag{61}$$

and contains only one auxiliary self-dual spin–2, u_{ra}, and one (self-dual) vector auxiliary field v_r, in addition to the fundamental physical spin–4 carrier $w_{r\bar{a}\bar{b}\bar{c}}$.

In conclusion, we have been able to uniquely construct self-dual spin–3 and 4 actions where auxiliary fields also appear in a self-dual form (including scalars) and where coupling terms are next-neighbours. In both cases we needed one self-dual auxiliary field of spin $s-2$, $s-3$, up to spin–1.

Since spin–4 clearly is the higher-spin case, we may conjecture that this self-dual picture exists for arbitrary integer spin, where the unique non uniform structure is the final layer fixing the good spin–0 behaviour.

An additional interesting question is what should be the higher spin structure of topologically massive theories. We are inclined to think that all of them will be of third-order, as it is the case for gravity and spin–3.

It would also be interesting to see what is the connection between the present self-dual spin–3, and 4 formulations and the recently proposed [13] anionic relativistic actions for spin-j real, since this scheme consistently contains the self-dual abelian vector case.

However, as we mentioned in the beginning, whether this Dirac-like bosonic structures can be consistently coupled either to Abelian vectors or to gravity is a worthwhile question which deserves further analysis.

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