The polygon representation of flat three-dimensional spacetimes

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ABSTRACT. An important set of solutions of $2+1$ dimensional gravity consists of spacetimes which contain an open neighborhood which admits a slicing $\Sigma_{g,N,b} \times (0,1)$ in a family of genus g Riemann surfaces with N punctures and b asymptotic regions. We define a set of polygons $\mathbb{P}$, an equivalence relation R, and a bijective map from $\mathbb{P}/R$ to this set of spacetimes. The construction of a spacetime from the corresponding polygon leads to a natural extension beyond singular surfaces. We discuss the existence of a global spacelike foliation, and generalize to de Sitter manifolds, or $2+1$–dimensional gravity with a cosmological constant.

RESUMEN. Un conjunto importante de soluciones de las ecuaciones de Einstein en $2+1$ dimensiones es el de los espacio-tiempos que contienen una vecindad abierta que admite una foliación $\Sigma_{g,N,b} \times (0,1)$ en una familia de superficies de *genus* g con N punturas (partículas) y b fronteras, o regiones asintóticamente planas. Definimos un conjunto de polígonos $\mathbb{P}$, una relación de equivalencia R y una biyección de $\mathbb{P}/R$ sobre el conjunto de soluciones. La construcción del espacio-tiempo a partir del polígono correspondiente lleva a una extensión natural de las soluciones más allá de las superficies singulares. Consideramos la existencia de una foliación global regular y generalizamos a soluciones del tipo de Sitter, que corresponden a la gravedad en $2+1$ dimensiones con constante cosmológica.

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INTRODUCTION

An important problem in $2+1$ gravity is to find a convenient parametrization of the reduced phase space. Achucarro and Townsend showed that the action of $2+1$ gravity is related to the Chern-Simons invariant [1], and Witten identified the reduced phase space implicitly as the component of the moduli space of flat ISO$(2,1)$ connections with maximal Euler class [2]. Moncrief used the ADM formalism to reduce the phase space explicitly to the cotangent bundle of Teichmuller space [3]. Carlip [4] and Moncrief have showed the equivalence of the Moncrief and Witten approaches in the case of genus one. A thorough study of the solutions of $2+1$ gravity was carried out by Mess [5] using Thurston's geometric structures [6]. Our aim in this article is to provide a simple parametrization for the set of solutions.

We will introduce a set of polygons $\mathbb{P}$, which generalize Poincaré's 1882 "fundamental polygon" [7], and an equivalence relation R together with a bijective map from $\mathbb{P}/R$ into the desired set of three-manifolds. This work formalizes a recent solution of $2+1$ gravity [8].

After a review of Poincaré's fundamental polygon (Sect. 1), our polygon representation of a three-dimensional manifold is constructed in Sect. 2. We then define a map from a well-defined set of polygons (Sect. 3) to the set of solutions of 2 + 1 gravity, and show that the map is surjective (Sect. 4). The bijective map is achieved by modding out the translation and mapping class groups, in Sect. 5. In the last two sections we discuss the existence of a global spacelike foliation, and the generalization from ISO(2,1) to SO(3,1) and SO(2,2).

1. Poincaré's polygon and flat SO(2,1) connections

In the process of studying Fuchsian functions, Poincaré defined a $6g-6$ dimensional set of polygons on the upper half plane. He proved that any polygon in this set is the fundamental domain for a discrete subgroup of SL(2, ℝ), and vice-versa, any discrete subgroup of SL(2, ℝ) defines an equivalence relation that tiles the upper half plane with a lattice of equivalent polygons. The identified polygon, defined to be the quotient of the upper half plane by the discrete subgroup of SL(2, ℝ), is a compact genus g surface with a Riemannian structure inherited from the constant curvature metric on the upper half plane, and all genus g Riemann surfaces can be represented in this way (Poincaré's theorem), so the $6g-6$ parameters that define a "polygon" are labels for the space of hyperbolic structures on a genus g Riemann surface.

We review Poincaré's fundamental polygon following a modern approach due to Maskit [9]. In Minkowski space, one constructs a family of hyperbolic surfaces defined by $t^2 - x^2 - y^2 = \tau^2$, and $4g$ timelike planes through the origin which define a cone with polygonal cross-section, as shown in Fig. 1. The intersection of the filled cone with any one of the hyperbolic surfaces is a "fundamental polygon" if the two conditions are satisfied:

i) The polygon is bound by pairs of equal length segments.

In that case, the two segments in a pair are matched by the Lorentz transformation which matches the two corresponding timelike planes. Define the "identified polygon" (Maskit, [8]) as the polygon with boundary edges identified in pairs. All of the corners of the polygon are identified, and will be referred to collectively as the "vertex of the identified polygon".

ii) The "cycle transformation" is the product of all these Lorentz transformations in the order which corresponds to a small loop around the vertex (Fig. 1), and must be the identity. In the standard basis $\{u_i, v_i\}$, where the only intersections are those of u_i with v_i $(i = 1, \dots g)$,

$$u_1 v_1 u_1^{-1} v_1^{-1} \dots u_g v_g u_g^{-1} v_g^{-1} = 1.$$

This guarantees that the sum of the interior angles of the polygon is a multiple* of 2π;

*Since the hyperbolic surface is invariant under SO(2,1), the identified polygon is differentiable

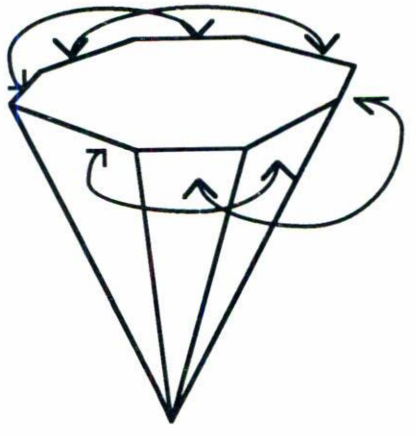

FIGURE 1. A spacetime which admits a slicing into a family of genus two surfaces, is represnted as an octagonal cone in Minkowski space, with eight walls that are identified in pairs. This spacetime is locally Minkowskian everywhere, and yet parallel transport around a non-contractible loop is not trivial. The identifications are hyperbolic elements of the Lorentz group (boosts), and are uniquely determined by the pairs of walls that are identified. They generate a "fuchsian" group of identifications, *i.e.* one that divides the future light cone in a lattice of equivalent cells —the octagonal cone being one such cell. The dotted line represents a circle around the vertex.

it is further required that the multiplicative factor be equal to one (so that there is no deficit angle at the vertex).

Poincaré's theorem states that the identified polygon is a genus g surface, and that, conversely, any genus g surface can be cut up and unfolded to become such a fundamental polygon (the proof in the present context is due to Maskit [9]). Milnor [10] considered the three-dimensional manifold which is the cone with polygonal sections, with walls identified two by two, and pointed out that the curvature vanishes at every point, *i.e.*, that it is a solution of Einstein's equations in $2+1$ dimensions. In the "time evolution" picture, one thinks of each section of the cone by a hyperbolic surface $t^2 - x^2 - y^2 = \tau^2$, as the universe at time τ. It is easy to see that the various slices are homothetic, and therefore the Teichmüller parameters are preserved in time.

Milnor also noted that there are other solutions of $2+1$ gravity, which can be understood as follows. One should consider not only Lorentz identifications, but more generally Poincaré identifications, since that is the isometry group of Minkowski space. In $2+1$ dimensions, there are as many translation generators as Lorentz generators, so this generalization implies doubling the number of parameters to $12g-12$. The additional $6g-6$ variables are the velocities of the Teichmüller parameters, roughly speaking. One would like to have a generalization of Poincaré's construction to include these non-stationary universes. The task at hand, ironically, is to generalize Poincaré's fundamental polygons to the Poincaré group.

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at the identified edges and the extrinsic curvature vanishes at the point. Any spacetime curvature singularity at the vertex must be a singularity of intrinsic curvature. The cycle being equal to the identity matrix, the intrinsic curvature singularity (surplus angle) must be a multiple of 2π.

2. Construction of a polygon from a given three-manifold

DEFINITION 2.1. Let $\mathbb{M}_{g,N,b}$ be the set of three-dimensional manifolds $\mathcal{M}$ with Lorentzian metric of vanishing curvature, that contain an open submanifold $\mathcal{N} \subset \mathcal{M}$ which admits a foliation $\Sigma_{g,N,b} \times (0,1)$ by a family of orientable genus g Riemann surfaces with N punctures and b asymptotic regions. We will denote the $\mathbb{M}$ the union of the sets $\mathbb{M}_{g,N,b}$ over all positive integers $\{g, N, b\}$.

THEOREM 2.1. ($N = 0$) Given a manifold $\mathcal{M} \in \mathbb{M}_{g,0,b}$, a surface $\Sigma_{g,0,b} \subset \mathcal{N}$ and a point P on $\Sigma_{g,0,b}$, there exist $2g+b$ closed segments $\gamma_i(s)$, $s \in [0,1]$, based at $P = \gamma_i(0) = \gamma_i(1)$, that are geodesic in $\mathcal{M}$ except at P, and form a basis of the fundamental group $\pi_1(\Sigma_{g,0,b})$.

Proof. Consider a set of $2g + b$ non-contractible loops $\gamma(s)$ on the surface $\Sigma_{g,0,b}$, based at P, which form a basis of $\pi_1(\Sigma_{g,0,b})$. We first show that any such loop $\gamma(s)$ can be smoothly deformed to be $\mathcal{M}$-geodesic except at P (this deformation generally draws the loop out of the surface, but $P \in \Sigma_{g,0,b}$ is held fixed). We will do this in two steps: First we define a variational principle which can only admit a geodesic as extremum, then we show that the extremum is attainable by smoothly deforming the original loop. Among all loops homotopically equivalent to a given $\gamma(s)$, we define a subset $\mathbb{L}_\gamma$ as those loops that are differentiable in the open interval $s \in (0,1)$ and such that the tangent at $S \to 0+$ is the same as the tangent at the end of the loop ($s \to 1-$) parallel-transported back to $s = 0$ along the path. Let $\mathbb{L}_\gamma^\epsilon$ be the set of loops $\gamma(s)$ minus their intersection with an ϵ-ball at P; we choose a parametrization $s \in [\epsilon, 1-\epsilon]$ for the truncated loops. We denote by $\mathrm{Diff}(D^\epsilon)$ the subgroup of diffeomorphisms which respects this condition on the parametrization. We will need the following non-negative functional on $\mathbb{L}_\gamma^\epsilon \times \mathrm{Diff}(D^\epsilon)$. The geodesic equation is the vector equation

$$G^a = \frac{du^a}{ds} + \Gamma^a{}_{bc} u^b u^c = 0,$$

where $u^a(s)$ is the tangent vector at $\gamma(s)$. Let

$$L^\epsilon[\gamma; \text{`coordinates'}] = \int_D \left\{ \left(G^0\right)^2 + \left(G^1\right)^2 + \left(G^2\right)^2 \right\} ds,$$

over the domain $D = [\epsilon, 1-\epsilon]$. Choosing ϵ small enough and given the differentiability of the curve, the boundary conditions defined above state that the tangent at $s = \epsilon$ is the same as the tangent at $s = 1-\epsilon$, parallel-transported back along the path. Given these boundary conditions, one easily shows that the variational principle based on the functional $L^\epsilon[\gamma]$ has a minimum at $L[\gamma_{\min}] = 0$, *i.e.*, $\gamma_{\min}$ is geodesic for $s \in [\epsilon, 1-\epsilon]$. Thus although the integrand is not covariant, the variational equation (geodesic equation) is covariant. Since ϵ can be chosen arbitrarily small, in the limit we obtain a geodesic except at $s = 0$, where it generally has a discontinuous tangent. We need to show that the minimum can be attained (*i.e.*, that the loop $\gamma_{\min}$ exists in $\mathbb{L}$). Consider all paths within the same homotopy class and which satisfy the boundary condition stated above; since it is non-negative, $L[\gamma]$ has a lower bound on this set of loops; suppose this lower bound were nonzero, and is attained with the loop γ_{min}. Consider the open curve γ_0 which is γ_{min} minus its end-point. There exists an open coordinate patch $\mathcal{O} \supset D$ which contains γ_0 and is isomorphic to an open neighborhood in $2+1$ dimensional Minkowski space. Since γ_{min} is not a stationary point of L, there is a loop deformation within this open neighborhood which leads to a smaller value of L, contradicting the assumption that L was at a minimum. Thus the lower bound must be zero. ■

THEOREM 2.2. ($b=0$) Given a manifold $\mathcal{M} \in \mathbb{M}_{g,N,0}$, a surface $\Sigma_{g,N,0} \subset \mathcal{N}$ and a point P on $\Sigma_{g,N,0}$ there exist $2g$ closed segments $\gamma_i(s)$, $s \in [0,1]$, based at $P = \gamma_i(0)$ that are geodesic in $\mathcal{M}$ for $s \neq 0,1$ and form a basis of the fundamental group of $\Sigma_{g,0,0}$, and N geodesic segments which connect P to each puncture P_j.

Proof. The first part is a corollary to Theorem 2.1, for $b=0$. Given a choice of $2g$ basis loops on $\Sigma_{g,0,0}$, there is a unique curve from P to each puncture P_j that does not intersect any of the $2g$ loops. Each such curve can be smoothly deformed into a $\mathcal{M}$–geodesic segment (proof as for Theorem 2.1). ■

THEOREM 2.3. ($N \neq 0$, $b \neq 0$) Given a manifold $\mathcal{M} \in \mathbb{M}_{g,N,b}$, a surface $\Sigma_{g,N,b} \subset \mathcal{N}$ and a point P on $\Sigma_{g,N,b}$, there exist $2g$ closed segments $\gamma_i(s)$, $s \in [0,1]$, based at $P = \gamma_i(0)$, b points $P_j \in \Sigma_{g,N,b}$ and as many closed segments $\gamma_j(s)$ based at $P_j = \gamma_j(0)$, such that the $2g+b$ segments are geodesic in $\mathcal{M}$ for $s \neq \{0,1\}$ and form a basis of the fundamental group of $\Sigma_{g,0,b}$, and $N=b$ geodesics which connect P to the punctures and to each of the b points P_j, and all punctures and handles lie to the inside of the worldsheet generated by the b geodesics $\gamma_i(S)$.

Proof. Each of the "asymptotic regions" tends to a cone with helical shift [11], which can be given coordinates $r > r_0$, $\theta \in [0, 2\pi]$. We choose b points $P_i = (r_i, 0)$. The loops $\gamma_i(s) = (r_i, 2\pi s)$ can be smoothly deformed into $\mathcal{M}$–geodesics as in Theorem 2.1. We divide $\mathcal{M}$ into three nonintersecting regions: $\mathcal{M}^{\pm}$ is the part of $\mathcal{M}$ in the causal past or future of the geodesic curves $\gamma_i(s)$, $\mathcal{M}^{\text{out}}$ is the region outside the curves (which includes the asymptotic regions) and $\mathcal{M}^{\text{in}}$, the part of $\mathcal{M}$ inside the b curves. We can always choose r_i large enough so that the N punctures lie in the interior region $\mathcal{M}^{\text{in}}$. ■

THEOREM 2.4. Given a manifold $\mathcal{M} \in \mathbb{M}_{g,N,b}$ and a choice of $2g+N$ geodesic segments based at $P \in \Sigma_{g,N,b}$, (as constructed in Theorem 2.1), there exists a surface $\Sigma^*_{g,N,b}$ which contains them and inherits a positive definite differentiable metric from $\mathcal{M}$.

Proof. We construct a positive triangulated surface, from which the desired Riemannian surface is obtained by smoothing. We choose any triangular of $\Sigma_{g,N,b}$ which includes the $2g+b$ loops as links, the points P, $P_i (i = 1, \ldots b+N)$ as vertices, plus any number of other links and vertices, and is such that each triangle with its boundary removed is a topologically trivial open set. We can smoothly deform all links of this triangulation into spacetime geodesics, as in Theorem 2.1 This leads to a triangulated surface which includes the $2g+b$ geodesic loops and $b+N$ geodesic segments —we need to show that all segments of this triangulated surface are spacelike. Let $\gamma(s)$ be any $\mathcal{M}$–geodesic segment with its end-points on $\Sigma_{g,N,b}$. We project it on $\Sigma_{g,N,b}$ following a field of parallel timelike geodesics, and show that the projected segment could not be spacelike if $\gamma(s)$ were timelike. Specifically, consider a topologically trivial open neighborhood which includes the geodesic segment $\gamma(s)$. If we are dealing with a loop, the construction proceeds with the segments $s \in [0, 1-\epsilon]$. Since $\mathcal{N} = \Sigma^{x}_{g,N,b}(0,1)$, one can define a field of parallel timelike geodesics through $\gamma(s)$, $\forall s \in [0, 1-\epsilon]$, which intersect $\Sigma_{g,N,b}$ at $\gamma'(s)$; since the end-points belong to $\Sigma_{g,N,b}$ we have $\gamma'(0) = \gamma(0)$ and $\gamma'(1) = \gamma(1)$. The curve $\gamma'(s)$ lies on $\Sigma_{g,N,b}$ and therefore must be spacelike. Now, the region between $\gamma(s)$ and $\gamma'(s)$, for $s \in [0,1]$, can be mapped into Minkowski space (possibly with identified points); the mapped segments lie on a timelike plane, or $1+1$–dimensional Minkowski space. There cannot be spacelike curve in $1+1$–dimensional Minkowski space which connects two causally related points; therefore if $\gamma(s)$ were timelike, $\gamma'(s)$ would have to be timelike as well, which cannot be since $\gamma' \in \Sigma_{g,N,b}$. ■

Denoting $\mathcal{M}^{\text{in}}$ the region of $\mathcal{M}$ to the inside of the b lines, as in Theorem 2.3, the surface $\Sigma^{*}_{g,N,b} \cap \mathcal{M}^{\text{in}}$ can be cut up along the $2g+2b+N$ geodesics and unfolded in Minkowski space. In this map, it becomes a polygonal surface bound by $2g+2N+2b$ edges which are identified two by two, and b edges which are not identified to any other. All these edges are geodesic segments, therefore they are represented by straight segments in Minkowski space. In the next section we establish the converse, *i.e.*, define a class of polygons bound by straight segments and a map from this set to the set of three-manifolds, $\mathbb{M}_{g,N,b}$.

3. GENERALIZED POLYGON

DEFINITION 3.1. We will call *generalized polygon* $\mathcal{P}$ a surface embedded in $2+1$–dimensional Minkowski space, bound by straight edges, and a set of proper orthochronous Poincaré transformations, with the following properties:

i) Edges of the *first kind* come in pairs of equal length segments and are identified by a proper orthochronous Poincaré transformation. Edges of the *second kind* are not identified with any other edge.

ii) The identified polygon, obtained by identifying the matched edges, is differentiable at the identified edges.

iii) The induced metric from the Minkowski embedding is positive definite.

We will refer to the set of such polygons as $\mathbb{P}$, and its elements as $\mathcal{P} \in \mathbb{P}$.

PROPERTY 3.1. There exists a piecewise flat triangulated surface in Minkowski space which includes all edges of the polygon and is spacelike.

Proof. The edges of the polygon are spacelike geodesic segments in Minkowski space, by definition. Consider a triangulation of the surface which includes the edges and any number of other segments. There is no obstruction in Minkowski space to deforming each segment into a geodesic (straight segment). Suppose that it were to become timelike as a result of this deformation. This timelike segment can be projected vertically onto the polygon, and the projected segment is spacelike (as in Theorem 2.4). On the plane of the projection (1 + 1–dimensional Minkowski space) we would have a spacelike segment connecting two causally related points, which is absurd. ∎

PROPERTY 3.2. Let s be an edge of the first kind, identified to another edge s' by an ISO(2, 1) transformation $g(s)$. Denote the future of s by $X^+(s)$ and its past by $X^-(s)$. The remainder X^0 is further divided into two nonintersecting regions, to the inside and to the outside of the polygon: $X^0(s) = X^i(s) + X^0(s)$. Let $\mathcal{O}(s)$ be any open set which includes the edge, and consider the intersection of $\mathcal{O}(s)$ and $X^i(s)$. The image under $g(s)$ of $\mathcal{O}(s) \cap X^i(s)$ does not intersect with the corresponding region $X^i(s')$ to the inside of the matched edge. This statement, together with the existence of a spacelike triangulation and the choice of proper orthochronous identifications, is necessary and sufficient for the existence of a spacelike surface which rests on the polygon, is differentiable at the identified edges and is orientable. Such a surface will be said to respect the matching conditions.

Proof. The proof was given by Maskit [9] for SO(2, 1); it remains valid upon replacing SO(2, 1) by ISO(2, 1). ∎

DEFINITIONS 3.2. We will need some important definitions based on the following construction. Consider any corner of the polygon, and a surface which respects the matching conditions. On this surface, draw a circle centered at this corner, starting from the interior of the polygon. When the circle crosses an edge of the first kind, use the identifications to continue the circle from the identified edge, around the identified point. The procedure stops when the circle crosses an edge of the second kind, or when the circle closes. The corners of the polygon that lie within this circle are identified; the set of such points will be called *vertex of the identified polygon.* If the circle closes, we will talk of a *vertex of the first kind.* If circle is interrupted at an edge of the second kind, we will talk of a *vertex of the second kind.* The intersection of two edges of the second kind will be called a *vertex of the third kind.* The various points which are identified will be called *images* of the vertex. A vertex of the third kind has only one image, while a vertex of the second kind has at least two (it belongs to an edge of the first kind, which is identified with another edge). For a vertex of the first kind, consider the closed circle on the identified polygon constructed in the definition above. The product of all the Poincaré identifications, in the order in which the edges are crossed as one follows a circle around the vertex will be called the *cycle transformation* at the vertex. For a vertex of the second or third kind, we define the "cycle transformation" simply by setting it equal to the identity.*

EXAMPLE. The heptagon of Fig. 2 is composed of six edges of the first kind, *i.e.*, three pairs, and one edge of the second kind. Following a circle around the point 1 in a clockwise direction, we meet the edge 17, which is of the second kind. In the counterclockwise direction, we meet the edge 12, continue through the edge 34, then meet the edge 45,

*This represents the fact that the asymptotically conical region should be glued on without introducing any singularity.

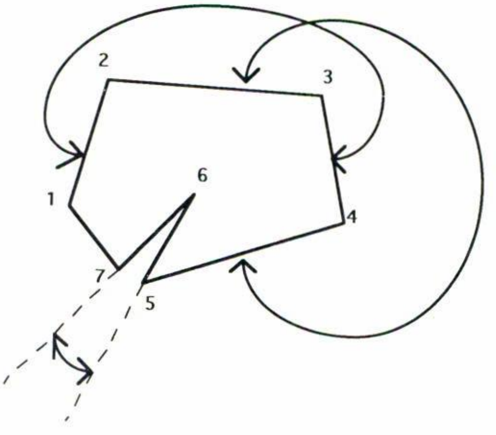

FIGURE 2. This heptagon embedded in 2 + 1 Minkowski space, with the identifications indicated by arrows, represents a surface with one handle, a puncture {6} and a boundary. The boundary is the only edge (17) which is not identified to any other. Upon "gluing" together the edges 56 and 67, a conical singularity appears at the point {6}. The edges 12 and 23, and their images 34 and 45, are the two generators chosen as basis for the homotopy group of the torus.

continue through the edge 23 then meet the edge 34, continue through the edge 12 then meet the edge 23, continue through the edge 45 then meet the edge 56, continue through the edge 67 then meet the edge 71, which is of the second kind. Starting from the point 6, we meet the edge 67 then continue through the edge 65 and close the circle. There are two vertices, $\{1,2,3,4,5,6\}$ and $\{6\}$. The cycle at the vertex $\{6\}$ is the Lorentz transformation which matches the sides 67 and 65, the other cycle is the identity by definition.

A smooth surface which respects the matching conditions has only one boundary, the edge 71. The identified polygon is a torus with a puncture, at $\{6\}$, and a boundary (Fig. 3).

PROPERTY 3.3. The cycle transformation at a vertex of the first kind is an elliptic element of the Lorentz group.

Proof. The cycle transformation is the holonomy of the Poincaré connection along the circle; such holonomies depend only on the homotopy class, since the loops considered are in Minkowski space (with identifications). Given any ϵ, we can choose a circle of radius smaller then ϵ. The holonomy must be a Lorentz transformation combined with a translation parallel to its axis [11]. Since the differentiable surface is spacelike, this axis must be timelike (a point like source with hyperbolic holonomy (tachyon) would imply the existence of closed timelike lines [12], which would contradict the fact that $\mathcal{M} \in \mathbb{M}_{g,N,b}$). If the translation parallel to the timelike axis were nonzero, one could choose ϵ small enough so that the circle would be timelike [12], in contradiction with the fact that it lies on a spacelike surface. Thus the cycle transformation must be a pure Lorentz transformation with timelike axis. ■

THEOREM 3.1. Properties 3.1, 3.2, 3.3 are necessary and sufficient for conditions (2) and (3) to be satisfied.

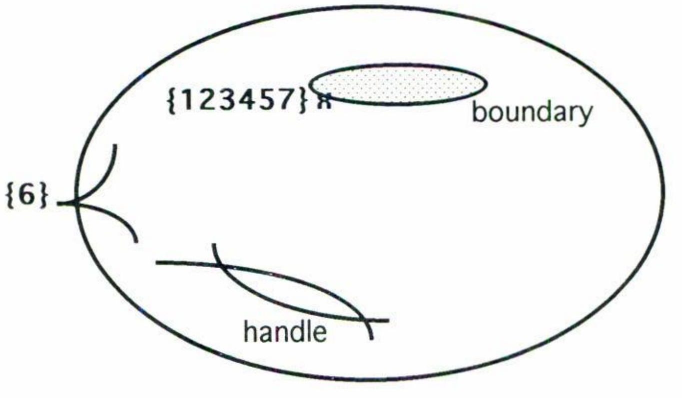

FIGURE 3. A smooth surface which rests on the polygon of Fig. 2 and respects the matching conditions, will remain smooth when the identified edges are glued together. Note the conical singularity and the boundary on the torus topology. To recover the heptagon, one would cut from the vertex {123457} following two intersecting loops around the torus, from this vertex to the vertex {6}, and also along a path which surrounds the boundary. The result is topologically equivalent to the heptagon, and can be made to be geometrically equivalent by deforming the paths into geodesic segments.

Proof. That the conditions are necessary was proved above for each property. The Property 3.3 is a consequence of 3.1, since the proof of the former depends only on the existence of a spacelike surface. That 3.1 together with 3.2 imply conditions (2) and (3) was proved by Maskit [9], as mentioned above. ■

The identified polygon is a surface with a positive definite, differentiable metric induced by the embedding in Minkowski space; what is the topology of this surface? A vertex of the first kind is a curvature singularity, unless the cycle happens to be the identity. The number of such vertices is therefore the number of punctures, N. To find the number of handles, or genus, consider the number of images for each vertex, $n(v)$. Then the number of handles is $g = \Sigma_v [n(v)/4]$ (consider a surface with g wormholes; the surface can be unwrapped by performing $2g$ cuts which intersect two by two, each intersection point is split into four images in the process. Vice-versa, for any set of four identified images there must be two pairs of identified edges, and one can draw two independent non-contractible loops which intersect at one point). Finally, the number of boundaries is equal to the number of vertices of the second kind.

4. THREE-MANIFOLD FROM A GIVEN POLYGON: SURJECTIVE MAP

DEFINITION 3.3. We define a map $f: \mathcal{P} \to \mathcal{M}$ by explicitly constructing a three-manifold from a given polygon, as follows. For each vertex of the first kind, choose an image of that vertex and construct the corresponding cycle transformation, an elliptic element of $SO(2,1)$. Draw a line through this image of the vertex and parallel to the axis of this $SO(2,1)$ transformation (the axis is timelike since the cycle is elliptic). Following the identifications, all other images of the vertex are related to the chosen one by a

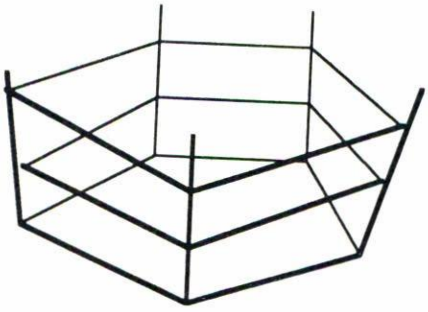

FIGURE 4. The three-manifold which is represented by a fundamental polygon (a pentagon in this example) is constructed by attaching to each corner of the polygon a timelike line, in such a way that these respect the identification conditions. A family of polygons is obtained by pushing the corners of the polygon along these timelike lines, again respecting the identification conditions, and the three-volume interior to the family of polygons, with the identifications, is the three-manifold with topology $\Sigma_{g,b,N} \times \mathbb{R}$ and vanishing three-curvature. In the example, $g = 1$, $N = 0$ and $b = 1$.

Poincaré identification; likewise, one obtains the images of the timelike line under these identifications. Every image of every vertex of the first kind is thus endowed with a timelike line. The cycle may be equal to the identity matrix, in which case the axis is ill-defined; in this case, and for all vertices of the second kind or third kind, the first image of the vertex is endowed with an arbitrary timelike line, say $(t, 0, 0)$; the lines through of other images of such a vertex are obtained by applying the corresponding identifications, as in the previous case. Each corner of the polygon is thus endowed with a timelike line; one can construct other polygons (or "slices") $\mathcal{P}(t)$ by moving each corner along its timelike line for a fixed proper time, connecting the corners with geodesic segments (like the edges of the original polygon), and choosing a surface $\mathcal{P}(t)$ which includes these segments and respects the matching conditions. We will show in Sect. 5 (Lemma 5.2) that the map f is a function $f: \mathbb{P} \to \mathbb{M}/R^*$, where two manifolds $\mathcal{M}, \mathcal{M}'$ are R^*-equivalent if they are isomorphic up to a possible singular surface. In this section, we will show that $\mathcal{P}(t)$ generates a three-manifold $\mathcal{M} \in \mathbb{M}$ and that any such manifold can be generated in this way.

THEOREM 4.1. Let $\mathcal{M}$ be the region of Minkowski space spanned by the family of polygons $\mathcal{M}(t)$ for $t \in (-\infty, +\infty)$, with the Poincaré identifications, and any singular surface removed. Then $\mathcal{M}$ is a three-manifold which belongs to the set M (Fig. 4).

Proof. Each $\mathcal{P}(t)$ has the identification rules of $\mathcal{P}(0)$, and therefore the same topology. The polygons are generated by sliding the corners along *timelike* lines, so one can attempt to construct surfaces $\mathcal{P}(t)$ such that there is no intersection between two polygons $\mathcal{P}(t)$ and $\mathcal{P}(t')$, for $t \neq t'$. This is clearly possible locally in t, to extend for all t, it is necessary to appeal to the mapping class group symmetry; we show in Theorem 5.4 that a timelike line, such as the world line of a corner of a polygon with intersections with singular surfaces removed, can be covered with a countable set of open neighborhoods $\mathcal{O}_i$, and that there exists a choice of generators of the fundamental group in each such neighborhood such that nonintersecting surfaces $\mathcal{P}(t)$ can be chosen in each interval. The region spanned by

$\mathcal{P}(t)$ minus the singular surfaces therefore has a regular slicing, and can be endowed with a map $\{\mathcal{O}^*_{\alpha i}\}$ generated from maps of the polygons $\mathcal{P}(t)$ (the map of $\mathcal{P}(t)$ is the set of open neighborhoods $\{\mathcal{O}^*_{\alpha i} \cap \mathcal{P}(t)\}$ for $t \in \mathcal{O}_i$). The region of Minkowski space spanned by $\mathcal{P}(t)$, with identifications, is a three-dimensional manifold. The curvature is identically zero except possibly at vertices. We have set the cycle equal to the identity at all vertices of the second kind and of the third kind, so that glueing the asymptotic regions does not introduce any curvature singularity. Curvature singularities occur along the worldliness of vertices of the first kind, which are the "punctures". It remains to show that there exists an open region $\mathcal{N} \subset \mathcal{M}$ which admits a slicing into a family of spacelike surfaces of genus g with N punctures and b boundaries. Since the family of polygons $\mathcal{P}(t)$ is continuous in t by construction and the polygon $\mathcal{P}(0)$ is spacelike, there exists and $\epsilon > 0$ such that the polygons $\{\mathcal{P}(t),\ t \in (-\epsilon, \epsilon)\}$ are spacelike. They have the same topology as $\mathcal{P}(0)$, as argued above, so these polygons generate an open neighborhood $\mathcal{N}_\epsilon$ with the required properties. ∎

THEOREM 4.2. The map $f: \mathcal{P} \to \mathcal{M}$ is a surjection onto $\mathbb{M}$.

Proof. Any flat three-dimensional manifold which contains an open neighborhood $\mathcal{N} \subset \mathcal{M}$ which admits a slicing in a family of genus g Riemann surfaces with N punctures and b asymptotic regions, for any g, N, b, can be represented by a generalized polygon as defined in the previous section. This is proved by construction, using the theorems of Sect. 2. From a point O of the three-manifold, one draws $2g$ closed spacetime geodesics which form a basis of the homotopy group of $\Sigma_{g,0,0}$, $N + b$ geodesics which connect the point O to the worldlines of the punctures and b points in the asymptotic regions along with b geodesic loops based at these points and such that the punctures lie to the inside of these loops, as explained in Theorem 2.3; altogether $2g + 2b + N$ geodesic segments. One chooses a smooth spacelike surface Σ^* which includes these geodesics (Theorem 2.4), then cuts this surface along the geodesics to obtain a surface with $4g + 2N + 3b$ edges and the topology of a disk; the cut surface is topologically trivial and can be mapped in Minkowski space, leading to a polygon $\mathcal{P}$. We must show that $\mathcal{P} \in \mathbb{P}$. Since all edges are spacetime geodesics, they become straight segments in Minkowski space. The polygon is bound by $4g+2N+2b$ edges of the first kind and b edges of the second kind; the identified polygon is the smooth spacelike surface Σ^*, so it is differentiable at the identified edges and inherits a positive definite metric from the Minkowski embedding, by construction, therefore $\mathcal{P} \in \mathbb{P}$. To show that $f(\mathcal{P}) = \mathcal{M}$, consider the vertices in $\mathcal{M}$ and timelike geodesics through each vertex; since $\mathcal{M}$ has the topology $\Sigma_{g,N,b} \times \mathbb{R}_s$, where $\mathbb{R}_s$ is the real line with points removed (s is the number of singular surfaces) the map in Minkowski space of these geodesics gives a timelike line at each image of each vertex of the polygon $\mathcal{P}$. We have constructed the map $f(\mathcal{P})$ from within $\mathcal{M}$, therefore necessarily $\mathcal{M} = f(\mathcal{P})$. ∎

5. EQUIVALENT POLYGONS; BIJECTIVE MAP $f: \mathcal{P} \in \mathbb{P}/R \to \mathcal{M}$

In this section we will define three equivalence relations in the set of polygons $\mathbb{P}$, related to translations of vertices in $\mathcal{M}$, to surface deformations and to the mapping class group.

The map $f: \mathcal{P} \to \mathcal{M}$ defined in Sect. 4 allows us to define a bijective map $f': \mathcal{P}' \in \mathbb{P}/R \to \mathcal{M} \in \mathbb{M}$ by choosing an element $\mathcal{P}'$ in each class. Finally, the mapping class group symmetry allows us to define a covering of the three-manifold with a countable set of open neighborhoods, and later to discuss the existence of a global spacelike foliation.

TRANSLATION OF A VERTEX. Given a polygon $\mathcal{P}$, one chooses an image of a vertex and constructs the corresponding cycle transformation. One draws a line through this image of the vertex and parallel to the axis of the SO(2,1) transformation. If the cycle is equal to the identity matrix one chooses an arbitrary timelike line, say $\gamma(t) = (t, 0, 0)$. All other images of the vertex are endowed with an image of this line following the identifications. One constructs the polygon $\mathcal{P}'$ by moving each image of the vertex along its timelike line for a fixed proper time, connecting the corners with geodesic segments (like the edges of the original polygon), and choosing a surface $\mathcal{P}'$ which includes these segments and respects the matching conditions. Note that $\mathcal{P} \in \mathbb{P}$ does not guarantee that $\mathcal{P}' \in \mathbb{P}$, for instance $\mathcal{P}'$ may not be spacelike.

DEFINITION 5.1. Two polygons $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$ are R_1-equivalent if there exists a set of translations of the vertices that take each image of each vertex of $\mathcal{P}$ to each image of each vertex of $\mathcal{P}'$.

EXAMPLE. The polygons $\mathcal{P}(t)$ which we constructed to define the map $f: \mathcal{P} \to \mathcal{M}$ are R_1-equivalent, for $t \in \mathcal{O}_i$.

THEOREM 5.1. If two polygons $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$ are R_1-equivalent, then they can be derived from manifolds $\mathcal{M}, \mathcal{M}' \in \mathbb{M}$ which admit isomorphic submanifolds: $\mathcal{N} \subset \mathcal{M}, \mathcal{N}' \subset \mathcal{M}'$, where $\mathcal{N} = \mathcal{N}'$ admits a slicing into a family of positive surfaces $\Sigma_{g,N,b} \times (0,1)$. Furthermore, $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic except possibly for their continuation beyond singular surfaces.

Proof. Assume first that the polygons are R_1-equivalent by timelike translations. Given the polygon $\mathcal{P} \in \mathbb{P}$, we construct a three-manifold as in section 4 by choosing timelike lines at the vertices that are parallel to the translations by which $\mathcal{P}$ is equivalent to $\mathcal{P}'$, when these are nonzero, and arbitrary otherwise. In this way we construct a three-manifold $\mathcal{M}$ which contains the slices $\mathcal{P}, \mathcal{P}'$ by construction and belongs to $\mathbb{M}$, so it admits a submanifold $\mathcal{N} \subset \mathcal{M}$ with the required properties (in this case $\mathcal{M} = \mathcal{M}'$). If the translations are not timelike then they are combinations of two timelike translations; the first (forward in time) generated a manifold $\mathcal{M}$ as before, which contains $\mathcal{P}$ and $\mathcal{P}''$. The second set of timelike translations leads to a manifold $\mathcal{M}'$ which includes $\mathcal{P}''$ and $\mathcal{P}'$. We must show that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic except possibly for different continuations beyond singular surfaces. Consider the manifold $\mathcal{M}$ and the vertices of the identified polygon $\mathcal{P}'' \in \mathcal{M}$. We can construct timelike geodesics in $\mathcal{M}$ at these vertices and parallel to the second set of timelike translations. Following these geodesics for proper times equal to those which defined the translations of the vertices of $\mathcal{P}''$ to those of $\mathcal{P}'$, one obtains a set of points of $\mathcal{M}$ which are connected to the vertices of $\mathcal{P}''$ by timelike geodesic segments. The map of this construction in Minkowski space shows a polygon $\mathcal{P}''$ and timelike geodesic segments from the corners of this polygon, with direction and length corresponding to the translations which took the polygon $\mathcal{P}''$ to $\mathcal{P}'$. We construct along these lines a family of polygons $\mathcal{P}'(t)$ where $\mathcal{P}'(0) = \mathcal{P}''$, $\mathcal{P}'(1) = \mathcal{P}'$ and each $\mathcal{P}'(t)$ represents a surface of the manifold $\mathcal{M}$. Since $\mathcal{P}'(t)$ generates $\mathcal{M}'$ (by definition of $\mathcal{M}'$), we conclude that the open neighborhood $\mathcal{N} = \{\mathcal{P}(t),\ t \in (0,1)\}$ is subset of both $\mathcal{M}$ and $\mathcal{M}'$. This shows that $\mathcal{M}$ and $\mathcal{M}'$ are evolving in time the same Cauchy data on an "initial surface" $\mathcal{P}_0$, where $\mathcal{P}_0$ is the identified polygon at any $t_0 \in (0,1)$. Therefore the time evolution generates a unique three-manifold up to possible singular surfaces. ■

COROLLARY. The manifolds constructed in Sect. 4 admit a submanifold $\mathcal{N} \subset \mathcal{M} \in \mathbb{M}$ which is independent of the timelike lines chosen for vertices of the second or third kind.

DEFINITION 5.2. Two polygons $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$ are R_2-equivalent if $\partial\mathcal{P} = \partial\mathcal{P}'$ and the sets of ISO(2,1) identifications of $\mathcal{P}$ and $\mathcal{P}'$ are the same.

THEOREM 5.2. If $\mathcal{P}$, $\mathcal{P}'$ are R_2-equivalent, then both can be derived from the same three-manifold $\mathcal{M} \in \mathbb{M}$.

Proof. Since $\partial\mathcal{P} = \partial\mathcal{P}'$, there is a compact submanifold $\mathcal{P} - \mathcal{P}'$ of 2 + 1–dimensional Minkowski space bound by $\mathcal{P}$ $\mathcal{P}'$. We construct the manifold $\mathcal{M} = f(\mathcal{P})$, as in Sect. 4. Since $\mathcal{P}, \mathcal{P}'$ are spacelike these lines intersect $\mathcal{P} - \mathcal{P}'$ only at the corners. The manifold $\mathcal{M}$ is the region of Minkowski space contained within the timelike walls defined by the corners and timelike lines, with identifications, so clearly $\mathcal{P}$ and $\mathcal{P}'$, which are spacelike, must be included in this manifold, *q.e.d.*. ■

DEFINITION 5.3. Two polygons $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$ are R_3-equivalent if the corresponding sets of ISO(2,1) identifications are related by a mapping class group transformation.

THEOREM 5.3. If two polygons $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$ are R_3-equivalent, then both can be derived from the same manifold $\mathcal{M} \in \mathbb{M}$.

Proof. Let $\mathcal{M} = f(\mathcal{P})$. The identified polygon is a positive surface $\Sigma_{g,N,b}$ embedded in $\mathcal{M}$. The mapping class group transformation with takes the ISO(2,1) identifications of $\mathcal{P}$ onto those of $\mathcal{P}'$ has a representation on $\Sigma_{g,N,b}$ which corresponds to changing the basis set of loops and segments which intersect at the vertices, holding these vertices fixed. The new loops and segments can be deformed smoothly to be geodesic except at the vertex, remain nonintersecting except at the vertices, and there exists a positive, differentiable surface $\Sigma^*_{g,N,b}$ which includes the new geodesic segments and loops (all of this was proved in Sect. 2). The map of $\Sigma^*_{g,N,b}$ in Minkowski space is a polygon $\mathcal{P}^*$ which has the same vertices and ISO(2,1) identifications as $\mathcal{P}'$ by construction, therefore they are R_2-equivalent, and the theorem becomes a corollary of Theorem 5.2 ■

COROLLARY. The manifolds constructed in Sect. 4 are independent of the choice of generators of the fundamental group.

THEOREM 5.4. Given a timelike geodesic $\gamma(t) \subset \mathcal{M}$, where $t \in \mathbb{R}_s$ is the real line with s points removed (where $\gamma(t)$ intersects singular surfaces), there exists a countable set of open neighborhoods $\mathcal{O}_i$ which covers $\gamma(t)$ and for each i there exists a basis of generators of the homotopy group at some $\gamma(t) \in \mathcal{O}_i$, that can be smoothly translated into nonintersecting generators of the homotopy group at $\gamma(t')$, $\forall t' \in \mathcal{O}_i$.

Proof. The proof is by construction with the help of the polygon representation. Since $\mathcal{M} \in \mathbb{M}$, it admits a surface $\Sigma_{g,N,b} \subset \mathcal{N}$ which can be smoothly deformed into the identified polygon $\mathcal{P}(0)$ whose map in Minkowski space is bound by straight segments. A family of polygons $\mathcal{P}(t)$ is constructed by associating a timelike line to each image of each vertex and pushing the corners of the polygon along these lines, as discussed in Sect. 4 Let $s_i(0)$ be the edges of the polygon $\mathcal{P}(0)$, and $s_i(t)$ the edges of the polygon $\mathcal{P}(t)$ which results from translating by a proper time t. Each $s_i(t)$ defines a "timelike wall" in Minkowski space (a timelike flat surface). As long as a corner of $\mathcal{P}(t)$ does not intersect with a wall $\gamma(t)$ (other than the two which is belongs to by construction), the construction just described is the required smooth translation of generators of the homotopy group within the first open neighborhood $\mathcal{O}_1$. If an intersection occurs at $t = t_1$, we must show that there exists an open neighborhood $\mathcal{O}_2$, where $\gamma(t_1) \in \mathcal{O}_2$ and a choice of geodesic generators of the homotopy group which do not intersect for $\gamma(t) \in \mathcal{O}_2$. We first show that the intersection point (a corner of $\mathcal{P}(t_1)$) must intersect the wall at the same t_1. If it did not, then $\gamma(t_1)$ and the intersection point would form a timelike triangle and $\mathcal{P}(t_1)$, which includes $\gamma(t_1)$ and the intersection point, could not be spacelike (proof as for Theorem 2.4); thus we would already have passed a singular surface at some $t < t_1$ and the open neighborhood $\mathcal{O}_1$ would be valid up to the singular surface, as required. Thus, the intersection point belongs to the geodesic segment $s \subset \partial\mathcal{P}(t_1)$. The corner lies on the geodesic segment s and as a corner of the polygon it is the intersection of two other geodesic segments, so the three geodesic segments are intersecting. Any choice of a basis of *non-intersecting* generators of the fundamental group can be deformed into spacelike geodesics as shown in Sect. 2; it is always possible to choose the basis so that these geodesic generators intersect only at their common base point, as long as the surface is not singular. This allows us to construct a new polygon $\mathcal{P}_2(t_1)$, R_3-equivalent to $\mathcal{P}(t_1)$, and a family of polygons $\mathcal{P}_2(t)$ which will be non-intersecting for some open neighborhood $\gamma(t) \in \mathcal{O}_2$. Repeating this procedure leads either to crossing a singular surface, covering of the line $\gamma(t)$ with a countable sequence of open neighborhoods, or convergence to an accumulation point of the series of intersections $\gamma(t_1), \gamma(t_2), \gamma(t_3), \ldots$. Such an accumulation point can only occur if for any choice of generators of the homotopy group and any ϵ, there is an N such that for $N' > N$ there is a corner of $\mathcal{P}(t_{N'})$ which lies at a distance less than ϵ from a segment $\gamma(t_{N'})$ to which it does not belong. This cannot happen if $\mathcal{P}(t)$ is a regular surface at the accumulation point, so $\mathcal{P}(t_\infty)$, would be a singular surface. ∎

DEFINITION 5.4. We define the subset $\mathcal{P}_s \subset \mathbb{P}$ by choosing a representative of each equivalence class of $\mathbb{P}$ modulo $R = R_1 U R_2 U R_3$.

LEMMA 5.1. Consider $\mathcal{P} \in \mathbb{P}$ and $\mathcal{P}'$, a spacelike surface with the topology of the disk and boundary $\partial\mathcal{P} = \partial\mathcal{P}'$. The three-manifold $\mathcal{M} = f(\mathcal{P})$ belongs to $\mathbb{M}_{g,N,b}$, for some g, N, b. The surface $\mathcal{P}'$ represents a genus g Riemann surface in $\mathcal{M}$, with N punctures and b asymptotic regions.

Proof. The three-manifold $\mathcal{M}$ is the region of Minkowski space bound by timelike walls $s_i(t)$, where $s_i(t)$ are the edges of the polygons $\mathcal{P}(t)$ constructed in Sect. 4, and the walls generated by segments of the first kind are identified. Since $\mathcal{P}'$ is spacelike and the walls are generated by timelike translations of the corners, the surface $\mathcal{P}'$ intersects the walls only at the edges $s_i(0)$ of $\partial\mathcal{P} = \partial\mathcal{P}'$ and $\mathcal{P}'$ is contained in $\mathcal{M}$. $\mathcal{P}'$ and $\mathcal{P}$ are disks with the same identifications on the boundary, therefore they have the same topology. ∎

DEFINITION 5.5. $\mathcal{M}, \mathcal{M}' \in \mathbb{M}$ are R'-equivalent if and only if there exist isomorphic open neighborhoods $\mathcal{N} \subset \mathcal{M}$, $\mathcal{N}' \subset \mathcal{M}'$, where $\mathcal{N}$ and $\mathcal{N}'$ satisfy the conditions stated in Definition 2.1

LEMMA 5.2. f is a function $f: \mathbb{P} \to \mathbb{M}/R^*$.

Proof. The lemma is a direct consequence of Theorems 5.1, 5.2 and 5.3. ∎

LEMMA 5.3. Let $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$ represent two slices of the same open neighborhood $\mathcal{N} \subset \mathcal{M} \in \mathbb{M}_{g,N,b}$ such that $\mathcal{P}$ represents a genus g surface with N punctures and b boundaries and $\mathcal{N}$ admits a foliation into spacelike surfaces. Then $\mathcal{P}'$ has the same topology as $\mathcal{P}$.

Proof. Since $\mathcal{P}, \mathcal{P}'$ belong to $\mathbb{P}$, they represent spacelike surfaces $\Sigma_{g,N,b}$ and Σ' in $\mathcal{M}$. The corners of $\mathcal{P}$ are images of vertices $P_i \in \Sigma_{g,N,b} \subset \mathcal{M}$. The map $f(\mathcal{P}) = \mathcal{M}$ (lemma 5.2) associates a timelike geodesic to each vertex P_i. Let P_i' denote the intersections of these timelike geodesics with Σ' (which exist since $\mathcal{M}$ has the topology $\Sigma_{g,N,b} \times \mathbb{R}$. Since $\mathcal{M}$ has the topology $\Sigma_{g,N,b} \times \mathbb{R}_s$ and $\mathcal{N}$ admits a spacelike foliation, there exists a spacelike surface $\Sigma^*_{g,N,b}$ which includes these vertices and has the same topology as $\Sigma_{g,N,b}$. Consider a basis of $2g + b$ loops and $N + b$ segments on Σ^*, which intersect only at the vertices. By Theorem 2.3 we can smoothly deform them into spacelike paths $\gamma_\mu(s)$ which are geodesic for $s \notin \{0, 1\}$ and intersect only at the vertices. We can smoothly deform Σ' holding the vertices fixed so that the deformed surface Σ'' includes the $2g + 2b + N$ geodesic segments. Cutting Σ'' and Σ^* along these segments we obtain polygons $\mathcal{P}''$ and $\mathcal{P}^*$ with $\partial\mathcal{P}'' = \partial\mathcal{P}^*$, so Lemma 5.1 tells us that Σ'' has the same topology as $\Sigma_{g,N,b}$, and also Σ' by transitivity. ∎

THEOREM 5.5. The restriction of f to $\mathbb{P}_s$, $f': \mathcal{P}_s \to \mathcal{M}$, is a bijection from $\mathbb{P}_s$ to $\mathbb{M}/R^*$.

Proof. It was shown in Sect. 4 that f is surjective onto $\mathbb{M}$, the restriction f' is also surjective as a result of Theorems 5.1 and 5.2. Given that f' is surjective on $\mathbb{M}$, it is also surjective on $\mathbb{M}/R^*$. We need to show that it is also injective. Suppose two polygons $\mathcal{P}, \mathcal{P}'$ lead to the same three-manifold $\mathcal{M}$. The identified polygons would be two spacelike slices Σ, Σ' of $\mathcal{M}$, and therefore must have the same topology (lemmas). Therefore, there is a mapping class group transformation which takes $\mathcal{P}$ to $\mathcal{P}''$, where $\mathcal{P}''$ is R_3-equivalent to $\mathcal{P}$ and is such that its boundary can be smoothly deformed to that of $\mathcal{P}'$; we will show that the displacement of corners is a combination of translations of vertices, and therefore that $\partial\mathcal{P}''$ is R_1-equivalent to $\partial\mathcal{P}'$, so $\mathcal{P}$ is equivalent to $\mathcal{P}'$ by transitivity and R_2-equivalence. Consider one corner P of $\mathcal{P}''$ and the corresponding corner P' of $\mathcal{P}'$; there is a translation which takes P onto $\mathcal{P}'$. Consider another corner which is an image of the first under the identification $g(P)$, where $g \in \mathrm{ISO}(2,1)$. The image $g(P')$ must be the corresponding corner of $\mathcal{P}'$ (if it were not, then the two corners of $\mathcal{P}'$ would not be related by the given ISO(2, 1) identification, which would contradict the fact that $\mathcal{P}'$ belongs to $\mathcal{M}$). The transformation of all such images of a vertex for all vertices of $\mathcal{P}''$ shows that $\mathcal{P}''$ is R_1-equivalent to $\mathcal{P}'$. ∎

6. Global existence of a spacelike foliation

The boundary of a given polygon consists of a one-dimensional closed figure in Minkowski space formed of straight edges. Consider a triangulation such that its segments consist exclusively of sums or differences of the boundary segments. We have seen that the space-time $\mathcal{M}$ can be constructed by sliding the corners of the polygon along timelike geodesics, leading to the family $\mathcal{P}(t)$ of polygons Likewise, this generates a family of triangulated surfaces. An important question is under what circumstances can the polygons $\mathcal{P}(t)$ be chosen to be spacelike for all t (no singular surface). It is clearly sufficient that the triangulated surfaces be spacelike for all t. To determine whether this is true, note that the edges of the polygon $\mathcal{P}(t)$ are three-vectors in Minkowski space which depend linearly on t; let us denote these vectors by $\mathbf{E}_\mu$. For segments of the first kind, these are identified to a Lorentz rotated three-vector $\mathbf{E}_{-\mu} = \mathbf{M}_\mu^{-1}\mathbf{E}_\mu$, where $\mathbf{M}_\mu$ is the Lorentz projection of the ISO(2, 1) identification (the minus sign and inverse are conventional). The area vector of a triangle is the antisymmetrized product of two of its edges; the condition that all such area vectors remain timelike is thus expressible as a set of conditions on the initial vectors and their velocities.

Theorem 6.1. Let $\mathcal{P}(0)$ be a polygon representing a genus g surface in the standard way (one vertex and $4g$ edges identified in pairs). It represents a manifold $\mathcal{M}$ which admits a regular spacelike foliation in a family of genus g surfaces if and only if the vectors $\{\mathbf{E}_\mu$ $(\mu = 1, \ldots, 2g)$, $\mathbf{E}_1+\mathbf{E}_2$, $\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_{-1}$, $\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_{-1}+\mathbf{E}_{-2}$, $\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_{-1}+\mathbf{E}_{-2}+\mathbf{E}_3, \ldots\}$, are spacelike for all t.

Proof. The given vectors form a spacelike triangulation of the polygon, so the condition is sufficient. To show that it is necessary, note that each of these vectors connect two images of the single vertex of the identified polygon, *i.e.* they are closed geodesic segments. A manifold $\mathcal{M}$ which admits a closed timelike geodesic segment could not admit a spacelike triangulation (see theorem 2.4). ∎

Example (genus two). Consider the genus two surface represented as an octagon in Minkowski space (Fig. 5). The eight images O_i of the vertex of the identified polygon are endowed with timelike lines respecting the identification conditions, namely $\mathbf{N}$ at O_1, $M_1M_2M_1^{-1}\mathbf{N}$ at O_2, $M_2M_1^{-1}\mathbf{N}$ at O_3, $M_1^{-1}\mathbf{N}$ at O_4, $M_2^{-1}M_1M_2M_1^{-1}\mathbf{N}$ at O_5, $M_3^{-1}\mathbf{N}$ at O_6, $M_4M_3^{-1}\mathbf{N}$ at O_7 and $M_3M_4M_3^{-1}\mathbf{N}$ at O_8. The boundary segments are the vectors $O_i - O_j$, for instance $\mathbf{E}_1(t) = \mathbf{E}_1(0) + (\mathbf{M}_1\mathbf{M}_2\mathbf{M}_1^{-1} - \mathbf{1})\mathbf{N}t$. It is spacelike if

$$\big(\mathbf{E}_1(0)\big)^2 + \big((\mathbf{M}_1\mathbf{M}_2\mathbf{M}_1^{-1} - \mathbf{1})\mathbf{N}t\big)^2 + 2\big(\mathbf{E}_1(0)\cdot(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_1^{-1} - \mathbf{1})\mathbf{N}t\big) > 0 \;\forall t.$$

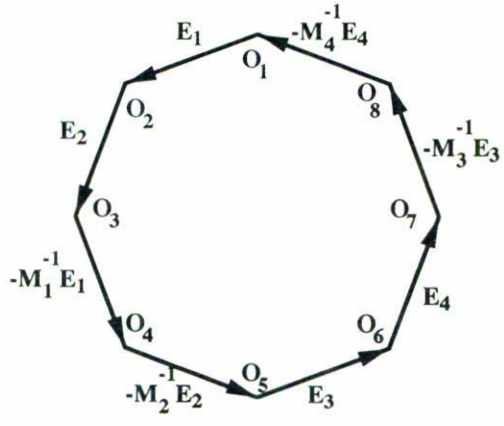

FIGURE 5. The octagon corresponds to a genus two surface, a slice of the spacetime represented in Fig. 1. The points O_i, $i = 1, \ldots, 8$ are the eight images of the same vertex of the identified polygon. The edges of the polygon are straight segments in Minkowski space which we represent by three-vectors, such as $\mathbf{E}_1$ and its identified partner $\mathbf{M}_1^{-1}\mathbf{E}_1$.

The statement is necessarily true at $t = 0$, it will be true for all t if and only if the characteristic determinant for the second order polynomial in t is negative, *i.e.*, iff

$$\Delta = \big(\mathbf{E}_1(0)\cdot(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_1^{-1} - 1)\mathbf{N}\big) - \big(\mathbf{E}_1(0)\big)^2\big((\mathbf{M}_1\mathbf{M}_2\mathbf{M}_1^{-1} - 1)\mathbf{N}t\big)^2 < 0.$$

A similar calculation for the other segments of the triangulation gives the complete set of conditions of the initial values $\mathbf{E}_\mu(0)$ and $\mathbf{M}_\mu$ (the latter are independent of t, so they are also "initial conditions").

THEOREM 6.2. Let $\mathcal{P}(0)$ denote any polygon in the set $\mathbb{P}$. It admits a spacelike triangulation $\mathcal{T}(0)$ whose vertices are corners of the polygon, by definition of $\mathbb{P}$. Given a choice of timelike lines at each vertex of the identified polygon and the images of these timelike lines at each corner of the polygon uniquely defined by the identifications, we obtain the triangulated surface $\mathcal{T}(t)$ by sliding the corners of the triangulated polygon $\mathcal{T}(0)$ along these timelike lines. If $\mathcal{T}(t)$ is positive, then the manifold $\mathcal{M} = f(\mathcal{P})$ admits a global foliation in a family of positive definite differentiable surfaces $\Sigma_{g,N,b}$ and $\mathcal{M}$ is independent of the choice of timelike lines.

Proof. The foliation can be constructed directly by a smoothing of the triangulated surfaces $\mathcal{T}(t)$, holding the boundary edges fixed. Note that the condition is sufficient but not necessary, since if a given triangulation fails there may exist another which remains spacelike. It is however necessary that all closed segments of the triangulation be spacelike at all t, as explained in Theorem 6.1. That $\mathcal{M}$ is independent of the choice of timelike lines is a direct consequence of the corollary of Theorem 5.1 and the fact that the surfaces are positive definite. ∎

7. Cosmological constant

The construction of a flat three-manifold from a generalized polygon can be extended to three-manifolds of constant curvature. We give the construction without proofs, as they are straightforward extensions of the proofs given in the previous section. Consider Minkowski space in $3+1$ or $2+2$ dimensions, and the $2+1$ the de Sitter spacetimes which are the hypersurfaces defined by $g_{ab}X^aX^b = \pm 1$ (the sign is $+$ or $-$ if the Minkowski metric g_{ab} has signature $\{-,+,+,+\}$ or $\{-,-,+,+\}$, respectively). These hypersurfaces are invariant under SO(3,1) and $SO(2,2)$, respectively. To construct a polygon, choose N points on the hypersurface and a set of $2g+N-1$ identification matrices in the corresponding group, that are a faithful representation of the fundamental group of a genus g surface with N punctures. The images of these points under the identification matrices provide the other corners of the polygon. The identification matrices are restricted by the requirement that the polygon satisfy the topological condition stated in Property 3.2, to guarantee the existence of an orientable surface which respects the matching conditions, and that there exist a positive surface which includes these points (Property 3.1), and that the N cycle transformations have a timelike invariant plane.

The plane which is invariant under the cycle transformation at a vertex of the identified polygon intersects the constant curvature hypersurface along a timelike line which is a geodesic of the hypersurface (de Sitter space) and is invariant under the cycle transformation. Each corner of the polygon is endowed with such a timelike line, as in Sect. 4. The polygon can be pushed forward in time along these timelike lines, leading to a $2+1$-dimensional region in the hypersurface, which is $2+1$ de Sitter space, with identified walls. The three-manifold obtained by identifying the walls two by two is locally de Sitter by construction, and has the same topology as in the case of zero cosmological constant. The introduction of boundaries is straightforward, the only difference being in the homotopy group of which the identification matrices are a representation. To prove that all $2+1$ locally de Sitter spacetimes can be represented in this way, one proceeds by construction as before, the only difference being that the geodesic cuts are not represented by three-vectors but by geodesic segments in de Sitter space.

8. Conclusions

We have generalized Poincaré's fundamental polygons, from the isometry group $\mathrm{SL}(2,\mathbb{R})$ to the group ISO(2,1), and implicitly to the homogeneous groups SO(3,1) and SO(2,2). While Poincaré's polygons parametrize the moduli space of flat $\mathrm{SL}(2,\mathbb{R})$ connections, the generalized polygons parametrize the moduli spaces of flat Poincaré connections, or its homogeneous generalizations. Rather than a representation of Riemann surfaces, or of stationary spacetimes, the generalized polygons provide a representation of a large set of flat spacetimes, stationary or not.

The limitations of our approach are the following. Only point like sources can be represented in this picture. No progress is made towards the purpose of representing any of these solutions in the form of a metric tensor, in some coordinate system. The residual discrete "braid" symmetry was only mentioned; it is a difficult but important symmetry to impose at the level of a quantized theory.

On the other hand, the approach has the potential to be applied to other problems of interest: What has been done here in $2+1$ dimensions can likely be generalized to higher dimensions, *i.e.*, to give a parametrization of flat (or de Sitter) manifolds with simple topologies; for instance the extension is trivial for separable topologies $\Sigma \times \mathbb{R} \times \mathbb{R}$, or $\Sigma \times \mathbb{R} \times \mathbf{S}^1$. Another possible field of generalization is to the supersymmetric extensions of the group $ISO(2,1)$.

Perhaps of greatest interest to the physicist is the quantization of $2+1$ gravity, considering that to this day, none of the conceptual problems of quantum gravity posed by DeWitt [13] have yet been convincingly resolved. We can only hope that the explicit representation of all classical solutions will prove to be a useful step toward this goal.

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