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Effective elastic properties of the medium with an array of thin inclusions

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ABSTRACT. Applications of a new version of the effective (self-consistent) field method for evaluation of elastic properties and microstresses in homogeneous medium with thin inclusions are presented. Two type of thin inclusions are considered: stiffer than the medium (hard flakes), and softer than the medium (quasicracks and cracks). Some results concerned with the influence of space distribution of thin inhomogeneities on effective elastic moduli of composites are discussed.

RESUMEN. En este trabajo se presentan las aplicaciones de una nueva versión del método de campo efectivo (autoconsistente) para estimar propiedades elásticas y microtensiones en un medio homogéneo con inclusiones delgadas. Se consideran dos tipos de inclusiones delgadas: más rígidas que el medio (hojuelas duras) y más suaves que el medio (casi-grietas y grietas). También se discuten algunos resultados que consideran la influencia de la distribución espacial de inhomogeneidades delgadas en el módulo elástico efectivo de materiales compuestos.

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1. INTRODUCTION

Thin inclusions represent an important class of inhomogeneities in real materials. On the one hand, cracks, microcracks and thin flaws in solids can be considered as such inclusions. Modeling of physical properties of materials with cracks and quasicracks is a problem of considerable interest for mechanics of fracture, geophysics and material science [1,2].

On the other hand, stiff thin inclusions are known as the most effective fillers for increasing the elastic moduli of plastics, rubbers etc. [3,4]. With the same volume concentrations and moduli of elasticity of the filler, the stiffness of composites reinforced with thin stiff inclusions is higher than when filled with fibers or quasispherical particles.

The attention of most authors was focused on the problem of elastic properties of cracked solids. (See, *e.g.*, reviews [5,6] were one can find a number of references to this problem). As it was emphasized in [6] this problem has some particularities. For example, the bounds for elastic moduli of cracked solids degenerate: neither the upper nor the lower non-trivial bounds can be established in this case. The simplest self-consistent schemes allow to obtain some solutions, but these solutions are not satisfactory from many points of view. For instance, elastic moduli of cracked solids obtained by effective medium method [7] tend to zero when concentration of cracks is not very large. The simplest variant of the method of effective field in the Mori-Tanaka's form gives the result which coincides with the case of non-interacting cracks [8]. (Note that in Mori-Tanaka's method every inclusion is assumed to be in constant and the same for all inclusions local external field). Thus interactions between cracks can not be taken properly into account when simplest self-consistent approaches are used. The same problem appears if one applies the above mentioned methods to the medium with thin inclusions, *i.e.*, in a more general situation.

In this article, the advanced self-consistent method is applied to the evaluation of elastic moduli in solids with thin inclusions of various types. This method differs from the above mentioned one in that the local external field which affects every inclusion in the composite is assumed to depend on the orientation of the inclusion. Such a modification of the effective field method permits to describe interactions between the inclusions in more details; it also takes into account the particular nature of the space distribution of thin inclusions.

The proposed method is based on the solution of a "one particle problem". It is an elastic problem for an isolated inclusion in infinite homogeneous medium under the action of constant external field of arbitrary structure. Inclusions with properties strongly different from the properties of the medium are of particular interest to the mechanics of composites. This is the reason for considering them in this article. Thus there are two small parameters in the problem under consideration: geometrical —the ratio of characteristic linear dimensions of the inclusion—, and physical —the ratio of characteristic elastic moduli of the medium and the inclusion, or its inverse. In this work only the main terms of the expansion of the solution in the series over these parameters are taken into account. The problem of constructing these terms is considered in the Sect. 2 of the article. Then in Sect. 3, the general scheme of the new version of effective field method and the solution of averaging problem is developed. In Sects. 4 and 5, this scheme is applied for obtaining the effective elastic moduli of the medium with thin stiff and soft inclusions. Some generalizations of the method are discussed in the conclusion.

2. A THIN ISOLATED INCLUSION IN HOMOGENEOUS ELASTIC MEDIUM (ONE PARTICLE PROBLEM)

Let us examine a homogeneous medium with tensor of elastic moduli C_0 and with a single inclusion having moduli tensor C. The inclusion perfectly fits into the undeformed medium and occupies finite volume V. One characteristic length parameter of this volume h is much smaller than two other (of the order l). Thus the ratio $\delta_1 = h/l$ is small. The external loading in the medium is represented by body forces and stresses at infinity.

Thin inclusions with elastic moduli essentially different from the moduli of the medium are of prime interest for us. In this case the ratio of characteristic moduli of elasticity of the inclusion and the medium ($\delta_2 = O(CC_0^{-1})$) is either small (soft inclusions), or large (stiff inclusions).

It should be noted that the most valuable information about the stress fields in the vicinity of thin inclusion is contained in the main terms of the asymptotic expansion of these fields over the parameters δ_1 and δ_2. In order to construct these terms, it is necessary to find the limiting solution of the suggested elastic problem when $\delta_1 \to 0$, $\delta_2 \to 0$ (or $\delta_2 \to \infty$), and the ratio δ_1/δ_2 (or product $\delta_1\delta_2$) remains constant and equal to its value for the given inclusion. Such asymptotic expansion describes the behavior of elastic fields at distances from the surface of the inclusion larger than its characteristic transverse size h, and this is precisely of interest for the mechanics of solids with thin inclusions.

Let us begin with thin soft inclusions when the parameter δ_2 is small. The middle surface of inclusion Ω will be considered as a smooth Lyapunov surface with given continuous field of its normal vector $n(x)$. The surface Ω is bounded by a closed contour Γ and $x(x_1, x_2, x_3)$ is a point of the medium. It has been proved in Ref. [9] that the main terms of strain $\varepsilon(x)$ and stress $\sigma(x)$ of the asymptotic expansions in δ_1, δ_2 in the medium with thin soft inclusion can be written as

$$\varepsilon_{ij}(x) = \varepsilon_{0\,ij}(x) + \int_\Omega K_{ijkl}(x - x')C_0^{klmn} n_m(x') b_n(x')\, d\Omega',$$

$$\sigma^{ij}(x) = \sigma_0^{ij}(x) + \int_\Omega S^{ijkl}(x - x') n_k(x') b_l(x')\, d\Omega', \tag{2.1}$$

$$K_{ijkl}(x) = -\left[\nabla_i \nabla_k G_{jl}(x)\right]_{(ij)(kl)},$$

$$S^{ijkl}(x) = C_0^{ijmn} K_{mnpq}(x) C_0^{pqkl} - C_0^{ijkl}\delta(x),$$

where $\varepsilon_0(x)$ and $\sigma_0(x)$ are the external strain and stress fields that would have existed in the medium without the inclusion under the given external loading, $G(x)$ is Green's function for the homogeneous medium with moduli C and $\delta(x)$ is the 3-D delta-function. The vector field $b(x)$ is the density of the potentials in the r.h.s. of (2.1); it satisfies on Ω the following equation [9]:

$$\lambda^{ij}(x) b_j(x) + \int_\Omega T^{ij}(x, x') b_j(x')\, d\Omega' = n_j \sigma_0^{ij}(x),$$

$$\lambda^{ij}(x) = \frac{1}{h(x)} n_k(x) C^{kijl} n_l(x),$$

$$T^{ij}(x, x') = -n_k(x) S^{kijl}(x - x') n_l(x'), \tag{2.2}$$

where $h(x)$ is the transverse dimension of the inclusion in the direction of normal vector $n(x)$ at the point $x \in \Omega$.

Notice that the operator T in (2.2) can be written as an integral operator only conventionally because the appropriate integral diverges for every smooth function $b(x)$ ($T(x, x') \sim |x - x'|^{-3}$ as $x' \to x$). Regularization of this operator for functions with continuous first derivatives was obtained in [10].

Let us consider the case of stiff inclusions, *i.e.*, when the parameter δ_2 is large. The main terms of asymptotic strain and stress (the limits of $\varepsilon(x)$ and $\sigma(x)$ when $\delta_1 \to 0$,

$\delta_2 \to \infty$, $\delta_1\delta_2 = 0(1)$) can be written in this case as follows [9]:

$$\varepsilon_{ij}(x) = \varepsilon_{0\,ij}(x) - \int_\Omega K_{ijkl}(x - x')q^{kl}(x')\,d\Omega',$$
$$\sigma^{ij}(x) = \sigma_0^{ij}(x) - \int_\Omega S^{ijkl}(x - x')C^{-1}_{0\,klmn}q^{mn}(x')\,d\Omega', \tag{2.3}$$

where $q(x)$ is a tensor on the surface Ω given by

$$n_j(x)q^{ji}(x) = 0, \qquad \Theta^{ij}_{kl}q^{kl}(x) = q^{ij}(x).$$

$\Theta(x)$ is an operator of orthogonal projection on the tangent to the Ω plain:

$$\Theta(x) = \Theta(n) = E_1 - 2E_5(n) + E_6(n), \qquad n = n(x),$$

where $E_i(n)$ are the elements of the following tensor basis:

$$E_{1\,ijkl} = \delta_{i(j}\delta_{k)l}, \qquad E_{2\,ijkl} = \delta_{ij}\delta_{kl}, \qquad E_{3\,ijkl} = n_1 n_j \delta_{kl},$$
$$E_{4\,ijkl} = \delta_{ij}n_k n_l, \qquad E_{5\,ijkl} = n_{(i}\delta_{i)(k}n_{)l}, \qquad E_{6\,ijkl} = n_i n_j n_k n_l, \tag{2.4}$$

and δ_{ij} is a Kronecker's symbol.

The field $q(x)$ satisfies on Ω the following integral equation:

$$\mu_{ijkl}(x)q^{kl}(x) + \int_\Omega U_{ijkl}(x, x')q^{kl}(x')\,d\Omega' = \Theta^{kl}_{ij}(x)\varepsilon_{0\,kl}(x),$$

$$\mu_{ijkl}(x) = \frac{1}{h(x)}(x)\Theta^{mn}_{ij}(x)C^{-1}_{mnpq}\Theta^{pq}_{kl}(x),$$

$$U_{ijkl}(x, x) = \Theta^{mn}_{ij}(x)K_{mnpq}(x - x')\Theta^{pq}_{kl}(x').$$

Regularization of the operator U in this equation has also been obtained in Ref. [9]. Methods of numerical solutions for Eqs. (2.2), (2.4) have been discussed in Ref. [11].

Let the inclusion be a thin ellipsoid with semiaxes a_1, a_2, h ($h/a_1, h/a_2 \ll 1$). Then Ω is a plane elliptical surface, and the normal to it will be denoted by m. It was shown in Ref. [10] that in the case of constant external field $(\varepsilon_0, \sigma_0)$, the solutions of Eqs. (2.2) and (2.4) have the form

$$b_i(x) = B_{ij}(m)m_k\sigma_0^{kj}z(x), \quad q^{ij}(x) = Q^{ijkl}(m)\varepsilon_{0\,kl}z(x),$$
$$z(x) = \left[1 - \left(\frac{x_1}{a_1}\right)^2 - \left(\frac{x_2}{a_2}\right)^2\right]^{\frac{1}{2}}, \tag{2.5}$$

where x_1, x_2 are Cartesian coordinates along the major axes of the ellipsoid. Here the tensors B and Q are expressed in terms of the absolutely converging integrals

$$\begin{aligned} B_{ij}(m) &= \left[h^{-1} m_k C^{kijl} m_l + S_0^{ij}\right]^{-1}, \\ Q^{ijkl}(m) &= \left[h^{-1} \Theta_{ij}^{mn}(m) C_{mnpq}^{-1} \Theta_{kl}^{pq}(m) + U_{0\,ijkl}\right]^{-1}, \\ S_0^{ij} &= \int m_k S^{kijl}(x) m_l [z(x) - 1]\, d\Omega, \\ U_{0\,ijkl} &= \int \Theta_{ij}^{mn} K_{mnpq}(x) \Theta_{kl}^{pq} [z(x) - 1]\, d\Omega, \end{aligned} \tag{2.6}$$

where integration is over the plane x_1, x_2 and the function $z(x)$ vanishes outside of Ω. These integrals are expressed in terms of elliptical functions. In case of isotropic medium and thin spheroids ($a_1 = a_2 = a$), tensors B and Q have the following forms:

$$B_{ij} = B_1(\delta_{ij} - m_i m_j) + B_2 m_i m_j,$$

$$B_1 = \frac{1}{\mu_0}\left[\xi + \frac{\pi}{8}(2\kappa_0 + 1)\right]^{-1}, \quad B_2 = \frac{1}{\mu_0}\left[\xi \frac{2(1-\nu)}{(1-2\nu)} + \frac{\pi}{2}\kappa_0\right]^{-1}, \quad \xi = \frac{a\mu}{2h\mu_0},$$

$$Q = Q_1 P_2(m) + Q_2\left[P_1(m) - \tfrac{1}{2} P_2(m)\right],$$

$$Q_1 = a\mu_0\left[\zeta \frac{1-\nu}{1+\nu} + \frac{\pi}{8}(2 - \kappa_0)\right]^{-1}, \; Q_2 = 2a\mu_0\left[\zeta + \frac{\pi}{16}(4 - \kappa_0)\right]^{-1}, \; \zeta = \frac{a\mu_0}{2h\mu}, \tag{2.7}$$

where μ, ν are the shear modulus and Poisson ratio for the inclusion, μ_0, ν_0 respectively, are their values for the matrix, $\kappa_0^{-1} = 2(1 - \nu_0)$; P_i are the elements of tensor basis related to the basis (2.4) by the equations

$$\begin{aligned} P_1(m) &= E_1 - 2E_5(m) + E_6(m), & P_2(m) &= E_2 - E_3(m) - E_4(m) + E_6(m), \\ P_3(m) &= E_3(m) - E_6(m), & P_4(m) &= E_4(m) - E_6(m), \\ P_5(m) &= E_5(m) - E_6(m), & P_6(m) &= E_6(m). \end{aligned} \tag{2.8}$$

3. MEDIUM WITH AN ARRAY OF THIN INCLUSIONS

Let us consider an infinite medium in which a random set of thin inclusions is homogeneously distributed. The external field applied to the medium (*i.e.*, ε_0 or σ_0) is assumed to be constant. The middle surface of the i-th inclusion Ω_i is the plane with normal m_i. Thus the inclusions are space oriented objects. Let us introduce local external field on the i-th inclusion. This field is composed of external field ε_0 and the fields induced by the surrounding inclusions. It is obvious that the orientation of the i–th inclusion in relation to the total external field and surrounding inhomogeneities influences the local external field on this inclusion. Method of effective field is used here for description of the interaction between the inclusions [12]. Due to the special shape of inclusions under consideration, the main hypothesis of the method should be reformulated. We assume that every inclusion in the composite is in a local homogeneous external field ε_* which depends on the orientation of this inclusion m. Using this hypothesis, the expressions for strain and stress tensors in the medium with thin inclusions can be represented in the form analogous to (2.3) (in the following the subscripts will be dropped for simplicity):

$$\varepsilon(x) = \varepsilon_0 - \int K(x - x')q(x')\,dx',$$

$$\sigma(x) = \sigma_0 - \int S(x - x')C_0^{-1}q(x')\,dx', \tag{3.1}$$

$$q(x) = \Lambda(x)\varepsilon_*(m)\Omega(x), \qquad \Omega(x) = \sum_i \Omega_i(x).$$

$\Omega_i(x)$ in (3.1) is a generalized function concentrated on the surface of i–th inclusion. The function $m(x)$ coincides with the normal m to the surface Ω_i when x belongs to $\Omega_i(x)$. The function $\Lambda(x)$ is found when $x \in \Omega_i$ from the solution of the elastic problem for the isolated thin inclusion in homogeneous external field $\varepsilon_*(m)$. For thin ellipsoidal inclusions, in particular, the expression for the function $\Lambda(x)$, $x \in \Omega_i$, has the form

$$\Lambda(x) = \Lambda(m_i)z_i(x), \tag{3.2}$$

where the function $z_i(x)$ in the basis of the main axes of ellipse is defined by the relation (2.5). The tensor $\Lambda(m)$ depends on the orientation m_i of the inclusion, and elasticity moduli of the latter and those of the medium. In the case of thin stiff inclusions the tensor $\Lambda(m)$ coincides with tensor Q in Eqs. (2.5)–(2.7). For thin soft inclusions $\Lambda(m)$ has the form

$$\Lambda(m) = -C_0M(m)C_0, \qquad M_{ijkl} = m_{(i}B_{j)(k}m_{l)}, \tag{3.3}$$

where tensor B is defined in Eqs. (2.6) and (2.7). Let us introduce the function

$$\Omega(x; x') = \sum_{i \neq j} \Omega_i(x'), \quad \text{when} \quad x \in \Omega_j. \tag{3.4}$$

It allows to express the local external field at point x placed on the middle surface of an arbitrary inclusion in the following form:

$$\varepsilon_*(x) = \varepsilon_0 - \int K(x - x')\Lambda(x')\varepsilon_*(m')\Omega(x; x')\,dx, \qquad x \in \Omega. \tag{3.5}$$

Let us average this equality with the condition that point x is placed on the middle surface of the inclusion with normal m. This averaging is denoted as $\langle \cdot | x, m\rangle$. If the mean $\langle \varepsilon_*(x)|x, m\rangle$ is identified with an effective field acting on the inclusion of orientation m,

$$\langle \varepsilon_*(x)|x, m\rangle = \varepsilon_*(m),$$

one can obtain from (3.5) the expression for $\varepsilon_*(m)$

$$\varepsilon_*(m) = \varepsilon_0 - \int K(x - x')\langle \Lambda(x')\varepsilon_*(m')\Omega(x; x')|x, m\rangle\, dx'. \tag{3.6}$$

Let us consider the mean integrand in this relation. Assuming that elastic properties of the inclusion are statistically independent from their space positions, one can obtain the expression for this mean taking into account (3.1)

$$\langle \Lambda(x')\varepsilon_*(m')\Omega(x, x')|x, m\rangle = \langle \Lambda^0(m')\varepsilon_*(m')\rangle \Psi_{m'}(x - x'), \tag{3.7}$$

$$\Lambda^0(m) = \langle z(x)\Omega(x)\rangle\Lambda(m), \qquad \Psi_m(x - x') = \frac{\langle \Omega(x; x')|x, m\rangle}{\langle \Omega(x)\rangle}. \tag{3.8}$$

The mean $\langle \Lambda^0(m)\varepsilon_*(m)\rangle$ is calculated over the ensemble of distributions by orientations and properties of inclusions. The function $\Psi_m(x)$ describes geometrical particularities of the inhomogeneities distribution in composite material. It follows from definition (3.4) of the function $\Omega(x; x')$ that the function $\Psi_m(x)$ has a property

$$\Psi_m(x) = 0, \quad \text{when} \quad x = 0. \tag{3.9}$$

Due to the weakening in geometrical linkage between positions of the inclusions when the distances increase between them, the following relations take place

$$\Psi_m(x) \to 1 \quad \text{when} \quad |x| \to \infty. \tag{3.10}$$

The function $\Psi_m(x)$ defines the shape of a "correlation hole" inside which a typical inclusion of the orientation m is located. Let us assume that there is a linear transformation of x-space that rearranges the function $\Psi_m(x)$ into a spherically symmetric one:

$$y = \alpha(m)x, \qquad \Psi_m\big(\alpha^{-1}(m)y\big) = \Psi_m(|y|).$$

In this case, ellipsoid A given by the equation

$$|\alpha(m)x| = 1,$$

with semiaxes α_1, α_2, α_3 describes the form of the correlation hole.

After substituting (3.7) in (3.6) and calculating the appropriate integrals (see the Appendix) one can obtain the expression for $\varepsilon_*(m)$ in the form

$$\varepsilon_*(m) = \varepsilon_0 + A(m)\langle \Lambda^0(m)\varepsilon_*(m)\rangle, \tag{3.11}$$

$$A(m) = \int K(x)\big[1 - \Psi_m(x)\big]\, dx. \tag{3.12}$$

The external strain field ε_0 here is assumed to be fixed for the problem [see Eq. (A.4)].

Let us multiply both sides of (3.11) by the tensor $\Lambda^0(m)$ and average the result over the ensemble of random orientations and properties of inclusions. Solving the obtained equation for tensor $\langle \Lambda^0(m)\varepsilon_*(m)\rangle$, we have

$$\langle \Lambda^0(m)\varepsilon_*(m)\rangle = \big[E_1 - \langle \Lambda^0(m)A(m)\rangle\big]^{-1}\langle \Lambda^0(m)\rangle\varepsilon_0. \tag{3.13}$$

The expression for the effective field $\varepsilon_*(m)$ can be found if we substitute $\langle \Lambda^0(m)\varepsilon_*(m)\rangle$ from (3.13) to the r.h.s. of (3.11).

The mean values of strain and stress fields follow from (3.1) after averaging both parts of these relations over the ensemble of random sets of inclusions:

$$\varepsilon(x) = \varepsilon_0 - \int K(x - x')\langle \Lambda^0(m)\varepsilon_*(m)\rangle\, dx', \tag{3.14}$$

$$\sigma(x) = \sigma_0 - \int S(x - x')C_0^{-1}\langle \Lambda^0(m)\varepsilon_*(m)\rangle\, dx'. \tag{3.15}$$

In order to obtain (3.14), (3.15) we take into account the relation

$$\langle \Lambda(x)\varepsilon_*(m)\Omega(x)\rangle = \langle \Lambda^0(m)\varepsilon_*(m)\rangle. \tag{3.16}$$

Note that $\langle \Lambda^0(m)\varepsilon_*(m)\rangle$ is a constant tensor. Since the external strain field ε_0 has been assumed to be fixed, the regularization procedure of the integrals in (3.14), (3.15) has the form given in Appendix A.4, and we obtain the final result in the form

$$\langle \varepsilon\rangle = \varepsilon_0, \qquad \langle \sigma\rangle = C_*\langle \varepsilon\rangle,$$

$$C_* = C_0 + \big[E_1 - \langle \Lambda^0(m)A(m)\rangle\big]^{-1}\langle \Lambda^0(m)\rangle, \tag{3.17}$$

where C_* is the tensor of effective elastic properties of a composite with an array of thin inclusions which we are looking for.

Note that when inclusions do not interact, then $\varepsilon_* = \varepsilon_0$ and the expression for C_* takes the form

$$C_* = C_0 + \langle \Lambda^0(m)\rangle.$$

4. ELASTIC MEDIUM REINFORCED WITH STIFF FLAKES

In this section we apply the above results to the calculation of elastic moduli of composites reinforced with stiff inclusions having the shape of flattened spheroids. In this case the middle surface of every inclusion is a circular area of random radius a. The material of the inclusions is supposed to be isotopic. In this case, tensor $\Lambda(m)$ in (3.2) coincides with tensor $Q(m)$ given by the expression (2.7).

Let us consider tensor $A(m)$ [Eq. (3.12)], which is present in Eq. (3.17) for the effective elastic moduli tensor of the composite. Function $\Psi_m(x)$ is assumed to have the symmetry of an ellipsoid which is coaxial with the considered inclusion. Let the semiaxes of this ellipsoid be $\alpha_1 = \alpha_2 = \alpha$ and α_3. The axis x_3 is directed along the normal m to the middle surface of the inclusion Ω. In this case the tensor $A(m)$ has the form ($\gamma = \alpha/\alpha_3 > 1$)

$$A(m) = A_1 P_2 + A_2(P_1 - \tfrac{1}{2}P_2) + A_3(P_3 + P_4) + A_5 P_5 + A_6 P_6, \tag{4.1}$$

$$A_1 = \frac{1}{2\mu_0}\left[(1-\kappa_0)f_0 + f_1\right], \qquad A_2 = \frac{1}{2\mu_0}\left[(2-\kappa_0)f_0 + f_1\right], \qquad A_3 = -\frac{1}{\mu_0}f_1,$$

$$A_5 = \frac{1}{\mu_0}[1 - f_0 - 4f_1], \qquad A_6 = \frac{1}{\mu_0}\left[(1-\kappa_0)(1-2f_0) + 2f_1\right],$$

$$f_0 = \frac{1-g}{2(1-\gamma^2)}, \quad f_1 = \frac{\kappa_0}{4(1-\gamma^2)^2}\left[(2+\gamma^2)g - 3\gamma^2\right], \quad g = \frac{\gamma^2}{\sqrt{\gamma^2-1}}\arctan\sqrt{\gamma^2-1}.$$

Thus in this case, tensor A depends only on one scalar parameter γ. If the positions of the inclusions in space are statistically independent, γ has the order of $\langle a/h\rangle$, which is a mean aspect ratio of the inclusion of the orientation m.

Let us consider some particular cases.

1. Inclusions are thin spheroids (flakes) of the same orientation.

According to (3.8), tensor $\Lambda^0(m)$ has the form

$$\Lambda^0(m) = n_0\left\langle \tfrac{2}{3}\pi a^2 \Lambda(m)\right\rangle,$$

where n_0 is the numerical value of concentration of inclusions.

In this case the composite medium will be transversely isotropic with the axis of isotropy directed along the common normal m to the surfaces of the inclusions. According to (3.16), the tensor of the effective moduli of elasticity C_* takes the form

$$C_* = C_0 + \frac{\Lambda_1}{1 - 4A_1\Lambda_1}P_2(m) + \frac{\Lambda_2}{1 - A_2\Lambda_2}\left[P_1(m) - \tfrac{1}{2}P_2(m)\right], \tag{4.2}$$

$$\Lambda_1 = n_0\left\langle \tfrac{2}{3}\pi a^2 Q_1\right\rangle, \qquad \Lambda_2 = n_0\left\langle \tfrac{2}{3}\pi a^2 Q_2\right\rangle,$$

where coefficients Q_1, Q_2 and A_1, A_2 are given by (2.7) and (4.1). If $\gamma \gg 1$ then, to the accuracy of the terms of order γ^{-1}, the expression for A_1, A_2 has the form

$$A_1 = \frac{\pi(2-\kappa_0)}{16\mu_0\gamma}, \qquad A_2 = \frac{\pi(4-\kappa_0)}{16\mu_0\gamma}.$$

2. Inclusions are thin spheroids homogeneously distributed over the orientations.

In this case, the composite is macroscopically isotropic and the expression for the tensor C_* takes the form

$$C_* = k_* E_2 + 2\mu_*\left(E_1 - \tfrac{1}{3}E_2\right), \tag{4.3}$$

$$k_* = k_0 + \frac{4\Lambda_1}{3[3 - 4\Lambda_1(2A_1 + A_3)]}, \qquad \mu_* = \mu_0 + \frac{\Lambda_1 + 3\Lambda_2}{15 - 2[2\Lambda_1(A_1 - A_3) + 3\Lambda_2 A_2]},$$

where the coefficients Λ_1, Λ_2 are the same as in (4.2). Note that the case of non-interacting inclusions follows from (4.1) and (4.3) when $\gamma \to \infty$ ($A_1, A_2, A_3 \to 0$).

Let us compare the obtained theoretical results with the experimental data from measurements of the moduli of elasticity in plastics reinforced with mice flakes with high aspect ratio ($a/h \gg 1$) [13]. In this work, the so-called "flexural" modulus of elasticity of composites in the plane of the flakes was measured. The orientation of the flakes was approximately the same, at least near the surfaces of the sample, *i.e.*, in the regions whose properties largely determine the flexural modulus of elasticity.

More than 100 experimental values of the moduli of elasticity of composites with different ratios of the matrix and inclusions moduli, different aspect ratio of the flakes and different volume concentration [13,14], were used for analyzing the value of $\Delta = |E^{\rm t}_{11} - E^{\rm e}_{11}|/E^{\rm e}_{11}$ —the relative difference between the theoretical $E^{\rm t}_{11}$ and the experimental $E^{\rm e}_{11}$ values of the effective Young's modulus in the plane of the flakes. It turns out that the value of Δ does not strongly depend on the concentration of fillers. It is determined by the values of parameters γ and $\zeta = a\mu_0/2h\mu \approx aE_0/2hE_{11}$. The minimal value of Δ averaged over all samples is achieved for the value of γ approximately equal to $\langle a/2h \rangle$.

The graph of $\Delta(\zeta)$ is given by the solid line in Fig. 1. The "experimental" values of Δ are given by the points, the scale along ζ axis is logarithmic. The maximum value of the relative difference for all samples with the same value of parameter ζ, but different values of $\tau = \frac{4}{3}\pi n_0 \langle a^3 \rangle$ was selected as Δ. When $\zeta > 0{,}2$ the deviation of $E^{\rm t}_{11}$ from $E^{\rm e}_{11}$ did not exceed 10% up to $\tau = 200$ (the volume concentration of flakes attained the value of $p = 0.7$ ($p = \langle a/h \rangle^{-1}\tau$) [13]).

The error of the theoretical formulae obtained here has two main sources. First, the use of only dominant terms in the expansion of the solution of the one particle problem with the small parameters δ_1 and δ_2. The corresponding error is of the order of $\max(\delta_1, \delta_2)$. It should be noted that with relatively small aspect ratio ($\langle a/h \rangle < 5$) and high stiffness of the inclusion ($E_{11}/E_0 > 100$), when parameter ζ is small, the error due to substitution of real flakes of ellipsoidal shape increases (the elevation of the curve with small ζ in Fig. 1 is apparently related to this).

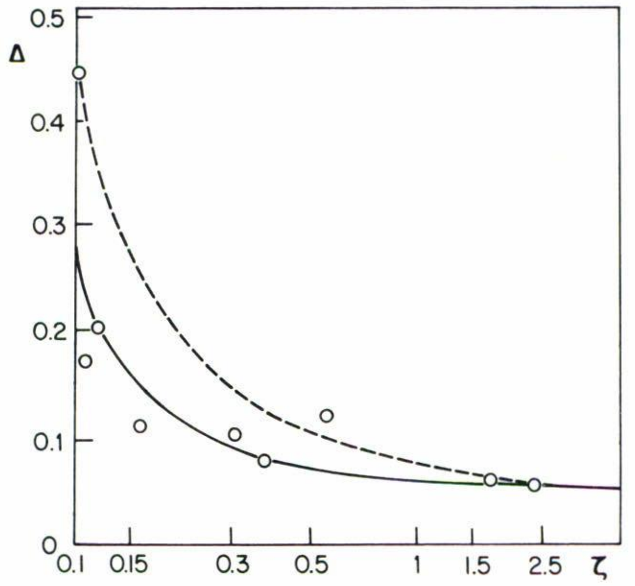

FIGURE 1. Dependence of the relative difference between the calculated and experimental values of the moduli of elasticity for composites reinforced with stiff flakes on parameters ζ characterizing the properties of the flakes.

Secondly, consideration of the interaction between inclusions is approximate. Analysis of experimental data shows that integral effect of the interaction in these composites is relatively weak. Within the framework of the proposed theory, the degree of interaction correlates with the value of parameter γ, which is determined by the space distribution of the inclusions. We note that total neglect of the interaction ($\gamma^{-1} = 0$) results in the increase in the relative error of Δ (the dashed curve in Fig. 1 corresponds this case).

The comparison of the theoretical and experimental data thus permits to draw the following conclusions. The area of applicability of the formulae obtained here can be estimated with the dimensionless parameter $\zeta = aE_{11}/hE_0$. When $\zeta > 0,15$ these formulae predict the elastic properties of plastics reinforced with thin stiff inclusions with the accuracy of 10 to 15%. This accuracy is acceptable for application in almost the entire range of change in concentration of the filler. The value of parameter γ is equal to half of the average aspect ratio of the reinforcing elements. We note that the value of γ should be determined from statistical analysis of the picture of the distribution of the filling elements in the bulk of the composite.

5. THIN SOFT INCLUSIONS AND CRACKS IN ELASTIC MEDIUM

Next we examine the case of thin inclusions with elastic moduli essentially smaller than the moduli of the medium ($CC_0^{-1} = O(\delta_2)$, $\delta_2 \ll 1$). We begin with considerations on the medium containing homogeneous random set of thin ellipsoidal inclusions. The method of effective field is used again for the description of the interaction between inclusions. As before, we assume that every inclusion of the orientation m is located in a constant local effective stress field $\sigma_*(m)(\sigma_*(m) = C_0\varepsilon_*(m))$. To the accuracy of dominant terms in the expansion of elastic fields with respect to small parameters of the problem, δ_1 and δ_2, the strain and stress fields in the medium with such inclusions are represented in the form which follows from (3.1),

$$\begin{aligned}\varepsilon(x) &= \varepsilon_0 - \int K(x-x')C_0 M(x')\sigma_*(m')\Omega(x')\,dx',\\ \sigma(x) &= \sigma_0 - \int S(x-x')M(x')\sigma_*(m')\Omega(x')\,dx',\end{aligned} \tag{5.1}$$

where function $M(x)$ is determined from the solution of the corresponding one particle problem. In case of flattened spheroids, this function on the inclusion middle surface with the orientation m has the form

$$M(x) = M(m)\left[1 - \left(\frac{|x|}{a}\right)^2\right]^{\frac{1}{2}}, \qquad |x| = \sqrt{x_1^2 + x_2^2},$$

where a is the largest radius of the spheroid. Tensor $M(m)$ is defined in (3.3).

The equation for effective stress field $\sigma_*(m)$ can be obtained in the same way as (3.5) and has the form

$$\sigma_*(m) = \sigma_0 - \int S(x-x')\langle M(x')\sigma_*(m')\Omega(x;x')|x,m\rangle\,dx', \tag{5.2}$$

where function $\Omega(x;x')$ is defined by (3.4). The mean integrand in this equation takes the form

$$\begin{gathered}\langle M(x')\sigma_*(m')\Omega(x;x')|x,m\rangle = \langle M^0(m)\sigma_*(m)\rangle\Psi_m(x-x'),\\ M^0(m) = \tfrac{2}{3}\pi a^2 n_0 M(m),\end{gathered}$$

where function $\Psi_m(x)$ is given by (3.8) and properties (3.9) and (3.10) hold for it. This function is supposed to have the symmetry of a ellipsoid coaxial to the inclusion of orientation m. Taking into account the properties of $\Psi_m(x)$ one can calculate the integral in the r.h.s. of Eq. (5.2) (see Appendix) and obtain the expression for $\sigma_*(m)$ in the form

$$\sigma_*(m) = \sigma_0 - D(m)\langle M^0\sigma_*(m)\rangle, \tag{5.3}$$

$$D(m) = \int S(x)\big[1 - \Psi_m(x)\big]\,dx, \qquad D(m) = C_0 A(m) C_0 - C_0,$$

where tensor $A(m)$ is defined in (4.1). The external stress field σ_0 is assumed to be fixed in the problem.

Let us multiply both sides of (5.3) by the tensor $M(m)$ and average the result over the ensemble distributions of orientations and properties of inclusions. The equation for the mean $\langle M^0(m)\sigma_*(m)\rangle$ may be obtained in such a way and its solution has the form

$$\langle M^0(m)\sigma_*(m)\rangle = \left[E_1 + \langle M^0(m)D(m)\rangle\right]^{-1}\langle M^0(m)\rangle\sigma_0.$$

After averaging the expressions (5.1) for strains and stresses and taking into account the relation

$$\langle M(x)\sigma_*(m)\Omega(x)\rangle = \langle M^0(m)\sigma_*(m)\rangle,$$

we obtain the result

$$\langle\varepsilon\rangle = \varepsilon_0 + \left[E_1 + \langle M^0(m)D(m)\rangle\right]^{-1}\langle M^0(m)\rangle\sigma_0, \qquad \langle\sigma\rangle = \sigma_0.$$

Since $\varepsilon_0 = C_0^{-1}\sigma_0$, one may rewrite these relations in the form

$$\langle\varepsilon\rangle = B_*\langle\sigma\rangle, \qquad B_* = C_*^{-1}, \tag{5.4}$$

$$B_* = C_0^{-1} + \left[E_1 + \langle M^0(m)D(m)\rangle\right]^{-1}\langle M^0(m)\rangle,$$

where C_* is the desired tensor of elastic moduli of the medium with thin soft inclusions.

Let the medium be isotropic and the symmetry of the function $\Psi_m(x)$ be determined by the spheroid with semiaxes $\alpha_1 = \alpha_2 = \alpha$, α_3. This ellipsoid is coaxial to the inclusion of orientation m. In this case the representation of tensor $D(m)$ in P–basis (2.8) has the form

$$D(m) = d_1P_2 + d_2\left(P_1 - \tfrac{1}{2}P_2\right) + d_3(P_3 + P_4) + d_5P_5 + d_6P_6, \tag{5.5}$$

$$d_1 = \mu_0[1 - 4\kappa_0 - 2(1 - 3\kappa_0)f_0 + 2\kappa_0f_1], \qquad d_2 = 2\mu_0[1 + (2 - \kappa_0)f_0 + \kappa_0f_1],$$

$$d_3 = 2\mu_0[(1 - 2\kappa_0)f_0 - 2\kappa_0f_1], \qquad d_5 = -4\mu_0[f_0 + 4\kappa_0f_1],$$

$$d_6 = -4\mu_0[(1 + 2\kappa_0)f_0 - 2\kappa_0f_1],$$

where functions $f_0(\gamma)$, $f_1(\gamma)$ ($\gamma = \alpha/\alpha_3 > 1$) are the same as in (4.1).

If $\gamma \gg 1$, then to the accuracy of the terms of order γ^{-1}, the coefficients d_i ($i = 1, 2, \ldots, 6$) in (5.5) are transformed to the following ones:

$$d_1 = \mu_0(1 - 4\kappa_0) + \frac{\pi\mu_0}{4\gamma}(7\kappa_0 - 2), \qquad d_2 = 2\mu_0 + \frac{\pi\mu_0}{4\gamma}(4 - \kappa_0),$$

$$d_3 = -\frac{\pi\mu_0}{2\gamma}(3\kappa_0 - 1), \qquad d_5 = -\frac{\pi\mu_0}{\gamma}(1 + 2\kappa_0),$$

$$d_6 = -\frac{\pi\mu_0}{\gamma}(1 + \kappa_0).$$

In the limit ($\gamma \to \infty$) we have

$$D(m) = -2\mu_0\big[P_1(m) - (1-2\kappa_0)P_2(m)\big].$$

The other limit ($\gamma \to 1$) corresponds to the correlation hole with the shape of a sphere. In that case $D(m)$ is an isotropic tensor

$$D(m) = D_0 = \tfrac{4}{9}\mu_0\big[(1-4\kappa_0)E_2 - \tfrac{3}{9}(5+4\kappa_0)(E_1 - \tfrac{1}{3}E_2)\big].$$

The case $\gamma = 1$ corresponds to the model of a random set of inclusions when there is a spherical area around every inclusion and the probability that other inclusions appear in this area is small.

Let us consider the expression for B_* (5.4) in some particular cases.

1. *Thin soft inclusion of the same orientation.* It follows from (5.4) and (5.5) that tensor B_* has the form

$$B_* = C_0^{-1} + \frac{4M_1^0}{4+M_1^0 d_5}P_5(m) + \frac{M_2^0}{1+M_2^0 d_6}P_6(m), \tag{5.6}$$

$$M_1^0 = \left\langle \frac{\tau}{2\mu_0}\left[\xi + \frac{\pi}{8}\frac{(2-\nu_0)}{(1-\nu_0)}\right]^{-1}\right\rangle, \quad M_2^0 = \left\langle \frac{\tau}{2\mu_0}\left[\xi\frac{2(1-\nu)}{(1-2\nu)} + \frac{\pi}{4(1-\nu_0)}\right]^{-1}\right\rangle,$$

$$\xi = \frac{a\mu}{2h\mu_0}, \qquad \tau = \tfrac{4}{3}\pi a^3 n_0,$$

where inclusions are supposed to be thin spheroids of random radius a.

In case of cracks ($k = \mu = 0$) and $\gamma \gg 1$ this expression for B_* becomes

$$B_* = C_0^{-1} + \frac{4\langle\tau\rangle}{\pi\mu_0(1+2\mu_0)}\left(1 - \frac{\langle\tau\rangle}{\gamma}\right)^{-1} P_5(m)$$

$$+ \frac{\langle\tau\rangle}{\pi\mu_0\kappa_0}\left[1 - \frac{\langle\tau\rangle(1+\kappa_0)}{\gamma\kappa_0}\right]^{-1} P_6(m). \tag{5.7}$$

The limit of $\gamma \to \infty$ here gives the formulas obtained in Ref. [15] which correspond to the case of non-interacting inclusions.

2. *Homogeneous distribution of inclusions over the orientations.* Tensor B in this case is isotropic and has the form

$$B_* = C_0^{-1} + \frac{M_2^0}{9(1+j_2)}E_2 + \frac{3M_1^0 + 2M_2^0}{15(1+j_1)}\big(E_1 - \tfrac{1}{3}E_2\big),$$

$$j_1 = \tfrac{2}{15}M_2^0(d_6 - d_3) + \tfrac{1}{10}M_1^0 d_5, \qquad j_2 = \tfrac{1}{3}M_2^0(2d_3 + d_6).$$

The expressions for bulk κ_0 and shear μ_0 moduli for composite follow from this result and take the forms

$$k_* = k_0\left[1+\frac{k_0M_2^0}{1+j_2}\right]^{-1}, \qquad \mu_* = \mu_0\left[1+\frac{2\mu_0}{15(1+j_1)}(3M_1^0+2M_2^0)\right]^{-1}.$$

The limit $\gamma\to\infty$ gives in the case of cracks ($k=\mu=0$):

$$k_* = k_0\left[1+\frac{\langle\tau\rangle k_0(3k_0+4\mu_0)}{\pi\mu_0(3k_0+\mu_0)}\right]^{-1}, \qquad \mu_* = \mu_0\left[1+\frac{4\langle\tau\rangle(2k_0+\mu_0)(3k_0+4\mu_0)}{3\pi(3k_0+2\mu_0)(3k_0+\mu_0)}\right]^{-1}.$$

If $\gamma=1$ these expressions take the forms

$$k_* = k_0\left[1-M_k(1+s_1M_k)^{-1}\right], \qquad \mu_* = \mu_0\left[1-M_\mu(1+s_2M_\mu)^{-1}\right],$$

$$M_k = k_0M_1^0, \quad M_\mu = \frac{2\mu_0}{15}\left(3M_1^0+2M_2^0\right), \quad s_1 = \frac{3k_0}{(3k_0+4\mu_0)}, \quad s_2 = \tfrac{1}{5}(3-s_1).$$

Note that within the framework of the effective field method, the stress intensity factor at the crack edge can also be estimated. According to the main assumption, every inclusion behaves as isolated in an effective field of external stress σ_*; in case of a circular crack, we have the expression for the coefficient K_I in the form

$$K_I = 2\sqrt{\frac{a}{\pi}}\,m_i\sigma_*^{ij}m_j.$$

The analysis of the accuracy of this formula is discussed in Ref. [16].

Let us consider the obtained formulae for the plain problem. The effective field method formalism can be reduced to the plain problem without any additional difficulties. The 2-D case is exceptionally interesting because there is a lot of experimental data for this situation, and there is also a number of exact solutions for the elastic plane with rectilinear cracks. Thus the opportunity appears to estimate the accuracy of the method.

For the plane with homogeneous array of thin soft elliptical inclusions, the tensor of elastic moduli is defined by the relation (5.8), where tensors M^0 and $D(m)$ have the forms

$$M^0(m) = M_1P_5(m) + M_2P_6(m), \tag{5.8}$$

$$M_1 = \frac{\pi l^2 n_0}{2\mu_0}(\xi+\kappa_0)^{-1}, \qquad M_2 = \frac{\pi l^2 n_0}{2\mu_0}\left(\xi\frac{2(1-\nu)}{(1-2\nu)}+\kappa_0\right)^{-1}, \qquad \xi = \frac{l\mu}{2h\mu_0},$$

$$D(m) = d_1P_1(m) + d_3(P_3(m)+P_4(m)) + d_5P_5(m) + d_6P_6(m), \tag{5.9}$$

$$d_1 = -4\mu_0\kappa_0(1-2f_0+3f_1), \qquad d_3 = -4\mu_0\kappa_0(f_0-3f_1), \qquad d_5 = 4d_3,$$

$$d_6 = -12\mu_0\kappa_0 f_1, \qquad f_0 = \frac{1}{(1+\gamma)}, \qquad f_1 = \frac{2+\gamma}{6(1+\gamma)^2}, \qquad \gamma = \frac{\alpha_2}{\alpha_1} > 1,$$

where m is normal to the middle line of an inclusion, l and h are semiaxes of the latter, and γ is an aspect ratio of a correlation hole for typical inclusions.

Let us consider the expression (5.4) for B_* in some particular cases.

1. *Array of inclusions of the same orientation* m:

$$B_* = C_0^{-1} + \frac{\langle M_1 \rangle}{1 + M_1 d_3} P_5(m) + \frac{\langle M_2 \rangle}{1 + M_2 d_6} P_6(m),$$

where M_1, M_2 and d_3, d_6 are defined in (5.7) and (5.8).

2. *Homogeneous distribution of inclusions by the orientations*:

$$B_* = C_0^{-1} + \frac{\langle M_2 \rangle}{4(1 + j_2)} E_2 + \frac{\langle M_1 + M_2 \rangle}{4(1 + j_1)} \left(E_1 - \tfrac{1}{2} E_2 \right),$$

$$j_1 = \tfrac{1}{8} \langle M_1 d_5 + 2M_2 (d_6 - d_3) \rangle, \qquad j_2 = \tfrac{1}{2} \langle M_2 (d_3 + d_6) \rangle.$$

If $\gamma \to \infty$, then $j_1, j_2 \to 0$ and this is the case of non- interacting inclusions. In the case of cracks ($k = \mu = 0$), the expressions for the Young's modulus E_* and Poisson's ratio ν_* follow from (5.6) and have the forms

$$E_* = \frac{E_0}{1 + \tau}, \qquad \nu_* = \frac{\nu_0}{1 + \tau}, \qquad \tau = \pi n_0 \langle l^2 \rangle, \tag{5.10}$$

where E_0 is Young's modulus of the medium.

If $\gamma = 1$, the expression for B_* for cracks takes the form

$$B_* = C_0^{-1} + \frac{\tau}{4\mu_0 \kappa_0 (2 - \tau)(4 - \tau)} \left[8E_1 + \tau (E_2 - 4E_1) \right],$$

where E_1, E_2 are the elements of the basis (2.4) for plane problem.

It follows that E_* and ν_* have the forms

$$\frac{E_*}{E_0} = \left[1 + \frac{\tau(8 - 3\tau)}{(2 - \tau)(4 - \tau)} \right]^{-1}, \tag{5.11}$$

$$\frac{\nu_*}{\nu_0} = \frac{E_*}{E_0} \left[1 - \frac{\tau^2}{\nu_0 (2 - \tau)(4 - \tau)} \right]. \tag{5.12}$$

The curves 1, 2 and 3 shown in Fig. 2 represent, respectively, functions (5.10), (5.11) and (5.12). They are compared with experimental data cited in Ref. [17]. Experiments were carried out on thin rubber sheets with a set of rectilinear through slits ($\nu = 0.5$). Experimental data are approximated by the dash line with small circles for (E_*/E_0) and dash-dot line for (ν_*/ν_0).

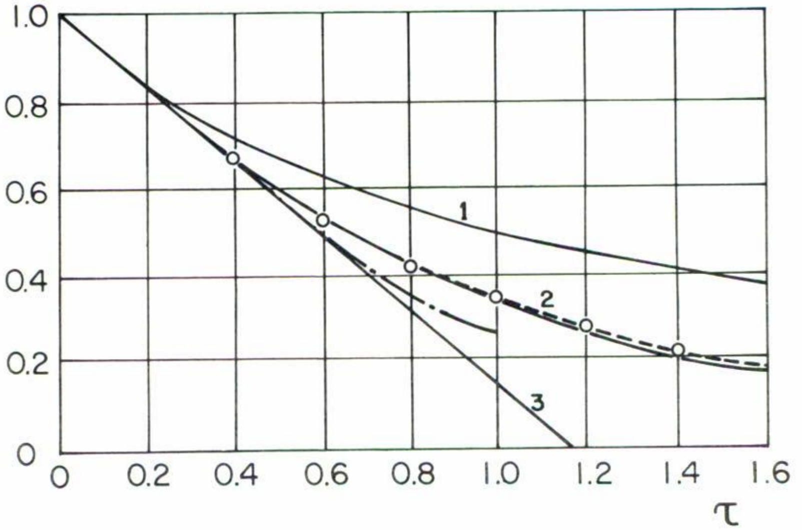

FIGURE 2. Theoretical and experimental dependence of the relative Young's modulus [(E_*/E_0)-curves (1), (2)] and Poisson's ratio [(ν_*/ν_0)-curves (1), (3)] of the plain with set of cracks *vs.* the parameter τ characterizing density of cracks.

The statistical analysis shows that the set of cracks investigated in Ref. [17] is satisfactorly defined by the model with the restriction on crack intersection. The formulas for the effective elastic constants obtained for this model are in good agreement with experimental data.

It turned out that formula (5.10) for Poisson statistically independent set of cracks ($\gamma \to \infty$) correctly describes well some experiments and results of computer simulation of the elastic properties of cracked solids, when the coordinates of the crack centers are statistically independent and homogeneously distributed in the plane [6]. We mentioned before that the limit $\gamma \to \infty$ corresponds to the assumption of non-interacting inclusions. If γ is finite, interactions between the inclusions take place. According to the obtained formulas, interactions always reduce the stiffness of the medium with soft inclusions. The same general statement was made by the authors of Ref. [18].

6. CONCLUSION

In this article, only the first step of generalization of the effective field method was considered. The effective field is assumed to depend on the orientation of the inclusion. The next step is to accept that the effective field randomly changes from one inclusion to an other. The outlining scheme for such a generalization was considered in Refs. [12,19]. This generalization permits to obtain not only the effective properties of composites but to estimate fluctuations of the elastic microfields. Moreover, this method allows to take into account the details of many particle interaction between inclusion [19] and to describe the non-local properties of inhomogeneous medium [12].

APPENDIX

Here we give a short survey of the properties of generalized functions connected with the second derivatives of the Green's function $G(x)$ for homogeneous elastic medium. This function is the solution of the equation

$$\nabla_i C_0^{ijkl} \nabla_k G_{lm}(x) = -\delta_m^j \delta(x),$$

where ∇_i is del-operator.

Fourier transform of $G(x)$ has the form

$$G(k) = L^{-1}(k), \qquad L^{ij}(k) = k_l C_0^{lijm} k_m.$$

Fourier transforms of functions $K(x)$ and $S(x)$ in Eqs. (2.1) and (2.3) take the forms

$$K_{ijml}(k) = \left[k_i G_{jm} k_l\right]_{(ij)(ml)}, \qquad S^{ijml}(k) = C_0^{ijpq} K_{pqrs}(k) C_0^{rsml} - C_0^{ijml}.$$

Thus $K(k)$ and $S(k)$ are homogeneous functions of zeroth degree in k.

Let us consider the actions of functions $S(x)$ and $K(x)$ on a function of class $S(R^3)$ (as $|x| \to \infty$, they decrease more rapidly than any negative power of $|x|$). These functions are generalized homogeneous functions of power -3 and their regularizations on function Φ of the mentioned class have the forms [20,21]

$$\begin{aligned}
\int S(x)\Phi(x)\,dx &= D_0\Phi(0) + \det\left(\alpha^{-1}\right) \fint S\left(\alpha^{-1}y\right)\Phi\left(\alpha^{-1}y\right) dy, \\
\int K(x)\Phi(x)\,dx &= A_0\Phi(0) + \det\left(\alpha^{-1}\right) \fint K\left(\alpha^{-1}y\right)\Phi\left(\alpha^{-1}y\right) dy,
\end{aligned} \tag{A.1}$$

where α is an arbitrary non-degenerate linear transformation of x–space. The integrals on the right hand side exist and are taken in the sense of Cauchy's principal value. The constant tensors D_0 and A_0 are expressed in terms of Fourier transforms $S(k)$ and $K(k)$ of functions $S(x)$ and $K(x)$ by the formulas

$$D_0 = \frac{1}{4\pi} \int_{\Omega_1} S(\alpha k)\,d\Omega, \qquad A_0 = \frac{1}{4\pi} \int_{\Omega_1} K(\alpha k)\,d\Omega, \tag{A.2}$$

where Ω_1 is the surface of the unit sphere in the k-space of Fourier transforms. Note that if $\Phi(\alpha^{-1}y) = \Phi(|y|)$ the integrals in the r.h.s. of (A.1) vanish.

Let us now consider the effect of the action of the convolution operator with the kernels $S(x)$ and $K(x)$ on the constant m_0. the respective integrals formally diverge at zero and infinity. Note that the integrals

$$\int S(x-x')m_0\,dx', \qquad \int K(x-x')C_0m_0\,dx'$$

have the meaning of internal stresses and strains, respectively, in homogeneous medium containing dislocation moments of constant density m_0 [22]. If the deformation of the medium is not constrained at infinity, such distribution of dislocations does not result in the appearance of internal stresses but induces an additional constant ("plastic") deformation of the medium of magnitude equal to m_0. Consequently in this case

$$\int S(x-x')m_0\,dx' = 0, \qquad \int K(x-x')C_0m_0\,dx' = m_0. \tag{A.3}$$

However, if the deformation of the medium is constrained at infinity, the result is obviously different:

$$\int S(x-x')m_0\,dx' = -C_0m_0, \qquad \int K(x-x')C_0m_0\,dx' = 0. \tag{A.4}$$

Another way to obtain these formulae was proposed in Ref. [21].

These formulas allow to find the action of generalized function $S(x)$ and $K(x)$ on the function $\Psi_m(x) = 1 - \Phi_m(x)$, where $\Phi_m(x)$ is a function of $S(R^3)$ class. Note that the pair correlation $\Psi_m(x)$ considered in this paper is a function of that type.

REFERENCES

1. J.R. Boehler (ed.), "Yielding, Damage and Failure of Anisotropic Solids", *Mechanical Engineering Publications*, London (1987).
2. J.B. Walsh, *Jour. Geophys. Res.* **70** (1965) 381.
3. G.C. Hiwley, *Plastic World* **34** (1976) 36.
4. J. Raxer and E. Andersen, *Polym. Eng. Sci.* **19** (1979) 1.
5. H.Horii and K. Sahasakmontri, in *Micromechanics and Inhomogeneity* (G. Weng *et al.*, eds.), Springer-Verlag (1990), p. 137.
6. M. Kachanov, *Appl. Mech. Rev.* **45** (1992) 304.
7. B. Budiansky and R.J.O'Connell, *Int. Jour. Solids and Structure* **12** (1976) 81.
8. Y. Benveniste, *Mech. Res. Comm.* **13** (1986) 193.
9. S.K.Kanaun, *Appl. Math. and Mech. (PMM)* **48** (1984) 81.
10. S.K.Kanaun, *Proc. Acad. Sci. of the USSR, Mechanics of Solids (MTT)*, No. 3 (1984) 74.
11. E.N. Wilchevskaja and S.K.Kanaun, *Appl. Math. and Mech. (PMM)* **56** (1992) 275.
12. S.K.Kanaun, *Appl. Math. and Mech. (PMM)* **46** (1982) 520.
13. K. Okuno and R.T. Woodhams, *Polym. Eng. Sci.* **15** (1975) 308.
14. K. Okuno, *Kobushi Ronbunshu* **37** (1980) 789.
15. V.M. Levin, *Proc. Acad. Sci. of the USSR, Mechanics of Solids (MTT)*, No. 6 (1976) 137.
16. S.K.Kanaun, *Applied Mechanics (Prikladnaja Mekhanika)* **16** (1980) 36.

17. A.S. Vavakin and R.L. Salganik, *Proc. Acad. Sci. of the USSR, Mechanics of Solids (MTT)*, No. 3 (1975) 65.
18. T. Gottesman, Z. Hashin and M.A. Brull, in *Advances in Composite Materials proc. ICCM* **3** (Bunsel *et al.* eds.) Pergamon Press, Oxford, N.Y. (1980) p. 743.
19. S.K.Kanaun, *Mechanics of composite materials (Mekhanika kompositnikh materialov)*, (1990) 702.
20. I.M. Gel'fand and E.M. Shilov, *Generalized Functions*, Academic, N.Y. (1964).
21. I.A. Kunin, *Elastic Medium with Microstructure,* **2**, Springer-Verlag, N.Y., London, Heidelberg, Tokio (1983).
22. I.A. Kunin, "Method of tensor analysis in the theory of dislocations". A supplement to the Russian translation of J.A. Schouten: *Tensor Analysis for Physicists*, Nauka, Moscow (1965) p. 374.