Conformal curvature and spherical symmetry

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Special Note

This article was written together with Robert Boyer in June of 1966. Robert Boyer came to work with me during my first five years in Mexico.

After returning to the University of Austin, he was shot by a madman, together with some other twenty people, close to the rectory tower. Of course, I don't need to tell what was my moral state upon hearing this news and this paper was set aside.

Recently I found it digging in my drawers and I decided to ask the Editors of the Revista Mexicana de Física whether it would be possible to publish it. They accepted. I think that the article deals with a basic issue. At that time, very few people were aware of the conformal structure of the interior Schwarzschild solution. This article is to honor the memory of my friend.

Jerzy Plebański
May 19, 1993.

ABSTRACT. Spherical symmetric space-times are found to have simple type D conformal curvature. We classify all such conformally flat spaces, showing in which cases conformal flat spaces, showing in which cases conformal flatness becomes strict flatness. We show how to obtain explicitly the conformal factor of such spaces. As special cases, it appears that the interior Schwarzschild and the Friedmann space-times are conformally flat. These are studied in more detail. We also give results on the embedding of space-times in five and six flat dimensions.

RESUMEN. Se encuentra que los espacio-tiempos simétricamente esféricos tienen una curvatura conforme simple tipo D. Clasificamos todos estos espacios conformalmente planos, mostrando en cuáles casos llegan a ser estrictamente planos. Mostramos cómo obtener explícitamente el factor conforme de tales espacios. Como casos especiales, se encuentra que el interior de los espacio-tiempos de Schwarzschild y Fridmann son conformalmente planos; estos se estudian en más detalle. También damos resultados sobre la inmersión de espacio-tiempos en cinco y seis dimensiones planas.

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1. INTRODUCTION

In this paper, we wish to gather together some results on space-times in which the spherical symmetry interacts with the curvature tensor. The most immediate result is that all such space-times have conformal curvature of type D with one independent invariant. This is perhaps, not too surprising, as spherical symmetry picks out unique space-like two elements at each event, and the two null normals might be supposed to be good candidates for double Debever vectors. What is less obvious is that the coefficient of the conformal curvature is greatly simplified in spatially isotropic coordinates, and contains only the *spatial* derivatives of *one* structural function. Thus the classification of conformally flat spaces with spherical symmetry is easy, and we also give formulas to exhibit the conformal flat mess explicitly.

Our experience has been that to study the Einstein curvature of spherically symmetric space it is better to use standard coordinates. If we use the field equations and suppose the material sources to be hydrodynamical, we find curious relations between the fluid variables and the single conformal invariant which are reminiscent of thermodynamics, and which may be useful in non-static collapse problems. But the most direct outcome of our study of the connection between the fluid variables and the conformal invariant is the recognition that the familiar interior Schwarzschild metric is conformally flat, a fact which certainly cannot be well known. Another by-product is the better-known result that all the Friedmann universes are conformally flat. We display explicitly the conformal flatness in all these cases.

Finally, we give some results on embedding space times in flat spaces of higher dimension. It is easy to see that all spaces of spherical symmetry can be embedded in six flat dimensions. One of us has already worked out the details of the embedding of all conformally flat spaces in six flat dimensions. And we conjecture that all conformally flat spaces of spherical symmetry can be embedded in five flat dimensions.

2. THE CONFORMAL CURVATURE OF SPHERICALLY SYMMETRIC SPACES

In this section we examine the curvature of a normal V_4 the riemannian space-time of signature $(+---)$ which contains O(3) as the sub-group of its (possibly larger) group of symmetries.

By an appropriate coordinate transformation the line element of such a space can always be brought to the form*

$$V_4\colon \quad ds^2 = \phi^{-2}(x^0, u)[\psi^2(x^0, u)\,(dx^0)^2 - dx^a\,dx^a] \tag{1.1}$$

This statement requires some justification. Let the metric of an arbitrary spherically symmetric space-time be written as

$$V_4\colon \quad ds^2 = h_{AB}(x^C)\,dx^A\,dx^B + H(x^C)\,(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2), \tag{1.1a}$$

*Latin letters will run from 1 to 3, greek from 0 to 3.

where $x^1 = r$ and $A, B = 0, 1$. We seek a transformation to $\bar{x}^0 = g(x^A)$ and $\bar{x}^1 = f(x^A)$ which will reduce h_{AB} to say, $\|\bar{h}_{AB}\| = Hf^{-2}\|\operatorname{diag}(\psi^2 - 1)\|$, so that we shall have (1.1) with $\phi = fH^{-1/2}$. Since $\bar{h}^{AB} = \frac{\partial \bar{x}^A}{\partial x^C}\frac{\partial \bar{x}^B}{\partial x^D} h^{CD}$, we must have

$$-\frac{f^2}{H} = h^{AB} f_{,A} f_{,B}, \tag{1.1b}$$

$$0 = h^{AB} f_{,A} g_{,B}, \tag{1.1c}$$

$$\frac{f^2}{H\psi^2} = h^{AB} g_{,A} g_{,B}. \tag{1.1d}$$

Representing $f = f(x^A)$ as $\Phi(x^A, f) = 0$, (1.1b) is equivalent to the equation

$$h^{AB}\Phi_{,A}\Phi_{,B} + \frac{f^2}{H}\Phi_{,f}^2 = 0, \tag{1.1e}$$

which always admits solutions locally because h_{AB} is hyperbolic normal. Equation (1.1c) then determine g as a function orthogonal to f, and (1.1d) serves to determine ψ.

The coordinates $x^\mu = (x^0, x^a)$ of (1.1) will be called spatially isotropic coordinates. We shall see that they are very convenient in the study of conformal curvature. We shall leave the topology of the manifold M_4 covered by V_4 unspecified in this section; we assume merely that locally M_4 is a differentiable manifold.

The space V_4 is conformally equivalent to

$$V_4'\colon \quad ds^2 = \psi^2(x^0, u)\,(dx^0)^2 - dx_a\,dx^a, \tag{1.2}$$

the conformal curvature tensor $C^\alpha{}_{\beta\alpha\delta}$ being the same for both. But since V_4' contains only one arbitrary function ψ, its curvature quantities are very simple.

In fact, let us consider a space V_4' slightly more general than V_4', with a more general function $\psi(x^\mu)$:

$$V_4''\colon \quad ds^2 = \psi^2(x^\mu)\,(dx^0)^2 - dx_a\,dx^a. \tag{1.3}$$

One easily finds that

$$V_4''\colon \quad \left\{ {0 \atop 0\ \mu} \right\} = \psi^{-1}\psi_{,\mu} \qquad \left\{ {a \atop 0\ 0} \right\} = \psi\psi_{,a}, \tag{1.4}$$

while all other Christoffel symbols vanish. One then quickly finds the Riemann tensor, and hence the Einstein and conformal curvature tensors:

$$V_4''\colon \quad G_{0\alpha} = 0, \qquad G_{ab} = -\psi^{-1}(\psi_{,ab} - \delta_{ab}\psi^{s}_{,s}), \tag{1.5}$$

$$V_4''\colon \quad C_{0bcd} = 0, \qquad C_{0a0b} = -\tfrac{1}{2}\psi(\psi_{,ab} - \tfrac{1}{3}\delta_{ab}\psi^{s}_{,s}), \tag{1.6}$$

$$C_{abcd} = -\psi^{-2}\epsilon_{abr}\epsilon_{cds}C_{0r0s}.$$

Notice the absence of temporal derivatives in all the curvature quantities of V_4''. Also observe that (1.6) imply that if V_4 is understood as a space conformally equivalent to V_4'' with proportionality factor ϕ^{-2}, then

$$V_4: \quad C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 2\phi^4\psi^{-2}\left[\psi_{,ab}\psi^{,ab} - \tfrac{1}{3}(\psi_{,ss})^2\right]. \tag{1.7}$$

One easily sees that $G_{\alpha\beta} = 0$ in V_4'' implies that V_4'' is flat. Indeed, from (1.5), $G_{\alpha\beta} = 0 \to \psi_{,ab} - \delta_{ab}\psi^{s}_{,s} = 0 \to \psi = a(x^0) + k_a(x^0)x^a$, with $a(x^0)$, $k_a(x^0)$ arbitrary. But this ψ used in (1.6) yields $C_{\alpha\beta\gamma\delta} = 0$. Therefore $R_{\alpha\beta\gamma\delta} = 0$, and the space

$$ds^2 = \left(a(x^0) + 2k^a(x^0)x^a\right)^2 \left(dx^0\right)^2 - dx^a\,dx^a \tag{1.8}$$

is flat.*

On the other hand, V_4'' can be conformally flat but not strictly flat. Indeed, according to (1.6):

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = 0 &\to \psi_{,ab} - \tfrac{1}{3}\delta_{ab}\psi_{,ss} = 0 \\ &\to \psi = a(x^0) + 2k^a(x^0)x^a + b(x^0)x^a x^a, \end{aligned} \tag{1.9}$$

with $a(x^0)$, $k_a(x^0)$, $b(x^0)$ arbitrary. With this ψ, the Einstein tensor becomes, according to (1.5)

$$G_{0\alpha} = 0, \qquad G_{ab} = -\tfrac{3}{2}b\psi^{-1}\delta_{ab}. \tag{1.10}$$

Thus, if $b(x^0) \neq 0$, then $G_{\alpha\beta} \neq 0$ and the conformally flat V_4'' with ψ from (1.9) is not strictly flat.

We include, for completeness, a method for displaying explicitly the conformal flatness of such a V_4'' satisfying (1.9). Let $f(x^0)$ be the (unique) solution of the integral equation

$$f(u) = \int_0^u f(x)H(s,u)\,ds + \Phi(u), \tag{1.10a}$$

where

$$\begin{aligned} H(s,u) &= \int_s^u \left[4k_a(s)k^a(t) - 2a(s)b(t) - 2a(t)b(s)\right] dt \\ \Phi(u) &= \int_0^u \left[Aa(t) + 2K^a k_a(t) + Bb(t)\right] dt \pm \sqrt{K_a K^a - AB} \end{aligned} \tag{1.10b}$$

* *E.g.*, metrics of the type $ds^2 = (x^0 - x^3)^2\,(dx^0)^2 - dx_a\,dx^a$ are simple examples of so-called "coordinate waves".

A, K_a, B are five arbitrary constants subject to $K_aK^a - AB \geq 0$. From

$$\phi(x^\mu) = \int f(x^0)\psi(x^0, x^a)\,dx^0. \tag{1.10c}$$

Then

$$ds^2 = \phi^{-2}\left[\psi^2\,(dx^0)^2 - dx_a\,dx^a\right] \tag{1.10d}$$

is flat. The proof, which is a straightforward application of a theorem of Schouten [1] will not be given here, as the details of a more practical method for the special case $k_a = 0$ will be given in Sect. 2.

Finally, we indicate how a general V_4'' can always be embedded (locally) in six flat dimensions. For constant ℓ let

$$\begin{aligned} x^1 &= x^1, \qquad X^0 = \ell\psi(x^\mu),\\ x^2 &= x^2, \qquad X^4 = \ell\psi(x^\mu)\sinh(x^0/\ell),\\ x^3 &= x^3, \qquad X^5 = \ell\psi(x^\mu)\cosh(x^0/\ell). \end{aligned} \tag{1.11}$$

Then

$$\begin{aligned} ds^2 &= \psi^2\,(dx^0)^2 - dx_a\,dx^a\\ &= (dX^0)^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2 + (dX^4)^2 - (dX^5)^2. \end{aligned} \tag{1.12}$$

Eliminating x^μ from (1.11), one sees that V_4'' is the intersection of the surfaces

$$(X^0)^2 + (X^4)^2 - (X^5)^2 = 0, \quad X^0 = \ell\psi\left[\ell\tanh^{-1}\left(\frac{X^4}{X^5}\right), X^1, X^2, X^3\right], \tag{1.13}$$

in the (flat) 6–dimensional space with metric (1.12).

This concludes our discussion of the more general V_4''. We now specialize to V_4', where $\psi(x^4) = \psi(x^0, u)$. One finds from (1.6)

$$V_4':\quad C_{0a0b} = -2u\psi\psi_{,uu}\left(\frac{x_a x_b}{u} - \tfrac{1}{3}\delta_{ab}\right). \tag{1.14}$$

Equivalently,

$$4u\psi_{,uu} = \psi_{,rr} - \frac{1}{r}\psi_{,r}, \tag{1.15}$$

which shows why we prefer to use $u = r^2$ in (1.14).

With (1.14) and (1.6) we can easily determine the algebraic type of the conformal curvature tensor of V_4' and hence of V_4.

Let

$$\begin{matrix} V_4': \\ V_4: \end{matrix} \quad k_\alpha = \left.\begin{matrix} 1 \\ \phi^{-1} \end{matrix}\right\} \left(\psi, -\frac{x^a}{r}\right), \qquad \ell_\alpha = \left.\begin{matrix} 1 \\ \phi^{-1} \end{matrix}\right\} \left(\psi, \frac{x^a}{r}\right). \tag{1.16}$$

Without the factor ϕ^{-1} we understand these vectors in the sense of V_4'. They satisfy

$$V_4, V_4': \quad k_\alpha k^\alpha = \ell_\alpha \ell^\alpha = 0, \quad k_\alpha \ell^\alpha = 2. \tag{1.17}$$

Remembering that $C^\alpha{}_{\beta\gamma\delta}$ in V_4 and V_4' coincide, but that operations with the metric require an additional factor in V_4, we find that

$$\begin{aligned} C_{\alpha\mu\beta\nu} k^\mu k^\nu &= -\tfrac{4}{3} u \phi^2 \psi^{-1} \psi_{,uu} k_\alpha k_\beta, \\ C_{\alpha\mu\beta\nu} \ell^\mu \ell^\nu &= -\tfrac{4}{3} u \phi^2 \psi^{-1} \psi_{,uu} \ell_\alpha \ell_\beta, \\ C_{\alpha\mu\beta\nu} k^\alpha \ell^\mu &= +\tfrac{8}{3} u \phi^2 \psi^{-1} \psi_{,uu} k_{[\beta} \ell_{\nu]}. \end{aligned} \tag{1.18}$$

This allows us to infer that $C^\alpha{}_{\beta\gamma\delta}$ is of Petrov type I degenerate (type D, or (22) in the Penrose notation) with k_α, ℓ_α as double Debever vectors. And when

$$\psi_{,uu} = 0 \rightarrow \psi = a(x^0) + b(x^0)u, \tag{1.19}$$

V_4 is conformally flat. This is of course, a special case of (1.9). We shall show in Sect. 2 that the important special case of (1.19) when a and b are constant is conformal to the interior Schwarzschild metric. Eq. (1.19) seems then to be as close a conformal counterpart as possible to the well-known Birkhoff theorem, which deals with spherical symmetry and Ricci-flatness.

We wish to discuss the spinorial description of the facts established above. Select as the Pauli matrices in V_4:

$$V_4: \quad g^{\mu\dot{A}B} = \phi \left[\psi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \tag{1.20}$$

where spinorial indices run over 0, 1. The vectors k_α, ℓ_α in V_4 can be written as

$$k^\mu = g^{\mu\dot{A}B} k_{\dot{A}} k_B, \qquad \ell^\mu = g^{\mu\dot{A}B} \ell_{\dot{A}} \ell_B, \tag{1.21}$$

$$\begin{aligned} k_A &= \left(\cos\vartheta/2 e^{i\vartheta/2}, \sin\vartheta/2 e^{-i\vartheta/2}\right), \\ \ell_A &= \left(-\sin\vartheta/2 e^{i\vartheta/2}, \cos\vartheta/2 e^{i\vartheta/2}\right). \end{aligned} \tag{1.22}$$

We have the angular variables ϑ, φ:

$$x^1 = r \sin\vartheta \cos\varphi, \quad x^2 = r \sin\vartheta \sin\varphi, \quad x^3 = r\cos\vartheta; \tag{1.23}$$

k_A and ℓ_A depend only on the angles and are normalized by

$$k_A \ell^A = \epsilon^{AB} k_A \ell_B = 1, \qquad \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{AB}. \tag{1.24}$$

Construct from the Pauli matrices the mixed object (spin tensor)

$$S^{\mu\nu AB} = \tfrac{1}{2} \epsilon_{\dot{R}\dot{S}} \big(g^{\mu \dot{R} A} g^{\nu \dot{S} B} - g^{\nu \dot{R} A} g^{\mu \dot{S} B} \big). \tag{1.25}$$

Then the conformal curvature can be represented by its spin-image

$$C^{\alpha\beta\gamma\delta} = S^{\alpha\beta AB} C_{ABCD} S^{\gamma\delta CD} + \text{c.c.} \tag{1.26}$$

Using (1.18) one finds after some work that

$$C_{ABCD} = C D_{ABCD}, \tag{1.27}$$

where

$$D_{ABCD} = K_{(A} K_B \ell_C \ell_{D)} \rightarrow D_{ABCD} D^{ABCD} = \tfrac{1}{6} \tag{1.28}$$

and

$$C = -u\phi^2 \psi^{-1} \psi_{,uu}. \tag{1.29}$$

D_{ABCD} depends only on the angles. In fact,

$$\begin{aligned} D_{1111} &= \tfrac{1}{8}(1 - \cos\vartheta) e^{2i\varphi}, \qquad & D_{2222} &= \tfrac{1}{8}(1 - \cos\vartheta) e^{-2i\varphi}, \\ D_{1112} &= -\tfrac{1}{8} \sin\vartheta e^{i\varphi}, & D_{2221} &= \tfrac{1}{8} \sin\vartheta e^{-i\varphi}, \end{aligned}$$
$$D_{1122} = \tfrac{1}{8}\big(\tfrac{1}{3} + \cos\vartheta\big). \tag{1.30}$$

The conformal curvature possesss in general four invariants, namely, the real and imaginary parts of the two complex quantities

$$\overset{2}{C} = C^{AB}{}_{CD} C^{CD}{}_{AB}, \qquad \overset{3}{C} = C^{AB}{}_{CD} C^{CD}{}_{EF} C^{EF}{}_{AB}. \tag{1.31}$$

(Strictly speaking, the real parts and the *squares* of the imaginary parts are scalars). However, in the (2-2) and (2-1-1) cases we have the condition

$$\big(\overset{2}{C}\big)^3 - 6\big(\overset{3}{C}\big) = 0, \tag{1.32}$$

so that $\overset{2}{C}$ contains all the information about curvature invariants.

In our spherically symmetric V_4 we find from (1.27–28):

$$V_4: \quad \overset{2}{C} = \tfrac{3}{16}(C)^2 = \text{real}, \tag{1.33}$$

so that only one conformal invariant characterizes the space. It follows that C given by (1.29) is invariantly defined, *i.e.*, that $|C|$ is a scalar.

A more direct interpretation of C in terms of the conformal curvature tensor can be obtained from (1.7) specialized to the case of $\psi = \psi(x^0, u)$. We find

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \left(\tfrac{8}{\sqrt{3}}\phi^2\psi^{-1}u\psi_{,uu}\right)^2 \geq 0. \tag{1.34}$$

Therefore

$$C = -u\phi^2\psi^{-1}\psi_{,uu} = \pm\tfrac{1}{8}\left(3C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\right)^{\frac{1}{2}} \tag{1.35}$$

We may justly interpret $|C|$ as the size, or the absolute value of the conformal curvature.

A remarkable property of C is the absence of temporal derivatives when isotropic coordinates are used. This property is not shared by other coordinate systems which are commonly used in cases of spherical symmetry. We give below, partly for purposes of comparison, and partly for their own sake, formulas in which standard coordinates are used. Namely, let

$$V_4: \quad ds^2 = e^{\nu}(dx'^0)^2 - e^{\lambda}dr'^2 - r'^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{1.36}$$

with ν and λ dependent on x'^0, r' (primes to distinguish these from isotropic coordinates). One can compute $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ and, consequently, C to be given by*

$$C = -\tfrac{3}{4}C^{0'r'}{}_{0'r'} = -\frac{1}{8}\left[\left(\nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}v'\lambda' - \frac{\nu' - \lambda'}{r'} + \frac{2}{r'^2}\right)e^{-\lambda} - \frac{2}{r'^2} - \left(\ddot{\lambda} + \tfrac{1}{2}\dot{\lambda}^2 - \tfrac{1}{2}\dot{\nu}\dot{\lambda}\right)e^{-\nu}\right] \tag{1.37}$$

The notation is that $C^{0'r'}{}_{0'r'}$ is the only independent of $C^{\alpha\beta}{}_{\gamma\delta}$ in the $x'^0, r', \vartheta, \varphi$ coordinates. Primes denote differentiation with respect to r', dots with respect to x'^0. The sign is so chosen that if ν, λ are static, the simple $r' = r'(r)$ transformation which bring (1.36) into (1.1) transforms (1.37) into (1.35).

Formulae (1.37) is rather complicated. However, when one expresses the components of the Einstein tensor [2] in terms of λ and ν, then one can eliminate the derivatives of these quantities, and express C in the form

$$C = -\tfrac{1}{4}\left[\frac{3}{r'^2}(e^{-\lambda} - 1) + G^{\alpha}_{\alpha} - 3G^2_2\right]. \tag{1.38}$$

*Formula (1.37) was first obtained by Mr. R. Pellicer.

Thus, if we assume the field equations

$$G^\alpha_\beta = \frac{8\pi K}{C^4} T^\alpha_\beta + \Lambda \delta^\alpha_\beta \tag{1.39}$$

(Λ is the cosmological constant introduced for generality, T^α_β is the energy-momentum tensor; in our signature $T^0_0 \geq 0$) we obtain

$$C = -\frac{1}{4}\left[\frac{3}{r'^2}(e^{-\lambda} - 1) + \frac{8\pi K}{C^4}(T^\alpha_\alpha - 3T^2_2) + \Lambda\right]. \tag{1.40}$$

It is interesting to specialize to the case of a perfect fluid:

$$T^\alpha_\beta = (\epsilon + P)u^\alpha u_\beta - P\delta^\alpha_\beta, \qquad u^\alpha u_\alpha = 1, \tag{1.41}$$

where P is the pressure and ϵ is the energy density in the rest frame of the fluid. Because of spherical symmetry we assume $u_\vartheta = u_\varphi = 0$, so that (1.41) used in (1.40) yields

$$C = -\frac{1}{4}\left[\frac{3}{r'^2}(e^{-\lambda} - 1) + \frac{8\pi K}{C^4}\epsilon + \Lambda\right]. \tag{1.42}$$

Later we shall consider some applications of this hydrodynamical formula.

For the moment we should like to point out that in the static case ($\dot{\lambda} = 0$) with $\lambda = -\nu$, (1.37) takes the form

$$C = -\frac{r'}{8}\frac{d^2}{dr'^2}\frac{e^{-\lambda} - 1}{r'}. \tag{1.43}$$

This applies, for example, to the case of a Reissner-Nordström space-time (with cosmological constant):

$$e^{-\lambda} = 1 - \frac{2m}{r'} + \frac{e^2}{r'^2} = \tfrac{1}{3}\Lambda r'^2 = e^\nu, \tag{1.44}$$

and yields

$$C = \frac{3}{2}\frac{m}{r'^3}\left(1 - \frac{e^2}{mr'}\right) \to C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 3\left(\frac{4m}{r'^3}\right)^2\left(1 - \frac{e^2}{mr'}\right)^2. \tag{1.45}$$

This formula is illustrated in Fig. 1.

We fell that formula (1.45) can be read in two ways. On the one hand, it shows that r' is algebraically related to the invariant of the conformal curvature, and can be understood to be invariantly defined. In the well-known Schwartzschild case, r' is proportional to $(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta})^{-1/6}$ when $e \neq 0$ it is a many valued function of this quantity. On the other hand, we see from (1.45) that $C^{\alpha\beta}{}_{\gamma\delta}$ and G^α_β of the metric (1.44) stay continuous when r' approaches critical values $e^{-\lambda} = 0$. Thus these values determine only coordinate

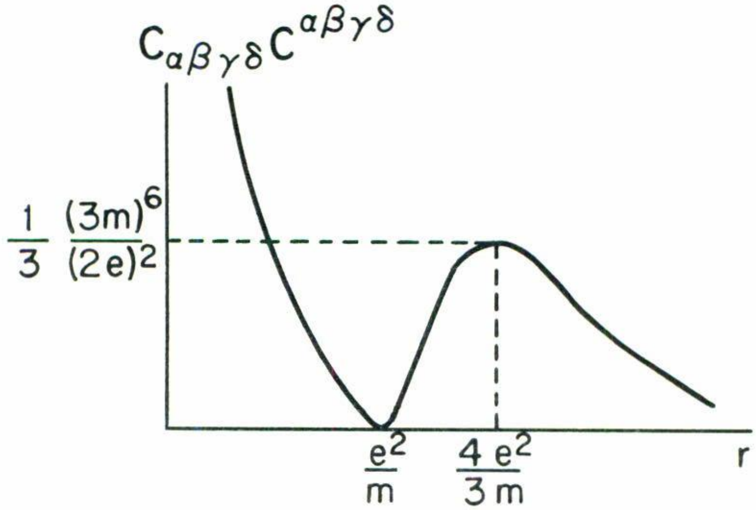

FIGURE 1.

singularities; this fact is responsible for the fact that analytic continuations of (1.44) exist [3].

That the constant Λ does not enter in (1.45) is not too surprising: the de Sitter space $e^{-\lambda} = 1 - \frac{\Lambda}{3}r'^2 = e^{\nu}$ is, of course, conformally flat. More generally, from (1.43) we infer that spaces with

$$e^{-\lambda} = 1 + ar' + br'^2 = e^{\nu} \quad (a, b = \text{constant}), \tag{1.46}$$

are conformally flat. We would point out that (1.19) is a stronger result (since time-dependence is permitted), and illustrates the superiority of spatially isotropic coordinates.

Now, let us return to (1.42) and examine the conformal curvature of space-time filled with hydrodynamical matter. Consider first the general non-static case. Then we have (among others) the field equations:

$$e^{-\lambda}\left(\frac{1}{r'^2} - \frac{\lambda'}{r'}\right) - \frac{1}{r'^2} = -\frac{8\pi K}{C^4}\left[(\epsilon + P)u^0 u_0 - P\right] - \Lambda,$$

$$e^{-\lambda}\frac{\dot{\lambda}}{r'} = -\frac{8\pi K}{C^4}(\epsilon + P)u^{r'}u_{0'}, \tag{1.47}$$

where $u_{0'}u^{0'} + u_{r'}u^{r'} = 1$. Using these in (1.42) we can show that

$$d(r'^3 C) = -\frac{2\pi K}{C^4}\left[r'^3\, d\epsilon + er'^2(\epsilon + P)u^{r'}(u_{0'}\, dx^{0'} + u_{r'}\, dr')\right]. \tag{1.48}$$

This general relation becomes particularly interesting if specialized to the case of differentials along the lines of current, *i.e.*, $dx^{\mu} = u^{\mu}\, ds$:

$$d\left\{r'^3\left[C + \frac{2\pi K}{C^4}\epsilon\right]\right\} = -\frac{2\pi K}{C^4}P\, d(r'^3). \tag{1.49}$$

On the other hand, if the differentials are orthogonal to the lines of current, *i.e.*, $u_\mu \, dx_\mu = 0$, then

$$d\{r'^3 C\} = -\frac{2\pi K}{C^4} r'^3 \, d\epsilon. \tag{1.50}$$

It is conceivable that the relations (1.48–49–50) may be of some use in the study of the thermodynamics of spherically symmetric collapse.

As a more modest application of (1.42) consider the case of a static fluid. We have $\dot{\lambda} = 0 = u_{r'}$ and $u_{0'}u^{0'} = 1$ so that using the first of (1.47) we obtain

$$e^{-\lambda} = 1 - \frac{\Lambda}{3} r'^2 - \frac{8\pi K}{r' C^4} \int_0^{r'} dr' \, r'^2 \epsilon. \tag{1.51}$$

The constant of integration has been chosen so that $e^{-\lambda}$ is finite at $r' = 0$, which is possible if we assume that $\epsilon r'^2 = O(r'^\epsilon)$ with $\epsilon \geq 0$. Now (1.51) used in (1.42) yields

$$C = \frac{2\pi K}{C^4} \left[\frac{1}{r'^3} \int_0^{r'} \epsilon \, dr'^3 - \epsilon \right]. \tag{1.52}$$

But since $\lim(r'^3 \epsilon) = 0$, we can integrate (1.52) by parts:

$$C = -\frac{2\pi K}{C^4} \frac{1}{r'^3} \int_0^{r'} r'^3 \frac{d\epsilon}{dr'} \, dr'. \tag{1.53}$$

This simple result applies independently of the equation of state to all static, spherically symmetric fluid configurations. It is also independent of the value of the cosmological constant.

Consider now the special case of ϵ constant inside a sphere of radius $r' = a$ and vanishing outside:

$$\epsilon = \epsilon_0 [1 - \theta(r' - a)]. \tag{1.54}$$

This should of course, describe the well-known Schwarzschild solution (interior + exterior), and in fact we find from (1.53)

$$C = \begin{cases} 0, & 0 \leq r' < a, \\ \dfrac{2\pi K a^3 \epsilon_0}{C^4} \dfrac{1}{r'^3}, & a < r' \leq \infty \end{cases}. \tag{1.55}$$

Thus the internal Schwarzschild solution with $\epsilon = \epsilon_0 =$ const. is *conformally flat*, a fact which does not seem to be well known. Thus in a sense the interior solution is simpler than the exterior. Moreover, at the boundary of the fluid, the conformal and Einstein curvatures exchange roles: $C_{\alpha\beta\gamma\delta} = 0$ inside and $G_{\alpha\beta} = 0$ outside. We shall examine various aspects of the interior Schwartzschild solution in Sect. 3.

We conclude this section by deriving the Schwartzschild metric components which follow from the choice (1.54) of ϵ. Used in (1.51) it results in

$$e^{-\lambda} = \begin{cases} 1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{3}\right) r'^2, & 0 \le r' \le a, \\[2ex] 1 - \dfrac{2m}{r'} - \dfrac{\Lambda}{3} r'^2, & a \le r' < \infty, \end{cases} \tag{1.56}$$

where $m = 4\pi K \epsilon_0 a^3 / 3C^4$. Comparing with (1.55), we see that for $r' \ge a$, $C = (\frac{3}{2}) m / r'^3$, in agreement with (1.45).

The e^{ν} corresponding to this $e^{-\lambda}$ is easily seen to be

$$e^{\nu} = \begin{cases} \left[\dfrac{\frac{3}{2}}{1 + \frac{\Lambda}{6}\frac{a^3}{m}} \sqrt{1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{3}\right) a^2} - \right. \\[2ex] \left. \dfrac{1}{2} \dfrac{1 - \frac{1}{3}\frac{\Lambda_a^3}{m}}{1 + \frac{\Lambda}{6}\frac{a^3}{m}} \sqrt{1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{3}\right) r'^2} \right]^2, & \text{if } 0 \le r' \le a, \\[2ex] 1 - \dfrac{2m}{r'} - \dfrac{\Lambda}{3} r'^2, & \text{if } a \le r' < \infty. \end{cases} \tag{1.57}$$

To this metric obeying the field equations (1.39) there belong the ϵ of (1.54) and the pressure

$$\frac{8\pi K}{C^4} P = \frac{2m}{a^3} \left(1 - \frac{1}{a^3} \frac{\Lambda a^3}{m}\right) \frac{\sqrt{1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{a}\right) r'^2} - \sqrt{1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{3}\right) a^2}}{\sqrt{1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{3}\right) a^2} - \left(\dfrac{1}{3} - \dfrac{\Lambda a^3}{9m}\right) \sqrt{1 - \left(\dfrac{2m}{a^3} + \dfrac{\Lambda}{3}\right) r'^2}}. \tag{1.58}$$

when $0 \le r' \le a$. When $r' = a$, P vanishes. When $\Lambda \to Q$, the formulas (1.56–58) reduce to the standard (interior + exterior) Schwartzchild formulas [4]. Their validity is, of course, restricted to such values of the parameters m, Λ, a that for $0 \le r' < a$ the pressure is positive and finite. This restriction will emerge more clearly in the next section.

Starting from (1.36) we studied various aspects of the conformal curvature —taking into account the field equations— in standard spherical coordinates. On the other hand, as is clear from (1.35), isotropic coordinates are very convenient in the study of the conformal curvature of spherically symmetric spaces. In the next section, several problems related to conformal curvature will be studied, making essential use of isotropic coordinates.

2. Spherically symmetric spaces in isotropic coordinates

In the previous section [Eq. (1.19)] we obtained the result that the spherically symmetric space

$$V_4\colon\quad ds^2 = \phi^{-2}(x^0, r)\big[(a(x^0) + b(x^0)r^2)^2\, dx^{02} - dx^a\, dx^a\big] \tag{2.1}$$

is conformally flat. We begin this section by showing the explicit coordinate transformation which displays the conformal flatness of V_4' and giving the conformal factor as determined by the arbitrary functions $a(x^0)$ and $b(x^0)$.

Consider the flat space

$$S_4\colon\quad ds^2 = dx'^{02} - dx'^a\, dx'^a = dx'^{02} - dr'^2 - r'^2\big[d\theta^2 + \sin^2\vartheta\, d\varphi^2\big], \tag{2.2}$$

and let α and β be arbitrary functions of a variable x^0. Now consider the coordinate transformation from $x^{0\prime}, r'$ to variables x^0, r defined by

$$x'^0 = \frac{\alpha}{\beta}\frac{1}{\alpha^2 - \beta^2 r^2} + \int \frac{d\beta}{\alpha\beta^2}, \qquad r' = \frac{r}{\alpha^2 - \beta^2 r^2}. \tag{2.3}$$

Alternatively,

$$x'^0 + r' = \frac{1}{\beta}\frac{1}{\alpha - r\beta} + \int \frac{d\beta}{\alpha\beta^2}, \qquad x'^0 - r' = \frac{1}{\beta}\frac{1}{\beta + r\beta} + \int \frac{d\beta}{\alpha\beta^2}. \tag{2.4}$$

Then, after an easy computation, one finds that

$$J = \frac{\partial(x'^0, r')}{\partial(x^0, r)} = \frac{\partial x'^0}{\partial x^0}\frac{\partial r'}{\partial r} - \frac{\partial x'^0}{\partial r}\frac{\partial r'}{\partial x^0} = (\alpha^2 - \beta^2 r^2)^{-2}\left(-\frac{\dot\alpha}{\beta} + \frac{\dot\beta}{\alpha}r^2\right), \tag{2.5}$$

where dots denote differentiation with respect to x^0.

More simply,

$$J = \frac{1}{2\alpha\beta}\frac{\partial}{\partial x^0}(\alpha^2 - \beta^2 r^2)^{-1}. \tag{2.6}$$

Therefore, when at least one of the variables α, β depends on $x^0(\dot\alpha^2 + \dot\beta^2 \neq 0)$, then the Jacobian does not vanish identically, and our coordinate transformation is legitimate.

Now, with the help of Eq. (2.4), one finds that

$$dx'^{02} - dr'^2 = (\alpha^2 - r^2\beta^2)^{-2}\left[\left(-\frac{\dot\alpha}{\beta} + \frac{\dot\beta}{\alpha}r^2\right)^2 dx_0^2 - dr^2\right]. \tag{2.7}$$

Therefore, we conclude that, as a result of our coordinate transformation,

$$dx'^{0\,2} - dx'^a\,dx'^a = dx'^{0\,2} - dr'^2 - r'^2\big[d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\big]$$
$$= (\alpha^2 - r^2\beta^2)^{-2}\left[\left(-\frac{\dot\alpha}{\beta} + \frac{\dot\beta}{\alpha}r^2\right)^2 dx^{0\,2} - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)\right]$$
$$= (\alpha^2 - r^2\beta^2)^{-2}\left[\left(-\frac{\dot\alpha}{\beta} + \frac{\dot\beta}{\alpha}r^2\right)^2 dx^{0\,2} - dx^a\,dx^a\right]. \tag{2.8}$$

Consequently,

$$\left[-\frac{\dot\alpha}{\beta} + \frac{\dot\beta}{\alpha}r^2\right]^2 dx^{0\,2} - dx^a\,dx^a = (\alpha^2 - \beta^2 r^2)^2(dx'^{0\,2} - dx'^a\,dx'^a). \tag{2.9}$$

x^0 and r appearing in the conformal factor on the right-hand side being understood as

$$x^0 = x^0(x'^0, r'), \qquad r = r(x'^0, r'), \tag{2.10}$$

i.e., the inversion of Eq. (2.3).

This shows that it is always possible to construct explicitly the conformally flat representation of any metric of the form (2.1). In the first step, with $a(x^0)$, $b(x^0)$ known, one solves the differential equations which determine $\alpha(x^0)$, $\beta(x^0)$:

$$\dot\alpha = -a\beta, \qquad \dot\beta = b\alpha. \tag{2.11}$$

The constants of integration can be chosen as convenient.*

In the second step, with $\alpha(x^0)$, $\beta(x^0)$ known, we seek the inverse transformation to (2.3), *i.e.*, (2.10). We can now apply the identity (2.9), obtaining

$$V_4'\colon\quad ds^2 = \phi'^{-2}(x'^0, r')\big[dx'^{0\,2} - dx'^a\,dx'^a\big], \tag{2.12}$$

where

$$\phi'(x'^0, r') = \phi(x^0, r)\big[\alpha^2(x^0) - \beta^2(x^0)r^2\big]^{-1}, \tag{2.13}$$

with x^0, r understood according to Eq. (2.10).

This procedure is general and works for all conformally flat spherically symmetric spaces. It is a practical method which leads to an explicit construction of a conformal factor. It should be easier to use in practice than the method given in Eqs. (1.10a–c), which

*Different choices of constants will lead to different conformal factors. This ambiguity is of course, related to the ambiguity of the conformal factor in the representation of flat space itself, which is governed by the 15-parameter conformal group.

applies to the slightly more general space V_4''. Indeed, as explained in Eqs. (1.1a–e), every spherically symmetric space can be brought to the form (1.1) in isotropic coordinates. If the space is conformally flat, then necessarily $\psi(x^0, r) = a(x^0) + b(x^0)r^2$. Then, one can apply the procedure of this section to construct the conformal factor according to Eq. (2.13). We should like to illustrate this with some examples.

Consider the Friedman universes given in stereographic coordinates by

$$ds^2 = dX^{0\,2} - R^2(X^0)\left(1 + \varepsilon\frac{r^2}{4}\right)^{-2} dx^a\, dx^a. \tag{2.14}$$

The spatial coordinates x^a are dimensionless, $R(X^0)$ is the evolutionary radius of the universe, and ε takes the values $+1, 0, -1$ in the open, flat, and closed models, respectively.

In the first step we introduce instead of X^0 the dimensionless x^0,

$$x^0 = \int \frac{dX^0}{R(X^0)} \to X^0 = X^0(x^0). \tag{2.15}$$

The function $R(x^0) = R\big[X^0(x^0)\big]$ we assume to be known. In terms of it:

$$X^0 = \int R(x^0)\, dx^0. \tag{2.16}$$

Then, we represent (2.14) in our canonical form:

$$ds^2 = \phi^{-2}(x^0, r)\left[\left(1 + \varepsilon\frac{r^2}{4}\right)^2 dx^{0\,2} - dx^a\, dx^a\right],$$

$$\phi = R^{-1}(x^0)\left(1 + \varepsilon\frac{r^2}{4}\right). \tag{2.17}$$

This is enough to see that all Friedmann universes are conformally flat [5]. The case of $\varepsilon = 0$ is trivial: already (2.17) gives the conformally flat representation. We assume $\varepsilon \neq 0$, and the Eqs. (2.11) become

$$\dot{\alpha} = -\beta, \qquad \dot{\beta} = \frac{\varepsilon}{4}\alpha. \tag{2.18}$$

In the case $\varepsilon = -1$ (open model) it is convenient to choose

$$\alpha = -2e^{\frac{1}{2}x^0}, \quad \beta = e^{\frac{1}{2}x^0} \to \int \frac{d\beta}{\alpha\beta^2} = \tfrac{1}{4}e^{-x^0}, \tag{2.19}$$

which, used in Eq. (2.4), yield

$$e^{-2x^0} = 16(x'^{0\,2} - r'^2), \qquad \left(\frac{r+2}{r-2}\right)^2 = \frac{x'^0 - r'}{x'^0 + r'}, \tag{2.20}$$

so that these equations are easily invertible. The resulting conformal factor is

$$\phi' = \sqrt{x'^{02} - r'^2}\, R^{-1}\big\{-\ln 4(x'^{02} - r'^2)^{1/2}\big\}. \tag{2.21}$$

In the case $\varepsilon = +1$ (closed model) we take

$$\alpha = 2\sin\frac{x^0}{2}, \quad \beta = -\cos\frac{x^0}{2} \to \int \frac{d\beta}{d\beta^2} = \tfrac{1}{2}\tan\frac{x^0}{2}. \tag{2.21a}$$

Used in Eq. (2.13), they yield, after some work, the conformal factor

$$\phi' = R^{-1}(x^0)\sqrt{x'^{02} + \big(x'^{02} - r'^2 - \tfrac{1}{4}\big)^2}, \tag{2.21b}$$

where

$$\sin x^0 = x'^0\big(x'^{02} + (x'^{02} - r'^2 - \tfrac{1}{4})^2\big)^{-1/2}.$$

Comparing (2.21) with Eq. (2.21b) we see that the conformal factor in the former case is a function of $x'^{02} - r'^2$ only, whereas in the latter case it is a function of both x'^0 and $x'^{02} - r'^2$. This is easily explained by the fact that the group of rigid motions of a 3-space of constant negative curvature (open model) is isomorphic to the homogeneous Lorentz group which preserves $x'^{02} - r'^2$, whereas the corresponding group in the closed model is quite different.

3. Some properties of the internal Schwartzchild solution

It was already observed in Sect. 1 that the interior Schwartzschild solution is conformally flat and as such is simpler than the exterior solution. The purpose of this section is: 1) to give the explicit conformally flat representation of this metric; and 2) to investigate some of its peculiar features.

For simplicity, we restrict ourselves to the solution without cosmological term:

$$ds^2 = \left[\tfrac{3}{2}\sqrt{1 - \frac{2m}{a}} - \tfrac{1}{2}\sqrt{1 - \frac{2m}{a}\left(\frac{r}{a}\right)^2}\right]^2 dx_0^2$$
$$- \frac{dr^2}{1 - \frac{2m}{a}\left(\frac{r}{a}\right)^2} - r^2\big[d\theta^2 + \sin^2\theta\, d\varphi^2\big], \qquad 0 \le r \le a. \tag{3.1}$$

This metric has the Einstein tensor

$$G_{\alpha\beta} = \frac{8\pi k}{C^4}\big[(\epsilon_0 + P)u_\alpha u_\beta - P\big], \tag{3.2}$$

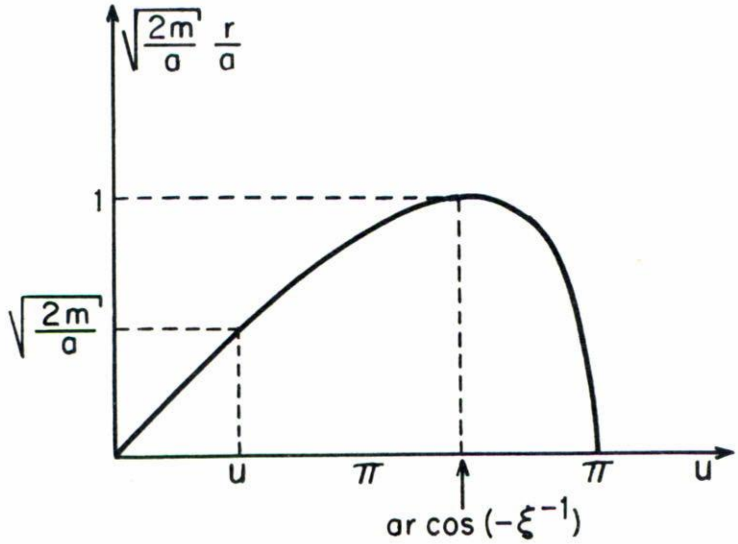

FIGURE 2.

where $\epsilon_0 = \text{const.}$ is related to m by $m = \frac{4\pi}{3}a^3\frac{\epsilon_0}{C^4}$ and

$$\frac{8\pi K}{C^4}P = \frac{2m}{a^3}\frac{\sqrt{1-\frac{2m}{a}\left(\frac{r}{a}\right)^2}-\sqrt{1-\frac{2m}{a}}}{\sqrt{1-\frac{2m}{a}}-\frac{1}{3}\sqrt{1-\frac{2m}{a}\left(\frac{r}{a}\right)^2}}. \tag{3.3}$$

The pressure is positive and finite for $0 \le r < a$ if

$$a > \tfrac{9}{4}m. \tag{3.4}$$

When $a = \frac{9}{4}m$ then $P(0) = \infty$. We shall confine the present discussion to the case $a > \frac{9}{4}m$ and discuss the singular case $a = \frac{9}{4}m$ separately.

We introduce for convenience the notation

$$\varepsilon \stackrel{\text{def}}{=} 3\sqrt{1-\frac{2m}{a}}, \tag{3.5}$$

so that

$$a > \tfrac{9}{4}m \leftrightarrow 3 > \varepsilon > 1. \tag{3.6}$$

(In the critical case $a = \frac{9}{4}m$, $\varepsilon = 1$.) Now introduce in Eq. (3.1) instead of r the new variable u:

$$\sqrt{\frac{2m}{a}}\,\frac{r}{a} = \frac{\sqrt{\epsilon^2-1}\sin u}{\epsilon + \cos u}. \tag{3.7}$$

The graph of $r = r(u)$ is given in Fig. 2.

As the u interval corresponding to $0 \leq r \leq a$ we choose $0 \leq u \leq u_a < \cos^{-1}(-\varepsilon^{-1})$. Notice that $a \geq 3m \to u_a \leq \frac{\pi}{2}$. Thus if, in addition to Eq. (3.4), we have $a > 3m$, $\cos u$ remains positive in the region of interest.

Defining

$$ds = \sqrt{\epsilon^2 - 1}\sqrt{\frac{a}{2m}}\, a\, d\sigma, \qquad \tau = \tfrac{1}{2}\sqrt{\epsilon^2 - 1}\sqrt{\frac{2m}{a}}\,\frac{x^0}{a} \tag{3.8}$$

the substitution of Eq. (3.7) into Eq. (3.1) yields the dimensionless expression

$$d\sigma^2 = (\epsilon + \cos u)^{-2}(d\tau^2 - d\omega_3^2), \tag{3.9}$$

where

$$d\omega_3^2 = du^2 + \sin^2 u\,(du^2 + \sin^2\vartheta\, d\varphi^2) \tag{3.10}$$

can be interpreted as the element of length in a closed 3-dimensional space of constant positive curvature.

Equation (3.9) can be understood as the maximum analytic extension of the space (3.1); indeed, with the obvious topology $-\infty \leq \tau < +\infty$, $0 \leq u \leq \pi$, $0 \leq \vartheta < \pi$, $0 < \varphi < 2\pi$, (3.9) gives (remembering that $\varepsilon > 1$) an expression for the metric analytic at all points. (In a sense, it can be regarded as the "Kruskalization" of the interior Schwarzschild solution: the lines $\tau = \pm u$, $\vartheta, \varphi =$ const. are null geodesics).

From Eq. (3.9) it is only a few steps to the explicit conformally flat representation of $d\sigma$. Indeed, let

$$\tau = \ln\frac{e^v - 1}{e^v + 1} \to d\tau = \frac{dv}{\operatorname{ch} v}. \tag{3.11}$$

Then

$$d\sigma^2 = \left[\operatorname{ch} v(\epsilon + \cos u)\right]^{-2}(dv^2 - \operatorname{ch}^2 v\, d_3\omega^2). \tag{3.12}$$

Therefore, when

$$\begin{aligned}
w^0 &= \sinh\ v,\\
w^1 &= \cosh v \sin u \sin\vartheta \sin\varphi,\\
w^2 &= \cosh v \sin u \sin\vartheta \sin\varphi,\\
w^3 &= \cosh v \sin u \cos\vartheta,\\
w^4 &= \cosh v \cos u,
\end{aligned} \tag{3.13}$$

then

$$d\sigma^2 = \left[w^4 + \epsilon\sqrt{1 + w^{02}}\right]^{-2} \left(dw^{0\,2} - dw^{1\,2} - \cdots - dw^{4\,2}\right), \tag{3.14}$$

with

$$-w^{0\,2} + w^{1\,2} + \cdots + w^{4\,2} = 1. \tag{3.15}$$

As the last step we parametrize the pseudosphere (3.15) (de Sitter space) by stereographic coordinates:

$$w^\mu = \frac{x'^\mu}{1 - \frac{1}{4}x'_\sigma x'^\sigma}, \qquad w^4 = \frac{1 + \frac{1}{4}x'_\mu x'^\mu}{1 - \frac{1}{4}x'_\sigma x'^\sigma} \tag{3.16}$$

($\mu = 0, 1, 2, 3$ and $x'_\mu x'^\mu = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$).

This used in Eq. (3.14) yields

$$d\sigma^2 = \left[1 + \tfrac{1}{4}x'_\sigma x'^\sigma + \epsilon\sqrt{x'^{0\,2} + \left(1 - \tfrac{1}{4}x'_\sigma x'^\sigma\right)^2}\right]^{-2} dx'_\mu\, dx'^\mu \tag{3.17}$$

(only in the region where $1 - \frac{1}{4}x'_\sigma x'^\sigma > 0$; where this is negative we must replace in Eq. (3.17) ϵ by $-\epsilon$).

This is the explicit conformally flat representation of (3.1). the region $0 \le r \le a$ corresponds to $1 \ge \cos u \ge u_a$; but

$$\cos u = \frac{w^4}{\sqrt{1 + w^{0\,2}}} = \operatorname{sign}\left(1 - \tfrac{1}{4}x'_\sigma x'^\sigma\right) \frac{1 + \frac{1}{u}x'_\sigma x'^\sigma}{\sqrt{x'^{0\,2} + \left(1 - \frac{1}{4}x'_\sigma x'^\sigma\right)^2}}. \tag{3.18}$$

When $a \ge 3m$, which we now in fact assume for simplicity, $\cos u_a \ge 0 \to 1 \ge \frac{1}{4}|x'_\sigma x'^\sigma|$, according to Eq. (3.18), and $\cos u \ge \cos u_a$; hence the choice of sign in Eq. (3.17) is correct.

Now, according to Eq. (3.18) the inequalities $1 \ge \cos u \ge \cos u_a$ are equivalent to

$$1 \overset{(1)}{\ge} \frac{1 + \frac{1}{4}x'_\sigma x'^\sigma}{\sqrt{x'^{0\,2} + \left(1 - \frac{1}{4}x'_\sigma x'^\sigma\right)^2}} \overset{(2)}{\ge} \cos u_a. \tag{3.19}$$

Noticing that $x'^{0\,2} + (1 - \frac{1}{4}x'_\sigma x'^\sigma)^2 = (1 + \frac{1}{4}x'_\sigma x'^\sigma)^2 + r'^2$, $r' = \sqrt{x'^s x'^s}$, we easily see that (1) holds automatically, becoming equality only when $r' = 0$. Thus $r = 0 \leftrightarrow r' = 0$. The inequality (2) is equivalent to $(1 + \frac{1}{4}x'_\sigma x'^\sigma)\sin u_a \ge \cos u_a r'$, which is in turn equivalent to

$$\frac{1}{\sin^2 u_a} + \left(\frac{x'^0}{2}\right)^2 \ge \left(\frac{r'}{2} + \operatorname{ctg} u_a\right)^2. \tag{3.20}$$

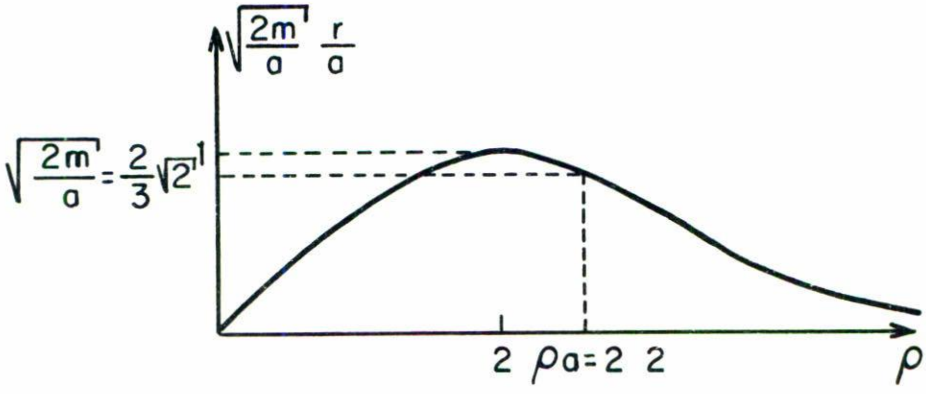

FIGURE 3.

Therefore, the events which in Eq. (3.1) are defined by $0 \leq r \leq a$ correspond to the events defined in the coordinates x'^{μ} by the condition

$$\frac{r'}{2} \leq \sqrt{\left(\frac{x'^0}{2}\right)^2 + \frac{1}{\sin^2 u_a}} - \operatorname{ctg} u_a \tag{3.21}$$

(for $x'^0 = 0$, $r' \leq 2\operatorname{ctg}(u_a/2)$).

Now we consider the critical case when $a = \frac{9}{4}m$, so that $\varepsilon = 1$ in Eq. (3.5). In Eq. (3.1) we introduce instead of r the dimensionless variables ρ:

$$\sqrt{\frac{2m}{a}}\,\frac{r}{a} = \frac{\rho}{1 + \frac{1}{4}\rho^2}. \tag{3.22}$$

The graph of $r = r(\rho)$ is given in Fig. 3. As the interval corresponding to $0 \leq r \leq a$ we choose $\infty \geq \rho \geq \rho_a = 2\sqrt{2}$.

Defining

$$ds = a\sqrt{\frac{a}{2m}}\,d\sigma, \qquad x_0 = a\sqrt{\frac{1}{2m}}\,x_0', \tag{3.23}$$

the substitution of Eq. (3.22) into Eq. (3.1) yields the dimensionless expression

$$d\sigma^2 = \left[1 + \tfrac{1}{4}\rho^2\right]^{-2}\left(dx'^{02} - d\rho^2 - \rho^2\,d\omega^2\right). \tag{3.24}$$

Thus, in coordinates $x'^1 = \rho \sin\vartheta \cos\varphi$, etc., Eq. (3.24) gives the explicit conformally flat representation:

$$d\sigma^2 = \left[1 + \tfrac{1}{4}x'^s x'^s\right]^{-2} dx'_{\mu}\,dx'^{\mu}. \tag{3.25}$$

The conformal factor is considerably simpler than that of the general case (3.17). The region $0 \leq r \leq a$ corresponds in x'^{μ} coordinates to the set of points

$$\infty \geq x'^{s} x'^{s} \geq 8. \tag{3.26}$$

Now we shall show that the spaces (3.1) is of class one, that is, it can be embedded in a 5–dimensional flat space without restrictions on the parameters. Indeed, (3.1) can be rewritten in the form

$$ds^2 = du^2 - dv^2 - dx^a\, dx^a, \tag{3.27}$$

where

$$\begin{aligned} u &= \left[\tfrac{3}{2}\sqrt{1-\frac{2m}{a}} - \tfrac{1}{2}\sqrt{1-\frac{2m}{a}\left(\frac{r}{a}\right)^2}\right]\sqrt{\frac{2a^3}{m}}\,\mathrm{sh}\sqrt{\frac{m}{2a^3}}\,x^0, \\ v &= \left[\tfrac{3}{2}\sqrt{1-\frac{2m}{a}} - \tfrac{1}{2}\sqrt{1-\frac{2m}{a}\left(\frac{r}{a}\right)^2}\right]\sqrt{\frac{2a^3}{m}}\,\mathrm{ch}\sqrt{\frac{m}{2a^3}}\,x^0, \end{aligned} \tag{3.28}$$

$$x^1 = r\sin\vartheta\cos\varphi, \quad x^2 = r\sin\vartheta\sin\varphi \quad , x^3 = \cos\vartheta.$$

Thus, the equation of the 4–dimensional surface in the 5–dimensional flat space of signature $(+----)$ is

$$v^2 - u^2 = \frac{2a^3}{m}\left[\tfrac{3}{2}\sqrt{1-\frac{2m}{a}} - \sqrt{1-\frac{2m}{a^3}x^s x^s}\right]^2. \tag{3.29}$$

The physical region consists of the points where $0 \leq x^s x^s \leq a^2$.

This 5–dimensional embedding of the interior Schwartzschild solution can easily be reconciled with known 6–dimensional embedding of the exterior solution [6]. This can be done in the form

$$ds^2 = du^2 - dv^2 - dw^2 - dx^a\, dx^a, \tag{3.30}$$

where x^1, x^2, x^3 are parametrically given by the same formulas as (3.28), whereas

$$\begin{aligned} u &= \sqrt{\frac{2a^3}{m}}\sqrt{1-\frac{2m}{r}}\,\mathrm{sh}\sqrt{\frac{m}{2a^3}}\,x^0, \\ v &= \sqrt{\frac{2a^3}{m}}\sqrt{1-\frac{2m}{r}}\,\mathrm{ch}\sqrt{\frac{m}{2a^3}}\,x^0, \quad (r > a) \\ w^{(r)} &= \int_a^r \sqrt{\frac{2m}{r}\frac{1-\left(\frac{a}{r}\right)^3}{1-\frac{2m}{r}}}\, dr. \end{aligned} \tag{3.31}$$