Investigación *Revista Mexicana de Física* **39**, *No. 6 (1993) 893–901*

Strong interaction radii of hadrons in the $1/d$–expansion

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Recibido el 4 de mayo de 1993; aceptado el 5 de agosto de 1993

Abstract. Within the framework of the nonrelativistic quark model and with the help of the leading term of the $1/d$–expansion we obtain analytical expressions for the quotients of baryon-to-meson strong interaction radii. Phenomenological relativistic corrections *à la* Povh and Hüfner are shown to lower the non relativistic estimate. Predictions for baryon radii are given and a comparison with experimental results (when available) is made. For some values of the quark masses, our estimate differs from the naive parton result in more than 20%.

Resumen. Se obtienen expresiones analíticas para el cociente entre radios de interacción fuerte de bariones y mesones en el marco del modelo no relativista de quarks y con ayuda del término principal de la solución de la ecuación de Schrödinger en potencias de $1/d$. Se muestra que las correcciones relativistas introducidas fenomenológicamente *à la* Povh y Hüfner disminuyen la estimación no relativista para el cociente de radios. Se presentan predicciones para los radios de algunos bariones y se hace una comparación con los resultados experimentales disponibles. Para algunos valores de las masas de los quarks, nuestra estimación difiere del modelo partónico primitivo en más del 20%.

PACS: 12.40.Qq, 14.20.–c, 14.20.–n

1. Introduction

A few years ago Povh and Hüfner[1–3] found that not only elastic scattering at zero momentum transfer, but also hadron-proton total cross sections at high energies can be interpreted along geometrical lines. By comparing the total cross sections for hadron-proton scattering with the corresponding slope parameters from elastic scattering ($b = \frac{d}{dt}(\ln \frac{d}{dt}\sigma_{\rm elastic})_{t=0}$, t is the Mandelstam variable) they obtained a linear relation between these magnitudes

$$b_{\rm hp} = \tfrac{1}{3}\left(R_{\rm p}^2 + R_{\rm h}^2\right), \qquad \sigma_{\rm total}^{\rm hp} = G R_{\rm p}^2 R_{\rm h}^2, \tag{1}$$

which shows that a "strong interaction radius" $R_{\rm h}$ can be assigned to each hadron according to the equations

$$R_{\rm p}^2 = \tfrac{3}{2} b_{\rm pp}, \qquad R_{\rm h}^2 = R_{\rm p}^2 \frac{\sigma_{\rm total}^{\rm hp}}{\sigma_{\rm total}^{\rm pp}}. \tag{2}$$

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In this way they found [3] the following values of squared hadronic radii (in fm^2):

$$R_\pi^2 = 0.41, \qquad R_{\rho,\omega}^2 = 0.52, \qquad R_K^2 = +0.35, \qquad R_\phi^2 = 0.21,$$
$$R_{J/\psi}^2 = 0.04 \qquad R_{\mathrm{p}}^2 = 0.67, \qquad R_{\Lambda,\Sigma}^2 = 0.58, \qquad R_\Xi^2 = 0.50, \qquad (3)$$

which exhibit the following properties:

i) the heavier the valence quark the smaller the radius, *i.e.* $R_\pi > R_K > R_\phi > R_{J/\psi}$ and $R_{\mathrm{p}} > R_{\Lambda,\Sigma} > R_\Xi$;

ii) the ratio $R_{\mathrm{baryon}}/R_{\mathrm{meson}}$ for systems of identical quarks is roughly equal to $\sqrt{3/2}$, as follows from a naive parton picture.

The importance of this method of determination of R_{h}^2 lies on the fact that the data on total cross sections are abundant, while a direct test of the internal structure of hadrons is difficult, because of their extremely short lifetimes. It is worthwhile mentioning that, when the comparison is possible, charge radii approximately coincide with the corresponding values of strong interaction radii given in Eq. (3).

In the present paper we aim at gaining in the qualitative understanding of the systematics of strong interaction radii presented above by stressing on the relations between baryon and meson radii. This is a natural extension of the ideas of paper [4] where we showed that the set of relations between baryon and meson mass spectra obtained within the framework of the nonrelativistic quark model and with the help of the $1/d$–method are satisfied by the existing experimental data on hadron masses.

Instead of doing numerical calculations, we perform a semiquantitative analysis based on analytic expressions for the quotients of radii. These expressions are obtained by means of the leading term of the $1/d$–expansion for the solution of the Schrödinger equation (for an introduction and some applications see Ref. [5] and references therein). The parameter d, which is taken as a free parameter and used for expanding the solution of the Schrödinger equation as a power series in $1/d$, arises when formally writing in d dimensions the Laplacians entering the Schrödinger equation. The first term of this series can be analytically computed. It reproduces qualitatively and semiquantitatively most of the properties of bound states in few-particle systems [5,6]. In particular, it can be used for the calculation of quotients between magnitudes corresponding to systems with different number of particles, as it is for example the ratio $R_{\mathrm{baryon}}/R_{\mathrm{meson}}$. A brief introduction to the $1/d$–method when applied to the calculation of radii is presented in the next section.

Phenomenological relativistic corrections *à la* Povh and Hufner [3] are considered in Sect. 3. They are shown to lower the values of the ratio $R_{\mathrm{baryon}}/R_{\mathrm{meson}}$ obtained from the non relativistic theory. In Sect. 4, some predictions for still unmeasured baryon radii are given and a comparison of our results with experimental data (when available) is made.

2. HADRONIC RADII IN THE $1/d$–PICTURE

The starting point is the nonrelativistic model of quarks [7], in which quarks with large effective masses interact through central forces which vary roughly as r at long distances and $-1/r$ at short distances. This forces are supposed to be flavor-independent. In addition, they satisfy the requirement

$$V_{qq}\Big|_{\text{in a baryon}} \simeq \tfrac{1}{2} V_{q\bar{q}}\Big|_{\text{in a meson}}, \tag{4}$$

which resembles a property of one-gluon-exchange forces, but is also satisfied by the long-range confining forces. In this approximation, the ground state of a hadron is a state with zero total angular momentum. We will be concerned with computations of root-mean-square radii in the ground state.

As mentioned above, for the computation of $R_{\text{baryon}}/R_{\text{meson}}$ we make use of the $1/d$–method [5,6]. In this method every physical magnitude characterizing the hadron is expressed as a power series in $1/d$, the first terms of which are easily computable. First, we consider hadrons made up from only one type of quarks, *i.e.* with quark structure $q\bar{q}$ and qqq, respectively. The root-mean-square separation between particles in the meson $q\bar{q}$ is written as

$$\left\langle r^2(g)\right\rangle_{q\bar{q}} = r_0^2 + \mathcal{O}(1/d), \tag{5}$$

where r_0 is the distance that minimizes the effective potential

$$U_{q\bar{q}}(g) = \frac{d^2}{8\mu_r r^2} + gV. \tag{6}$$

Let us make some comments concerning Eqs. (5) and (6). The $\mu_r = m_q/2$ is the reduced mass of the meson, g is the coupling constant and the system of units is such that $\hbar = 1$. The potential $U_{q\bar{q}}$ is the only term surviving in the meson Hamiltonian when we take the formal limit $d \to \infty$. It means that fluctuations are suppressed and $d \to \infty$ is a classical limit. In this limit, the physical system is endowed with a rigid structure characterized by the interparticle separation r_0. Higher terms in the series (5) take into account the contribution of fluctuations around the equilibrium distance r_0.

Let us consider a baryon with three identical quarks in a state with zero total angular momentum. The effective potential depends on the scalar combinations r^2, ρ^2 and $\gamma = \vec{r}\cdot\vec{\rho}/r\rho$ ($\vec{r} = \vec{r}_2 - \vec{r}_1$ and $\vec{\rho} = \vec{r}_3 - (\vec{r}_1 + \vec{r}_2)/2$, where $\vec{r}_i$ is the position of the i–th particle, are the two Jacobi vectors of the problem):

$$U_{qqq}(g) = \frac{d^2}{8(1-\gamma^2)}\left(\frac{1}{\mu_r r^2} + \frac{1}{\mu_\rho \rho^2}\right) + \frac{g}{2}\big(V_{12} + V_{23} + jV_{31}\big), \tag{7}$$

where $\mu_r = m_q/2$ and $\mu_\rho = 2m_q/3$. The minimization of U_{qqq} leads to a system of three equations for r_0^2, ρ_0^2, and γ_0, *i.e.* the leading terms of the series for $\langle r^2(g)\rangle_{qqq}$, $\langle \rho^2(g)\rangle_{qqq}$ and $\langle \vec{r}\cdot\vec{\rho}/r\rho\rangle_{qqq}$ [see Eq. (5)]. However, permutation symmetry helps to find the solution. We shall simply look for the solution in the class of configurations representing equilateral triangles (*i.e.* $\rho^2 = \frac{3}{4}r^2, \gamma = 0$) in which U_{qqq} takes the form

$$U_{qqq}(g) = 2U_{q\bar{q}}\big(\tfrac{3}{4}g\big), \tag{8}$$

i.e.,

$$\left\langle r^2(g)\right\rangle_{qqq} = \left\langle r^2\left(\tfrac{3}{4}g\right)\right\rangle_{q\bar{q}} + \mathcal{O}(1/d). \tag{9}$$

Let us note that equilateral triangles are expected to be the dominant configurations in spherically symmetric states. In states with high momentum, other configurations may appear to be the most favored [8]. Taking into account the evident geometrical relations between $\langle r^2\rangle$ and the r.m.s radii, R^2, we obtain

$$\frac{R^2_{qqq}(g)}{R^2_{q\bar{q}}\left(\frac{3}{4}g\right)} = \frac{4}{3} + \mathcal{O}(1/d). \tag{10}$$

From Eq. (10) and the evident relation $R_{q\bar{q}}\left(\frac{3}{4}g\right) > R_{q\bar{q}}(g)$, we obtain the bound $R_{qqq}(g)/R_{q\bar{q}}(g) > \sqrt{4/3}$, but in order to get more refined estimates, we shall pick a particular interquark potential. As we said above, in the non relativistic model of quarks the quark interaction through central forces varies as r at long distances and $-1/r$ at short distances. Thus a potential $V = r^\beta$ with $\beta \simeq 0$ should work as an average of the real interquark potential. The property $\beta \simeq 0$, coming from interpolation between $\beta \simeq 1$ at long distances and $\beta \simeq -1$ at short ones, means that the effective potential is nearly logarithmic. In fact, the Martin's fit [9], corresponding to $\beta = 0.1$, is shown to yield very good results for meson spectra and decay rates. With the help of a power like potential, we obtain

$$\frac{R^2_{qqq}(g)}{R^2_{q\bar{q}}(g)} = \left(\frac{4}{3}\right)^{\frac{\beta+4}{\beta+2}} + \mathcal{O}(1/d), \tag{11}$$

and taking into account that β is small we may write

$$\frac{R_{qqq}(g)}{R_{q\bar{q}}(g)} = \frac{4}{3} + \mathcal{O}(\beta, 1/d), \tag{12}$$

One shall note that the leading term in the right-hand side of (12) is independent of the quark mass. Corrections to this leading term come from higher order (in $1/d$) contributions as well as from deviations of the effective interquark potential from the $\beta \simeq 0$ behavior. We expect these corrections to be only a few percents ($< 10\%$) of the leading term, and thus will not be computed. A similar situation takes place when we compute the $1/d$–series for the quotient of baryon to meson energies [4]. A second source of corrections to Eq. (12), which are specially important in hadrons containing light quarks, come from the inclusion of relativistic effects. They will be discussed in the next section.

Next, we consider the less symmetric case in which the meson and baryon have respectively the quark structure $q\bar{Q}$ (or $Q\bar{q}$) and Qqq. In the meson, the effective potential surviving in the $d \to \infty$ limit is again (6), with $\mu_r = m_q m_Q/(m_q + m_Q)$. The r_0 is obtained from the minimization of $U_{Q\bar{q}}$. In the baryon, we get again Eq. (7), but with μ_r defined as above and $\mu_\rho = 2m_Q m_q/(2m_q + m_Q)$. Permutation symmetry leads only to one simplification: $\gamma = 0$. To obtain a relation between the meson and baryon interparticle distances we shall pick a particular potential. We choose $V = r^\beta$ with $\beta \simeq 0$ and then obtain that the "geometry" of the baryon ground state, *i.e.* the ratio $x = \langle\rho^2(g)\rangle_{Qqq}/\langle r^2(g)\rangle_{Qqq}$, becomes independent of the strength of the potential (the coupling constant) and is determined by the equation:

$$\frac{4\xi x^2}{2+\xi} = \tfrac{1}{2}\left(\tfrac{1}{2} + \left[\tfrac{1}{4} + x\right]^{1-\beta/2}\right), \tag{13}$$

which may be approximately rewritten as

$$x = \frac{\xi+2}{16\xi}\left(1 + \left[1 + \frac{24\xi}{\xi+2}\right]^{\frac{1}{2}}\right) + \mathcal{O}(\beta, 1/d), \tag{14}$$

where $\xi = m_Q/m_q$. On the other hand, by comparing the minimum equations for the meson and baryon it follows that

$$\left\langle\rho^2(g)\right\rangle_{Qqq} = \left(\frac{\xi+2}{2(\xi+1)}\left[1 + \frac{1}{4x}\right]^{1-\beta/2}\right)^{\frac{2}{\beta+2}} \left\langle r^2(g)\right\rangle_{Q\bar{q}} + \mathcal{O}(1/d), \tag{15}$$

which in the vicinity of $\beta = 0$ leads to

$$\left\langle\rho^2(g)\right\rangle_{Qqq} = \frac{\xi+2}{2(\xi+1)}\left(1 + \frac{1}{4x}\right)\left\langle r^2(g)\right\rangle_{Q\bar{q}} + \mathcal{O}(\beta, 1/d). \tag{16}$$

In order to compare with the radii found in Ref. [3] we shall construct a magnitude with the meaning of a geometrical radius. This radius should not contain contributions from very heavy particles and should reduce to the known answer in the equal mass case. These requirements are fulfilled if we define the radius through the density

$$n(\vec{r}) = \left\langle \sum_i \frac{1}{m_i}\delta(\vec{r}_i - \vec{r})\right\rangle \tag{17}$$

i.e.

$$R^2 = \frac{\int d^3r\, r^2 n(r)}{\int d^3r\, n(r)} = \frac{\sum_i \frac{1}{m_i}\left\langle r_i^2\right\rangle}{\sum_i \frac{1}{m_i}}.$$

The above definition gives for the meson and baryon radii the values:

$$R^2_{Q\bar{q}}(g) = \frac{1+\xi^3}{(1+\xi)^3}\left\langle r^2(g)\right\rangle_{Q\bar{q}}, \tag{18}$$

$$R^2_{Qqq}(g) = \frac{\xi}{1+2\xi}\left[\frac{1}{\xi}\left(\frac{2}{2+\xi}\right)^2 + 2\left(\frac{\xi}{2+\xi}\right)^2 + \frac{1}{2x}\right]\left\langle\rho^2(g)\right\rangle_{Qqq}, \tag{19}$$

respectively, and with the account of the approximate relation (16) we obtain for the squared radii the equation:

$$\frac{R^2_{Qqq}}{R^2_{Q\bar{q}}} = \frac{\xi(\xi+2)(\xi+1)^2}{2(1+2\xi)(1+\xi^3)}\left(1+\frac{1}{4x}\right)\left[\frac{1}{\xi}\left(\frac{2}{2+\xi}\right)^2 + 2\left(\frac{\xi}{2+\xi}\right)^2 + \frac{1}{2x}\right] + \mathcal{O}(\beta, 1/d). \quad (20)$$

As mentioned above, we expect β– and $1/d$–corrections to Eq. (20) to be only a few percent of the leading term. A more important contribution is made by the relativistic corrections [3], the effects of which is considered in the next section.

3. Phenomenological relativistic corrections

So far, we compared radii of baryons and mesons, calculated within the framework of non-relativistic quantum mechanics with effective heavy quarks thought as punctual objects. However, it is known that for light quarks, relativistic effects lead to a smearing of quark coordinates which adds a term to the radius of the hadron [10,11]. In Ref. [3] this effect was taken into account by assigning to u, d and s quarks effective sizes proportional to their inverse mass (a natural relativistic scale). The following expression was written for the squared radius of the i–th quark [3]:

$$\langle r_i^2 \rangle = \langle r_i^2 \rangle_{\text{wf}} + \frac{\eta}{m_i^2}, \quad (21)$$

where the coefficient η was seen to be $\eta \simeq 0.036$, and the subscript wf refers to radii calculated from the wave function of the system. From Eq. (21) and our definition of density it follows that

$$R^2 = R^2_{\text{wf}} + \frac{\sum_i \frac{\eta}{m_i^3}}{\sum_i \frac{1}{m_i}}. \quad (22)$$

Thus for the quotients of radii, we have

$$\frac{R^2_{\text{baryon}}}{R^2_{\text{meson}}} = \frac{R^2_{qqq} + \sum_{i=1}^{3} \frac{\eta}{m_i^3} \Big/ \sum_{i=1}^{3} \frac{1}{m_i}}{R^2_{Q\bar{q}} + \sum_{i=1}^{2} \frac{\eta}{m_i^3} \Big/ \sum_{i=1}^{2} \frac{1}{m_i}}, \quad (23)$$

where the wave function radii were written as R^2_{Qqq} and $R^2_{Q\bar{q}}$, like in the previous section.

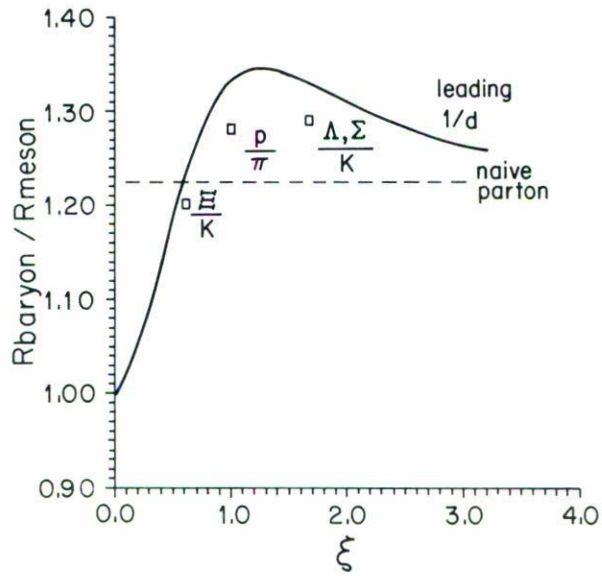

FIGURE 1. Comparison between the quotient of radii calculated from the non relativistic theory, Eq. (20) (solid line), and the experimental values (squares). The result of the naive-parton model (dashed line) is also given for reference.

The important point to notice in Eq. (23) is that the relativistic corrections lower the value of the ratio of radii given by Eq. (20) This result follows from the dependence of $R^2_{\rm wf}$ on the quark mass, $R^2_{\rm wf} \sim m^{-2/(2+\beta)}$. It may be easily verified for any particular values of the magnitude $\xi = m_Q/m_q$.

When $\xi \to 0$ Eq. (20) leads to $R^2_{Qqq}/R^2_{Q\bar{q}} \to 1$. Relativistic corrections are the same in both systems (η/m_Q^2), thus in our approximation $R^2_{\rm baryon}/R^2_{\rm meson} \to 1$ when $\xi \to 0$, independently of the quark mass m_Q.

When $\xi = 1$ we obtained above that $R^2_{qqq}/R^2_{q\bar{q}} \simeq \left(\frac{4}{3}\right)^2$, while relativistic corrections are in both cases equal to η/m_q^2. Thus, in general, $1 < R^2_{\rm baryon}/R^2_{\rm meson} < \left(\frac{4}{3}\right)^2$. The lowest value is reached when $m_q \to 0$ and relativistic corrections dominate.

Finally, when $\xi \to \infty$, $R^2_{Qqq}/R^2_{q\bar{q}} \simeq \frac{25}{18}$, while the relativistic contributions to the radii are η/m_q^2. Thus $1 < R^2_{\rm baryon}/R^2_{\rm meson} < \frac{25}{18}$.

Predictions for baryon radii based on these results and a comparison with experiment are presented in the next section.

4. Comparison with experimental results and discussion

We have summarized in Fig. 1 our results and the "experimental" points, *i.e.* the quotients of radii calculated from Povh and Hüfner estimates Eq. (3). Unfortunately, the experimental points are concentrated in the region $\xi \simeq 1$, so we can not test the $\xi \to 0$ and $\xi \to \infty$ regions.

The bold curve in Fig. 1 is drawn from Eq. (20), *i.e.* represents the non-relativistic estimate. Relativistic corrections lower this estimate in a way which is explicitly dependent on both m_q and m_Q (and not only on its combination, ξ). To locate the experimental points in the figure, we used the quark masses $m_{u,d} \simeq 0.3$ GeV, $m_s \simeq 0.5$ GeV. For comparison, the naive parton model result, in which the total cross section is simply proportional to the number of quarks in the system, that is $R_{\text{baryon}}/R_{\text{meson}} \simeq \left(\frac{3}{2}\right)^{1/2}$, is also represented (dashed line).

We predict values of $R_{\text{baryon}}/R_{\text{meson}}$ that differ from the naive parton result in more than 20% in the region $\xi \to 0$. For these values of ξ the fact $R_{Qqq}/R_{Q\bar{q}} \to 1$ has a simple interpretation. As the heavy quarks in the baryon become heavier the distance between them reduces to zero. Consequently, the light quark is attracted with twice the strength. When this effect is combined with the 1/2 of Eq. (4) we get the result $R_{Qqq}/R_{Q\bar{q}} \to 1$. Using this result, we can obtain the following bounds:

$$1 \le \frac{R_{\Xi_{bb}}}{R_B} \le \frac{R_{\Xi_{cc}}}{R_D}. \tag{24}$$

On the other hand, in the $\xi \gg 1$ region the non relativistic estimate is $R_{Qqq}/R_{Q\bar{q}} \simeq 5\sqrt{2}/6 \simeq 1.17$, which is still lowered by relativistic corrections. It leads to the bounds

$$\frac{R_{\Omega_c}}{R_{D_s}} \ge \frac{R_{\Lambda_c,\Sigma_c}}{R_D} \ge \frac{R_{\Lambda_b,\Sigma_b}}{R_B} \simeq 1.17. \tag{25}$$

Relations (24) and (25), giving the behavior of $R_{\text{baryon}}/R_{\text{meson}}$ in the $\xi \to 0$ and $\xi \to \infty$ limits, could be experimentally tested in the next future. In particular, all the resonances mentioned in Eq. (25) have been already (although not firmly) established [12], and the realization of the corresponding hadron-proton collision experiment is only a matter of time.

When $\xi \simeq 1$, $R_{qqq}/R_{q\bar{q}}$ is only a few percents over the naive parton result and should be lowered by relativistic corrections. The experimental points in Fig. 1 show this regularity. From Eq. (12) and the observed values of $R_{\rho,\omega}$, R_ϕ, $R_{J/\Psi}$ we can give estimates of the radii of the Δ, Ω^- and Ω_{ccc} baryons. They are respectively (in fm),

$$R_\Delta \le 0.96, \qquad R_{\Omega^-} \le 0.60, \qquad R_{\Omega_{ccc}} \le 0.27. \tag{26}$$

Finally, we would like to show an interesting application of Eq. (2) to the determination of the strong interaction radius of the photon. Data on total cross sections at $E_{\text{cm}} \ge 8$ GeV may be obtained from [12] or from the detailed compilation [13]. Taking the values $\sigma_{\gamma p} \simeq 0.12$ mb and $\sigma_{pp} \simeq 36$ mb we obtain $R_\gamma \simeq 0.04$ fm, a distance of the order of the limit of applicability of QED.

ACKNOWLEDGEMENT

Augusto González thanks to Dr. M. Moshinsky, the Director, Dr. O. Novaro, and the Physics Department for hospitality at the Instituto de Física at UNAM, where this work was completed. Financial support from CONACYT is gratefully acknowledged.

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