Magnetic field line hamiltonians for some perturbed MHD equilibria in cylindrical geometry

RICARDO LUIZ VIANA
Departamento de Física, Universidade Federal do Paraná
Caixa Postal 19081, Curitiba, 81531-970, Paraná, Brazil

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ABSTRACT. The magnetic field line equations for an axisymmetric plasma confining system can be cast in a hamiltonian form. We apply a general procedure for this analogy in the case of cylindrical symmetry and use the result in order to describe two typical MHD equilibria. The case of a general, nonintegrable perturbation is considered, resonances being treated with use of secular perturbation theory. We study the magnetic island formation near these resonances.

RESUMEN. Las ecuaciones de las líneas de campo magnético para un sistema axisimétrico de confinamiento de plasmas se pueden poner en una forma hamiltoniana. Aplicamos un procedimiento general para esta analogía en el caso de geometría cilíndrica y utilizamos el resultado para describir dos equilibrios MHD típicos. El caso de una perturbación general no-integrable es considerado, y tratamos las resonancias con el uso de la teoría secular de perturbaciones. Estudiamos la formación de islas magnéticas vecinas a las resonancias.

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1. INTRODUCTION

One of the necessary conditions for plasma confinement in the commonly operating schemes like Tokamaks and Stellarators in the existence of nested magnetic surfaces [1]; on which the magnetic field lines twist around the magnetic axis, in an equilibrium configuration. Although the topology of these surfaces must be toroidal, if the torus aspect ratio is large enough, one can use a (periodic) cylindrical approximation, and treat toroidicity effects as a first order correction.

The relationship between magnetic field line equation and Hamiltonian equations of motion is well known for a long time, since Kerst [2] as early as in 1962, applied this similarity to treat a simple situation, in which a quadrupole error field was superposed to a uniform plasma containing magnetic field. Over the subsequent years the subject has received few, although important, contributions [3]. Whiteman [4] has suggested a fairly general formulation for the problem, by using a contravariant curvilinear coordinate system; but no applications are found in his paper. This formulation was revisited, some years later, by Bernardin and Tataronis [5], which have used it to deal with Mercier coordinates, in order to study expantions near the magnetic axis of an equilibrium configuration.

In this note, Whiteman's method is applied for cylindrical coordinates, and the results resemble those obtained by Lichtenberg [6], who have derived a field line hamiltonian through a more direct approach. This hamiltonian is developed for two examples found in the plasma literature [7], introduced by means of its plasma current density profiles. The expressions so obtained turn to be of practical interest when combined with error fields or internal disturbing fields caused by plasma instabilities, since these effects can be successfully described by hamiltonian perturbation theory [11].

The paper is organized as follows: in the second section Whiteman's formulation is outlined, according to the presentation by Bernardin and Tataronis. The cylindrical symmetry is considered, results being compared with Lichtenberg's expression. The third section is devoted to the study of some current profiles. In the fourth section we consider the effects of a nonsymmetrical perturbation upon the equilibrium system, in order to investigate the motion in the neighborhood of the resonances. Section 5 develops the framework for treatment of various possible kinds of perturbing fields using the equilibria of Sect. 3. Our conclusions are left to the last section.

2. WHITEMAN'S FORMULATION FOR FIELD LINE HAMILTONIAN

In axisymmetric equilibria, as in toroidal and cylindrical geometries, there is an ignorable coordinate, which can play the role of a "time" variable. In fact, the resulting field line flow is not actually dynamical in the usual sense, but rather is a kind of "streaming" along the magnetic field lines, which is akin to the lagrangean description in fluid mechanics [8].

Adopting this description, it is a straightforward matter to put the magnetic field line equations in a form that resembles Hamilton equations of motion, after defining a convenient hamiltonian for field line flow. This analogy enables us to handle the powerful techniques developed over the past decades to deal with classical hamiltonian systems, like canonical perturbation theory, adiabatic invariance, the KAM theorem and so on.

In the following, we shall take the problem as proposed by Whiteman [4]. Let the magnetic field $\mathbf{B} = (B^1, B^2, B^3)$, written in terms of its contravariant components, such that $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential. Indices are raised or lowered in the curvilinear system of coordinates by means of the metric tensor g_{ij}. We may take as canonical variables to

$$q = x^1, \tag{2.1}$$

$$p = p(x^1, x^2, x^3), \tag{2.2}$$

$$t = x^3, \tag{2.3}$$

where p is the canonically conjugated momentum to the coordinate q; and x^3 is taken to be the ignorable coordinate.

Choosing a gauge such that $A_2 = 0$, the momentum is given by

$$p = \int dx^2 \sqrt{g}\, B^3(x^1, x^2) + \gamma(x^1, x^3), \tag{2.4}$$

where $g = \det g_{ij}$ and γ is an integration constant. If $g \neq 0$ the magnetic field line equations read

$$\frac{dx^1}{B^1} = \frac{dx^2}{B^2} = \frac{dx^3}{B^3}, \tag{2.5}$$

which can be cast in a hamiltonian form

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \tag{2.6}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{2.7}$$

if we define

$$H = \int dx^2 \sqrt{g}\, B^1(x^1, x^2) + \delta(x^1, x^3). \tag{2.8}$$

The integration constants appearing in (2.4) and (2.8) are related by the expression

$$\sqrt{g}\, B^2 + \frac{\partial H}{\partial x^1} + \frac{\partial p}{\partial x^3} = 0. \tag{2.9}$$

The axisymmetric equilibrium configurations are characterized by a x^3-independent hamiltonian, so that the problem has only one degree of freedom and is autonomous. According to the Liouville criterion these systems are always integrable, and one can expect regular behaviour of the field lines [9]. If a non-axisymmetric perturbation is superimposed to this equilibrium, the system turns to be near-integrable, if the perturbation strength is small enough. One of the novel features to be expected in this case is chaotic behaviour in space, since we are dealing with a lagrangean description with no real temporal variables.

It is a simple matter to write down the hamiltonian for cylindrical coordinates ($x^1 = \theta, x^2 = r, x^3 = z$), with metric tensor $g_{ij} = \mathrm{diag}(r^2, 1, 1)$ and $g = r^2$. In practice, one do not use contravariant field components directly, but rather the so-called "physical" components, given by (no sum)

$$B_{\langle i \rangle} = \sqrt{g_{ii}}\, B^i. \tag{2.10}$$

The canonical variables read

$$q = \theta, \tag{2.11}$$

$$p = \int dr\, r B_z(r, \theta) + \gamma(\theta, z), \tag{2.12}$$

$$t = z, \tag{2.13}$$

$$H = \int dr\, B_\theta(r, \theta) + \delta(\theta, z), \tag{2.14}$$

and (2.9) furnishes

$$\frac{\partial \delta(\theta, z)}{\partial \theta} + \frac{\partial \gamma(\theta, z)}{\partial z} = 0, \tag{2.15}$$

trivially satisfied if δ and γ vanish identically.

In the large aspect ratio approximation for Tokamak equilibria, it is customary to take

$$B_r = 0, \tag{2.16}$$

$$B_\theta = B_\theta(r), \tag{2.17}$$

$$B_z = B_0 = \text{const.} \tag{2.18}$$

in such a way that Eqs. (2.12) and (2.14) read

$$p = \frac{B_0 r^2}{2}, \tag{2.19}$$

$$H(p) = \int_0^p \frac{dp'}{\sqrt{2p'B_0}} B_\theta(p'). \tag{2.20}$$

The hamiltonian above was written in terms of the momentum only, so that we may state that p and q are actually action-angle variables for the system.

Some authors prefer to use the so-called "rotational transform" [1], defined as the θ-angle covered by a given field line in a complete turn over the cylinder

$$\iota(r) = \frac{R_0}{2\pi} \frac{d\theta}{dz}. \tag{2.21}$$

From Eqs. (2.17), (2.18) and (2.5), the hamiltonian (2.20) is simply

$$H(p) = \frac{1}{2\pi R_0} \int_0^p dp' \, \iota(p'), \tag{2.22}$$

where $2\pi R_0$ is the cylinder length. This result is essentially the same as that obtained by Lichtenberg [5], if one replaces the momentum by $\zeta = r^2/2$ and $\phi = z/R_0$ as the "time" variable.

3. HAMILTONIAN FUNCTIONS FOR SOME MHD EQUILIBRIA

Let us apply the formalism sketched in the previous section in two case examples found in the plasma literature. The former is the one-parameter parabolic model [7]. The plasma current profile is axisymmetric, and has only one nonvanishing component:

$$j_z(r) = j_0 \left(1 - \frac{r^2}{a^2}\right), \tag{3.1}$$

where j_0 is a positive constant and a is the plasma column radius. The application of Ampere's Law will give for the θ-component of the equilibrium field (in the following equations S.I. units are used):

$$B_\theta(r) = \frac{aB_\theta(a)}{r}\left[1-\left(1-\frac{r^2}{a^2}\right)^2\right], \tag{3.2}$$

where $B_\theta(a) = \mu_0 I_\mathrm{p}/2\pi a$ (I_p is the total plasma current).

The rotational transform for the magnetic surfaces related to this particular equilibrium is

$$\iota(p) = \frac{\pi R_0 a B_\theta(a)}{p}\left[1-\left(1-\frac{2p}{B_0 a^2}\right)^2\right]. \tag{3.3}$$

Putting Eq. (3.3) into (2.22) a simple integration gives

$$H(p) = \frac{2B_\theta(a)}{B_0 a}p\left(1-\frac{p}{2B_0 a^2}\right). \tag{3.4}$$

The second example to be considered is the "peaked model" [10], characterized by the current density profile

$$j_z(r) = j_0\left(1+\frac{\lambda r^2}{a^2}\right)^{-2}, \tag{3.5}$$

where j_0 and a are the same variables as in the preceding example, λ being an adjustable adimensional parameter, in order to fit realistic current profiles. The poloidal field generated by such a current distribution is

$$B_\theta(r) = \frac{aB_\theta(a)}{r}\left(\frac{1+\lambda}{\lambda}\right)\left[1-\left(1+\frac{\lambda r^2}{a^2}\right)^{-1}\right], \tag{3.6}$$

where $B_\theta(a) = \mu_0 j_0 a/2(1+\lambda)$. The total current is obtained by a straightforward calculation as

$$I_\mathrm{p} = \frac{\pi j_0 a^2}{1+\lambda}.$$

Instead of the rotational transform, one might prefer to work with the so-called safety factor [1], defined as $q(r) = 2\pi/\iota(r)$. It reads for this case

$$q(r) = q(a)\left(\frac{r^2}{a^2}\right)\left(\frac{\lambda}{1+\lambda}\right)\left[1-\left(1+\frac{\lambda r^2}{a^2}\right)^{-1}\right]^{-1}, \tag{3.7}$$

after defining its value at the plasma edge: $q(a) = aB_0/R_0B_\theta(a) = (1+\lambda)q(0)$. The hamiltonian (2.22) reads

$$H(p) = \frac{aB_0(a)}{2}\left(\frac{1+\lambda}{\lambda}\right)\ln\left(1+\frac{2\lambda p}{a^2B_0}\right). \tag{3.8}$$

4. Perturbation Theory

These analytically obtained hamiltonians are very useful when taking into account perturbation effects due to internal or external sources. The standard form of the perturbed hamiltonian for field line flow is

$$H(p,q,t) = \frac{1}{2\pi R_0}\int_0^p dp'\,\iota(p') + \epsilon H_1(p,q,t), \tag{4.1}$$

where p, q, t are the (action-angle-time) variables defined in Eqs. (2.1)–(2.3), and ϵ is an order parameter, such that we may set $\epsilon = 1$ at the end of the calculations.

As the perturbation term H_1 must be periodic in the angle variable $q = \theta$ as well as in the "time" $t = z$, we may Fourier expand it in order to write

$$H_1(p,q,t) = \sum_{j,k} A_{jk}(p)\exp\left[i\left(jq - k\frac{t}{R_0}\right)\right], \tag{4.2}$$

where $(i,j) \in Z$ are the mode numbers. It is customary to write

$$A_{jk}(p) = a_{jk}(p)\exp(i\chi_{jk}),$$

where a_{jk} are real Fourier coefficients and χ_{jk} are taken to be constants.

Let us suppose that the main harmonic excited by perturbation is labeled by mode numbers (m,n), coprime integers. According to Poincaré Birkhoff theorem [11], periodic ("magnetic") islands will be created around the rational magnetic surface with safety factor $q = m/n$. The location ($p_{m/n}$, in the phase space) of these surfaces is obtained through solution of

$$q(p_{m/n}) = \frac{m}{n}. \tag{4.3}$$

In order to put in evidence this fact, we rewrite (4.1) and (4.2) as

$$\begin{aligned} H(p,q,t) = {} & \frac{1}{2\pi R_0}\int_0^p \iota(p')\,dp' + \epsilon a_{mn}(p)\cos\left(mq - n\frac{t}{R_0} + \chi_{mn}\right) \\ & + \frac{\epsilon}{2}\sum_{(j,k)\neq(\pm m,\pm n)} a_{jk}(p)\exp\left[i\left(jq - k\frac{t}{R_0} + \chi_{mn}\right)\right], \end{aligned} \tag{4.4}$$

where we have used the symmetry property of Fourier coefficients, $A^*_{-m,-n}(p) = A_{mn}(p)$ (*i.e.*, $a_{-m,-n} = a_{mn}$ and $\chi_{-m,-n} = -\chi_{mn}$) and set $\epsilon \to \frac{\epsilon}{2}$ for later convenience.

Now, we are interested in studying the behaviour of this hamiltonian system at the neighborhood of the exact resonance $p_{m/n}$. Taking

$$p = p_{m/n} + \Delta p \tag{4.5}$$

we make a Taylor expansion of (4.4) up to second order in Δp for the unperturbed part, and evaluate perturbation only at resonance. The result is

$$\begin{aligned} H(p_{m/n} + \Delta p) = H_0(p_{m/n}) &+ \frac{n}{R_0 m}\Delta p + \frac{1}{4\pi R_0}\left.\frac{d\iota}{dp}\right|_{p_{m/n}} \Delta p^2 \\ &+ \epsilon a_{mn}(p_{m/n}) \cos\left(mq - n\frac{t}{R_0} + \chi_{mn}\right) \\ &+ \frac{\epsilon}{2}\sum_{j,k} a_{jk}(p_{m/n}) \exp\left[i\left(jq - k\frac{t}{R_0} + \chi_{jk}\right)\right]. \end{aligned} \tag{4.6}$$

As is well-known from perturbation theory [11], the presence of resonances leads to small denominations in the corresponding terms of canonical perturbation series. In order to circumvent this problem, secular perturbation theory prescribes the use of a "rotating frame" for description of the system near a given resonance. Mathematically, this is realized by a canonical transformation of variables $(p, q) \to (J, \vartheta)$, performed through a time-dependent generating function of the second kind:

$$F(J, q, t) = \left(mq - n\frac{t}{R_0} + \chi_{mn}\right) J. \tag{4.7}$$

The canonical transformation equations are [12]

$$p = \frac{\partial F}{\partial q} = mJ, \tag{4.8}$$

$$\vartheta = \frac{\partial F}{\partial J} = mq - n\frac{t}{R_0} + \chi_{mn}, \tag{4.9}$$

$$K(J, \vartheta, t) = H(p, q, t) + \frac{\partial F}{\partial t} = H - \frac{n}{R_0}J. \tag{4.10}$$

This procedure puts in evidence two variables: a "slow" angle ϑ (because $\frac{d\vartheta}{dt} = m\frac{dq}{dt} - \frac{n}{R_0} = 0$ at the exact resonance, so that the phase of cosine in (4.6) is stationary) and a "fast" variable t. Averaging over this fast variable [6], the hamiltonian in the rotating frame variables reads

$$K(J_{m/n} + \Delta J, \vartheta, t) = K_0(J_{m/n}) + \Delta K(\Delta J, \vartheta), \tag{4.11}$$

where

$$K_0(J_{m/n}) = H_0(J_{m/n}) - \frac{n}{R_0} J_{m/n} \tag{4.12}$$

and

$$\Delta K(\Delta J, \vartheta) = \frac{m^2}{2} (\Delta J)^2 \frac{1}{2\pi R_0} \left. \frac{d\iota}{dp} \right|_{p_{m/n}} + \epsilon a_{mn}(J_{m/n}) \cos \vartheta. \tag{4.13}$$

One recognizes (4.13) as the nonlinear pendulum hamiltonian, which describes the motion near an exact resonance $J_{m/n}$ in the $\Delta J \times \vartheta$ phase plane (actually is a Poincaré surface of section). The general features of this situation are fairly well-known [11]: there are two kinds of curves —librations (closed) and rotations (open)— and a separatrix between them, which characterizes a magnetic island. The singular points in this case are elliptical (locally stable) at $\vartheta = \pi$ and hyperbolical (locally unstable) at $\vartheta = 0, 2\pi$.

From (4.13) it is possible to evaluate islands half-widths $(\Delta J)_{m/n}$ by imposing the following condition:

$$\Delta K\big(\Delta J = (\Delta J)_{m/n}, \vartheta = \pi\big) = \Delta K\big(\Delta J = 0, \vartheta = 0\big); \tag{4.14}$$

giving the result

$$(\Delta J)_{m/n} = \frac{2}{m} \sqrt{2\pi R_0} \left| \frac{\epsilon a_{mn}(J_{m/n})}{\frac{d\iota}{dp}\Big|_{p_{m/n}}} \right|^{1/2}. \tag{4.15}$$

5. Analysis of some examples

Physically speaking, Eq. (4.15) says that the magnetic island amplitudes depend on: (i) the strength of perturbation, measured by its resonant Fourier coefficient; (ii) the "magnetic shear" of field lines $d\iota/dp$. Both quantities must be evaluated at resonance, *i.e.*, the position of the corresponding rational magnetic surface, as given by (4.3). It is straightforward to compute this value from the examples studied in Sect. 3.

For the one-parameter parabolic model defined in (3.1), the location of rational surfaces with $q = m/n$ is given by

$$p_{m/n} = \frac{n}{m} \left(\frac{B_0^2 a^3}{2} \right) \left(\frac{2m}{n B_0 a} - \frac{1}{R_0 B_\theta(a)} \right), \tag{5.1}$$

where we have used (3.3). Notice that this value is related to the actual radius of a cylindrical magnetic surface by Eq. (2.19). The magnetic shear of field lines is

$$\frac{d\iota}{dp} = \frac{\pi R_0 a B_\theta(a)}{p} \left[\left(1 - \frac{2p}{B_0 a^2} \right) \left(\frac{1}{p} + \frac{2}{B_0 a^2} \right) - \frac{1}{p} \right]. \tag{5.2}$$

Now for the peaked model (*cf.* 3.5), application of (4.3) furnishes the rational surface position in phase space as

$$p_{m/n} = \frac{a^2 B_0}{2q(a)} \left(\frac{1+\lambda}{\lambda} \right) \left(\frac{m}{n} - \frac{q(a)}{1+\lambda} \right). \tag{5.3}$$

From the rotational transform given by Eq. (3.7) is a simple matter to write down an analytical expression for magnetic shear

$$\frac{d\iota}{dp} = -\frac{\pi a^2 B_0}{q(a)} \left(\frac{1+\lambda}{\lambda} \right) \left(\frac{2\lambda}{a^2 B_0 + 2\lambda p} \right)^2. \tag{5.4}$$

In order to extract information about perturbing fields it is necessary to know its Fourier components. This can be accomplished in a quite direct way from the magnetic field line equations. Suppose that the perturbing fields are given by

$$\mathbf{B}_1 = (B_{1\theta}(p,q,t), B_{1r}(p,q,t), B_0),$$

i.e., the radial as well as angular components are superposed with the uniform (toroidal) field. This is the case, for example, of the ergodic magnetic limiter [13]. Assuming that these components are periodic in both variables, we Fourier-expand them:

$$B_{1r}(p,q,t) = \sum_{j,k} B_{jk}(p) \exp\left[i \left(jq - k\frac{t}{R_0} \right) \right], \tag{5.5}$$

$$B_{1\theta}(p,q,t) = \sum_{j,k} C_{jk}(p) \exp\left[i \left(jq - k\frac{t}{R_0} \right) \right]. \tag{5.6}$$

Substituting (5.5), (5.6) and (4.1), (4.2) into Eqs. (2.6), (2.7) and integrating (remember that $\frac{dp}{dt} = B_0 r \frac{dr}{dt}$) one obtains the following relations among the (complex) Fourier coefficients:

$$A_{mn}(p) = i\sqrt{\frac{2p}{B_0}}\,\frac{B_{mn}(p)}{m} = \int_0^p \frac{dp'}{\sqrt{2p'B_0}} C_{mn}(p'), \tag{5.7}$$

where we have used the magnetic field line equations in cylindrical coordinates:

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{dz}{B_z}. \tag{5.8}$$

In many situations of practical interest, real Fourier coefficients for fields have a power-law dependence

$$K \left(\frac{p}{p_0} \right)^{m/2}, \tag{5.9}$$

where p_0 is a positive constant (*e.g.*, equal to $B_0 r_0^2/2$, with r_0 some geometrical parameter) and K depends on some strength quantity, as an external or internal current density, and sometimes also on a mode number. Inserting (5.9) in some of the Eqs. (5.7) gives a power-law dependence in A_{mn} of the form const. $\times\, p^{(m+1)/2}$, up to a constant phase.

6. CONCLUSIONS

The magnetic field equations can be cast in a hamiltonian form, if the magnetic configuration has got any spatial symmetry. There is a systematic way to do this for an arbitrary curvilinear coordinate system. We carry out this procedure for analysing magnetic configurations with cylindrical symmetry, by using two given equilibrium profiles found in Tokamak research literature. In both cases, hamiltonian functions are found in terms of canonically conjugated variables.

These expressions are supposed to describe integrable systems, but is very important to consider small perturbations, which can appear as a result of MHD instabilities as well as externally applied magnetic fields. A typical feature of this situation is the presence of resonances, that are treated with techniques borrowed from secular perturbation theory. In the neighborhood of the exact resonances the Poincaré surface of section shows up a structure very similar to a pendulum, characterizing a magnetic island. We have obtained an expression to calculate magnetic island widths, from parameters of equilibrium and perturbation fields.

An important issue to be noted here is the onset of chaotic behaviour of the field lines characteristic of the border (near separatrix) region of a magnetic island. If two islands with different mode numbers interact, the magnetic (KAM) surfaces between them are progressively destroyed, and in the limit situation the chaotic region may occupy a significant portion of the available phase space. There are several practical applications of this idea, one of the most relevant being the ergodic magnetic limiter (EML) concept [13]. The application of the formalism presented in this paper to EML effects on field lines will be the subject of a forthcoming paper.

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