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Radial equation for the particle-antiparticle system with a Dirac oscillator interaction, and a qualitative application to mesons

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ABSTRACT. A relativistic equation for a particle-antiparticle system with a Dirac oscillator interaction was derived in previous publications and solved by a perturbative procedure which led to a spectrum of bound states. In this paper we derived a radial equation for the system in question and show that, contrary to the perturbative analysis, it gives rise to a continuous spectrum but in which there are resonant states, *i.e.*, those purely outgoing for certain complex energies, and we derived the latter by means of the $1/N$ expansion method. The spectrum of excitation energies, the strong interaction radii, and the decay widths of the resonant states of our model are calculated and compared with the corresponding experimental magnitudes for mesons with a pure quark composition and total angular momentum $J = L$. The emerging picture of Dirac oscillator mesons seems to be in qualitative agreement with meson phenomenology.

RESUMEN. Una ecuación relativista para un sistema de partícula-antipartícula con una interacción de oscilador de Dirac fue derivada en una publicación previa y resuelta por un método perturbativo que lleva a un espectro de estados ligados. En este artículo derivamos la ecuación radial del sistema en cuestión y mostramos que, en contradicción con el análisis perturbativo, da lugar a un espectro contínuo en el cual hay estados resonantes, esto es, aquellos puramente salientes para algunas energías complejas, y derivamos estos últimos por el método de expansión $1/N$. El espectro de energías de excitación, los radios de interacción fuerte y las anchuras de decaimiento fueron calculados y comparados con las magnitudes experimentales correspondientes para mesones de una composición pura de quarks y de momento angular $J = L$. La impresión que emerge es que los mesones basados en el oscilador de Dirac tienen un acuerdo cualitativo con la fenomenología mesónica.

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1. INTRODUCTION AND SUMMARY

At the present time we expect hadrons to be the asymptotic scattering state of quantum chromodynamics, the field theory of strong interaction. However, we are not able to compute the S-matrix of QCD. As a consequence, the existing approaches to obtain hadron properties are either very qualitative (for example, the $1/N_c$-expansion in QCD [1,2]), or

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approximate numerical (lattice calculation [3], for instance), or phenomenological QCD-based approaches. The latter constitute in fact the basis for the current classification of hadron resonances and computation of static properties of hadrons (see, for example, Ref. [4]).

In the present paper, we study a relativistic model in which the properties of mesons may be explicitly computed. The aim is not to describe real mesons, but to gain in the qualitative understanding of mesons properties. In fact, some aspects of the model are very unrealistic: the interaction is local and not field-mediated, the spin-orbit coupling is excessively strong, and there is no mixing between mesons with different quark composition.

Our starting point is a Poincaré invariant two-particle equation obtained previously in Refs. [5,6] for the particle-antiparticle system with a Dirac-oscillator interaction. The conserved quantum numbers in the theory are J —the total angular momentum—, and the parity. Besides, there is only one free parameter in the model, the oscillator frequency, ω (in units of the quark mass). For states with parity $(-1)^L = (-1)^J$ the above mentioned equation is reduced to one radial Schrödinger-like equation with an eigenvalue-dependent potential.

The solutions of this radial equation exhibit many of the properties of real mesons. Indeed, all states are shown to be resonances, with decay widths that decrease with the increasing of the quark mass, strong interaction radii for light mesons are proportional to the inverse quark mass and show a very soft dependence on the interaction potential, etc.

Few-particle systems with a Dirac-oscillator interaction were considered previously in Refs. [7-9,5,6]. The first example in which the Dirac oscillator was used was the case of three quarks, *i.e.* the mass spectra of baryons [7,8]. In Ref. [9] the two-body problem was introduced as an example for anomalous representation of the Poincaré group. The particle-antiparticle system was studied in [5,6] by means of a perturbative (in ω) procedure. As will be seen below, the small-ω region corresponds to heavy mesons for which relativistic effects are not so important. In the present paper we extend the analysis of Refs. [5,6] to the whole range of variation of ω by means of a nonperturbative method, the $1/N$-expansion, which was previously applied to other few-body problems [10].

We start in Sect. 2 by generalizing the Poincaré invariant equation for a system of n-bodies with Dirac oscillator interaction, and restrict it then to a particle-antiparticle system. In Sect. 3 we consider the equation in the frame of reference where the center of mass is at rest and reduce it from four to a single component. In Sect. 4 we obtain the corresponding radial equation in states with parity $(-1)^L = (-1)^J$, which is further solved approximately by means of the $1/N$-method in Sect. 5. Comparison with real mesons is presented in Sect. 6. Concluding remarks are given at the end of the paper.

2. Generalization of the Poincaré invariant n-body equation with Dirac oscillator interactions

In Ref. [9], to whose equations we shall refer with the addition of a roman I, we followed a reasoning developed by Barut *et al.* [11] to derive a single Poincaré invariant equation for n–particles with a Dirac oscillator interaction of the form

$$\left\{n^{-1}\sum_{s=1}^{n}\Gamma_s(\gamma_s^\mu P_\mu)+\sum_{s=1}^{n}\Gamma_s\left[\gamma_s^\mu(p'_{\mu s}-i\omega x'_{\mu s}\Gamma)+1\right]\right\}\psi=0, \tag{2.1}$$

which differs from Eq. (3.37I) by the fact that we take now relativistic units in which $\hbar$, the mass of the particle m and velocity of light c are 1, while the frequency of the Dirac oscillator ω appears explicitly.

As indicated in (3.28I) the Γ_s, Γ are defined by

$$\Gamma_s=(\gamma_s^\mu u_\mu)^{-1}\Gamma, \qquad \Gamma=\prod_{r=1}^{n}(\gamma_r^\mu u_\mu), \tag{2.2a, b}$$

where γ_s^μ ($\mu=0,1,2,3$; $s=1,2,\ldots,n$), are Dirac γ matrices associated with particle s, and u_μ is an appropriate unit time-like four vector. Repeated μ means summation over its values.

Furthermore

$$x'_{\mu s}=x_{\mu s}-X_\mu, \qquad p'_{\mu s}=p_{\mu s}-n^{-1}P_\mu, \tag{2.3a, b}$$

where

$$X_\mu=n^{-1}\sum_{s=1}^{n}x_{\mu s}, \qquad P_\mu=\sum_{s=1}^{n}p_{\mu s}, \tag{2.4a,b}$$

with P_μ, $\mu=0,1,2,3$, being the total energy-momentum four vector, and X_μ the center of mass position four vector (recall that all the masses are assumed to be equal to one).

We shall now generalize Eq. (2.1) in two ways. First we can replace the ω common to all the particles by ω_s, *i.e.*, a different frequency for each particle, which does not spoil the Poincaré invariance. Furthermore, to have a single time in our final equation in the center of mass frame we replace $x'_{\mu s}$ by its transverse part $\tilde{x}'_{\mu s}$, *i.e.*,

$$\tilde{x}'_{\mu s}=x'_{\mu s}-(P^\nu x'_{\nu s})P_\mu(P_\tau P^\tau)^{-1}, \tag{2.5}$$

so that finally we get the equation

$$\left\{n^{-1}\sum_{s=1}^{n}\Gamma_s(\gamma_s^\mu P_\mu)+\sum_{s=1}^{n}\Gamma_s\left[\gamma_s^\mu(p'_{\mu s}-i\omega_s\tilde{x}'_{\mu s}\Gamma)+1\right]\right\}\psi=0. \tag{2.6}$$

We now proceed as in (3.34I) giving to u_μ the dynamical meaning

$$u_\mu=(P_\mu/P), \qquad P=(-P_\mu P^\mu)^{1/2}, \tag{2.7}$$

which implies that the unit time like four vector u_μ takes the form $(1,0,0,0)$ in the frame of reference where the center of mass is at rest. By an analysis similar to the one given in Ref. [9], Eq. (2.6) in this frame takes the form

$$\left\{-P^0+\sum_{s=1}^{n}\left[\boldsymbol{\alpha}_s\cdot(\mathbf{p}'_s-i\omega_s\mathbf{x}'_sB)+\beta_s\right]\right\}\psi=0, \tag{2.8}$$

where we expressed the time-like component of the total four momenta in its contravariant form using the metric

$$g_{\mu\nu}=0,\quad \mu\neq\nu,\quad g_{11}=g_{22}=g_{33}=-g_{00}=1, \tag{2.9}$$

and the B is given by

$$B=\prod_{r=1}^{n}\beta_r \tag{2.10}$$

The eigenvalue of P^0 is the total energy of the n–particle system with Dirac oscillator interactions and we designate it by μ, as it can be interpreted as the mass associated with the states.

3. The particle-antiparticle system with a Dirac oscillator interaction

We now restrict ourselves to the two-body particle-antiparticle system. As we show in Refs. [5,6] this implies that if we take the frequency $\omega_1=\omega$ for the particle we have to take $\omega_2=-\omega$ for the antiparticle just as we have the charge e for the electron and $-e$ for the positron. Noting now that $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2$, $\mathbf{X}=\frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)$ and defining

$$\mathbf{x}'\equiv\tfrac{1}{\sqrt{2}}(\mathbf{x}_1-\mathbf{x}_2),\qquad \mathbf{p}'\equiv\tfrac{1}{\sqrt{2}}(\mathbf{p}_1-\mathbf{p}_2), \tag{3.1}$$

Equation (2.8) becomes

$$\left\{\tfrac{1}{\sqrt{2}}(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2)\cdot\mathbf{p}'-i\omega\left[\tfrac{1}{\sqrt{2}}(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)\cdot\mathbf{x}'\right]B+\beta_1+\beta_2\right\}\psi=\mu\psi, \tag{3.2}$$

where $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, β_1, β_2 are the direct products (4.2I) which can also be written in the 4×4 matrix form (4.4I).

The wave function ψ has then also four components as indicated in (4.3I), *i.e.*,

$$\psi=\begin{pmatrix}\psi_{11}\\ \psi_{21}\\ \psi_{12}\\ \psi_{22}\end{pmatrix}, \tag{3.3}$$

and thus Eq. (3.2) can be written as

$$i\omega^{1/2}\begin{pmatrix}\boldsymbol{\sigma}_1\cdot\boldsymbol{\eta} & \boldsymbol{\sigma}_2\cdot\boldsymbol{\xi}\\ \boldsymbol{\sigma}_2\cdot\boldsymbol{\xi} & \boldsymbol{\sigma}_1\cdot\boldsymbol{\eta}\end{pmatrix}\begin{pmatrix}\psi_{21}\\ \psi_{12}\end{pmatrix} = \begin{pmatrix}(\mu-2)\psi_{11}\\ (\mu+2)\psi_{22}\end{pmatrix}, \tag{3.4a}$$

$$-i\omega^{1/2}\begin{pmatrix}\boldsymbol{\sigma}_1\cdot\boldsymbol{\xi} & \boldsymbol{\sigma}_2\cdot\boldsymbol{\eta}\\ \boldsymbol{\sigma}_2\cdot\boldsymbol{\eta} & \boldsymbol{\sigma}_1\cdot\boldsymbol{\xi}\end{pmatrix}\begin{pmatrix}\psi_{11}\\ \psi_{22}\end{pmatrix} = \mu\begin{pmatrix}\psi_{21}\\ \psi_{12}\end{pmatrix}, \tag{3.4b}$$

where

$$\boldsymbol{\eta} = \tfrac{1}{\sqrt{2}}(\omega^{1/2}\mathbf{x}' - i\omega^{-1/2}\mathbf{p}'), \qquad \boldsymbol{\xi} = \tfrac{1}{\sqrt{2}}(\omega^{1/2}\mathbf{x}' + i\omega^{-1/2}\mathbf{p}'). \tag{3.5a, b}$$

Multiplying (3.4a) by μ and substituting in it (3.4b) we obtain, after some straightforward algebra that

$$\omega\begin{bmatrix}A & D\\ D & A\end{bmatrix}\begin{bmatrix}\psi_{11}\\ \psi_{22}\end{bmatrix} = \begin{bmatrix}(\mu^2-2\mu) & 0\\ 0 & (\mu^2+2\mu)\end{bmatrix}\begin{bmatrix}\psi_{11}\\ \psi_{22}\end{bmatrix}, \tag{3.6}$$

where

$$A = (p^2+r^2) - \mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2), \tag{3.7a}$$

$$D = -2(\mathbf{S}\cdot\mathbf{p})^2 + 2(\mathbf{S}\cdot\mathbf{r})^2 + p^2 - r^2, \tag{3.7b}$$

in which we use the variables

$$\mathbf{r} = \omega^{1/2}\mathbf{x}', \qquad \mathbf{p} = \omega^{-1/2}\mathbf{p}', \tag{3.8a, b}$$

while

$$\mathbf{L} = \mathbf{r}\times\mathbf{p}, \qquad \mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2), \tag{3.8c, d}$$

and extensive use was made of the relations between Pauli spin matrices,

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{3.9}$$

It is convenient, as in (4.7I), to substitute ψ_{11}, ψ_{22} by ϕ_+, ϕ_- through the relation

$$\begin{bmatrix}\psi_{11}\\ \psi_{22}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix}, \tag{3.10}$$

so Eq. (3.6) becomes

$$\omega\begin{bmatrix}A-D & 0\\ 0 & A+D\end{bmatrix}\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix} = \begin{bmatrix}\mu^2 & -2\mu\\ -2\mu & \mu^2\end{bmatrix}\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix}. \tag{3.11}$$

Writing the two equations in ϕ_+, ϕ_- explicitly and eliminating ϕ_- between them, we obtain for ϕ_+, which from now on we denote simply by ϕ, the equation

$$\left[\mu^4 - (4 + 2A\omega)\mu^2 + \omega^2(A + D)(A - D)\right]\phi = 0. \tag{3.12}$$

This is the equation that we obtained in Refs. [5] and [6] and were able to solve there by a perturbation procedure starting with $H_0 \equiv (4+2A\omega)$ whose eigenstates and eingenvalues can be determined exactly. We got then μ^2 as a power series in ω, and we stopped at the term with ω^2.

The impression we get, for $\omega \ll 1$ where the perturbative analysis is valid, is that we are dealing with bound states and, in fact, if we disregard the term ω^2 in (3.12) the eigenstates are those of the harmonic oscillator with spin [5,6]. We shall see though, when discussing the radial equation in the next sections, that the spectrum is continuous but for $\omega \ll 1$ there are resonant states with very small width.

4. The radial wave equation

From (3.7) we see that the wave Eq. (3.12) depends on the spherical coordinates r, θ, φ associated with $\mathbf{r}$, as well as on derivatives with respect to them and the components S_i, $i = 1, 2, 3$, of the total spin.

The solution ϕ of (3.12) will also be an eigenstate of the total angular momentum squared $\mathbf{J}^2$ and its projection J_3, where

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{4.1}$$

as the operator in the square bracket of (3.12) commutes with J_i, $i = 1, 2, 3$. Furthermore this operator is invariant under the reflection

$$\mathbf{r} \to -\mathbf{r}, \qquad \mathbf{p} \to -\mathbf{p}, \tag{4.2}$$

so that ϕ will also be characterized by its parity.

We denote now by the ket $|(L, S)Jm\rangle$ the angular and spin part of the function ϕ, *i.e.*,

$$|(L, S)Jm\rangle = \sum_{\mu,\sigma} \langle L\mu, S\sigma|Jm\rangle Y_{L\mu}(\theta, \varphi)\chi_{S\sigma}, \tag{4.3}$$

where $\langle\ \ |\ \ \rangle$ is a Clebsch-Gordan coefficient, $Y_{L\mu}$ (with $L = j \pm 1$ or J) is a spherical harmonic, and $\chi_{S\sigma}$ the spin part of the wave function where $S = 0$ or 1.

From the integrals of motion mentioned above, the Eq. (3.12) admits two types of solutions, one of the form

$$\phi = r^{-1}f_0(r)|(J, 0)Jm\rangle + r^{-1}f_1(r)|(J, 1)Jm\rangle, \tag{4.4}$$

which has parity $(-1)^J$, while the other is

$$\phi = r^{-1} f_+(r)|(J+1,1)Jm\rangle + r^{-1} f_-(r)|(J-1,1)Jm\rangle, \tag{4.5}$$

with parity $-(-1)^J$. The functions $f_0(r)$, $f_1(r)$ or $f_+(r)$, $f_-(r)$, will be determined by radial matrix equations, and we shall proceed to derive the one for $f_S(r)$, $S = 0, 1$.

We note that for our purpose we require the matrix elements

$$\langle (L', S')Jm|M|(L,S)Jm\rangle, \tag{4.6}$$

where M is either A, D given in (3.7). These matrix elements, besides being functions of r, will also depend on the derivative (d/dr) as the operator $\mathbf{p} = -i\nabla$ appears in A, D. We shall proceed to give all the matrix elements we require for the case when the parity of our state is $(-1)^J$, *i.e.*, when $L = L' = J$ and $S = 0$ or 1:

$$\big\langle (J,S')Jm\big|r^2\big|(J,S)Jm\big\rangle = \delta_{SS'} r^2, \tag{4.7a}$$

$$\big\langle (J,S')Jm\big|p^2\big|(J,S)Jm\big\rangle = \delta_{SS'}\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{J(J+1)}{r^2}\right], \tag{4.7b}$$

$$\big\langle (J,S')Jm\big|\mathbf{L}\cdot(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\big|(J,S)Jm\big\rangle = -\tfrac{1}{2}\left[1 - (-1)^{S+S'}\right]\left[J(J+1)\right]^{1/2}, \tag{4.7c}$$

$$\big\langle (J,S')Jm\big|(\mathbf{S}\cdot\mathbf{p})^2\big|(J,S)Jm\big\rangle = \delta_{S'1}\delta_{S1}\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{J(J+1)}{r^2}\right], \tag{4.7d}$$

$$\big\langle (J,S')Jm\big|(\mathbf{S}\cdot\mathbf{r})^2\big|(J,S)Jm\big\rangle = \delta_{S'1}\delta_{S1} r^2. \tag{4.7e}$$

The matrix element (4.7a) is trivial, the one in (4.7b) follows from the expression of the Laplacian in spherical coordinate and the definition (4.3) of the ket $|(L,S)Jm\rangle$. The value (4.7c) can be obtained from simple considerations of Racah algebra [5,6]. The matrix elements (4.7d, e) can be written as

$$\begin{aligned}&\big\langle (J,S')Jm\big|(\mathbf{S}\cdot\mathbf{w})^2\big|(J,S)Jm\big\rangle \\ &\quad = \sum_{\tau=-1}^{1}\big\langle (J,S')Jm\big|\mathbf{S}\cdot\mathbf{w}\big|(J+\tau,1)Jm\big\rangle\big\langle (J+\tau,1)Jm\big|\mathbf{S}\cdot\mathbf{w}\big|(J,S)Jm\big\rangle,\end{aligned} \tag{4.8}$$

where $\mathbf{w}$ is either $\mathbf{r}$ or $\mathbf{p}$. The intermediate states on the left hand side of (4.8) have to be of opposite parity as $\mathbf{w}$ is a polar vector, and thus are restricted to $|(J\pm 1,1)Jm\rangle$. The matrix elements of $(\mathbf{S}\cdot\mathbf{w})$ with $\mathbf{w} = \mathbf{r}$ or $\mathbf{p}$ are discussed in the Appendix and from them and (4.8) we get the results (4.7d, e).

Equation (3.12) becomes then a radial matrix equation of the form

$$\left\{\mu^4 - (4 + 2\mathbf{A}\omega)\mu^2 + \omega^2(\mathbf{A}+\mathbf{D})(\mathbf{A}-\mathbf{D})\right\}\begin{bmatrix} f_0(r) \\ f_1(r)\end{bmatrix} = 0, \tag{4.9}$$

where **A**, **D** are 2×2 matrices of the form

$$\mathbf{A} = \begin{bmatrix} \pi^2 + r^2 & \{J(J+1)\}^{1/2} \\ \{J(J+1)\}^{1/2} & \pi^2 + r^2 \end{bmatrix}, \tag{4.10a}$$

$$\mathbf{D} = (\pi^2 - r^2) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.10b}$$

where

$$\pi^2 = -\frac{d^2}{dr^2} + \frac{J(J+1)}{r^2}, \tag{4.11a}$$

as due to the factor r^{-1} appearing in (3.4) we see that

$$-\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\right)\frac{f_S(r)}{r} = -\frac{1}{r}\frac{d^2 f_S(r)}{dr^2}. \tag{4.11b}$$

Written out explicitly, (4.9) gives rise to the two coupled equations

$$\left\{(\mu^2 - 2\omega\pi^2)(\mu^2 - 2\omega r^2) - 4\mu^2 + \omega^2[J(J+1)]\right\} f_0(r)$$
$$= 2\omega[J(J+1)]^{1/2}(\mu^2 - 2\omega\pi^2) f_1(r), \tag{4.12a}$$

$$\left\{(\mu^2 - 2\omega r^2)(\mu^2 - 2\omega\pi^2) - 4\mu^2 + \omega^2[J(J+1)]\right\} f_1(r)$$
$$= 2\omega[J(J+1)]^{1/2}(\mu^2 - 2\omega r^2) f_0(r). \tag{4.12b}$$

We can use (4.12b) to express $f_0(r)$ in terms of $f_1(r)$, and substituting it in (4.12a) we finally get for $f_1(r)$ the equation

$$\left\{\mathcal{O} - \left[2\mu + \omega J^{1/2}(J+1)^{1/2}\right]^2\right\}\left\{\mathcal{O} - \left[2\mu - \omega J^{1/2}(J+1)^{1/2}\right]^2\right\} f_1(r) = 0, \tag{4.13}$$

where

$$\mathcal{O} = (\mu^2 - 2\omega r^2)(\mu^2 - 2\omega\pi^2). \tag{4.14}$$

As the operators in the two curly brackets of (4.13) commute, we see that $f_1(r)$ must satisfy the equation

$$\left\{(\mu^2 - 2\omega r^2)(\mu^2 - 2\omega\pi^2) - \left[2\mu \pm \omega J^{1/2}(J+1)^{1/2}\right]^2\right\} f(r) = 0, \tag{4.15}$$

where we suppressed the index 1 in $f_1(r)$ and have either a + or − sign in the last square bracket, and π^2 is given by (4.11a). Once $f(r)$, and in the process also μ^2, have been

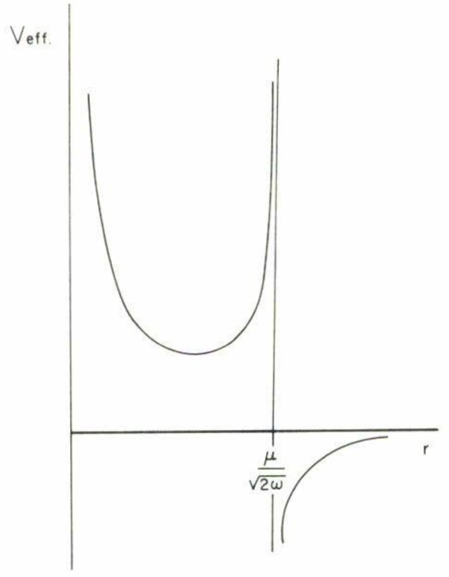

FIGURE 1. Effective potential entering Eq. (4.16). The Coulomb-like barrier at $r = \mu/\sqrt{2\omega}$ is penetrable. It means that for positive energies this potential supports only resonant states (the mesons).

determined, then $f_0(r)$ can be derived from (4.12b) and thus finally ϕ in (4.4) can be obtained explicitly.

A similar analysis can be carried out for the state (4.5) of parity $-(-1)^J$ but the results are more complicated and we leave them for a future analysis.

Equation (4.15) may be rewritten in the following Schrödinger-like form:

$$\left\{-\frac{d^2}{dr^2}+\frac{J(J+1)}{r^2}+\frac{\left[\mu/\omega \pm \frac{1}{2}\sqrt{J(J+1)}\right]^2}{\mu^2/2\omega - r^2}\right\}\psi = \frac{\mu^2}{2\omega}\psi, \tag{4.16}$$

where we substituted f by ψ and which has the interesting property that the effective potential depends on the eigenvalue of the equation. We shall look for solutions of (4.16) which go to zero as $r \to 0$. Let us suppose $\mu^2 > 0$. The effective potential entering (4.16) is schematically drawn in Fig. 1. The regions $r < \mu/\sqrt{2\omega}$ and $r > \mu/\sqrt{2\omega}$ are separated by a Coulomb-like barrier which is penetrable [12]. Consequently, the spectrum of the Hamiltonian for "positive energies", $\mu^2 > 0$, is continuous. Nevertheless, as in any barrier problem, there are solutions of (4.16) corresponding to resonant states. We shall identify these resońant state of a quark and an antiquark with the mesons. The resonant energies are commonly obtained from the asymptotic behaviour of the wave function at large distances:

$$\psi \approx A(k)e^{ikr} + B(k)e^{-ikr}, \qquad k^2 = \frac{\mu^2}{2\omega}, \tag{4.17}$$

as the complex solutions of the equation $B(k) = 0$.

The resonant states are mainly confined to the region $r^2 < \mu^2/2\omega$. So, the second interesting property of Eq. (4.16) to be stressed is that it predicts strong interaction radii of hadrons to be of the order of $\frac{1}{2}(\hbar c/mc^2)\mu/\sqrt{2\omega}$ (in ordinary units). Using that $\hbar c \approx 0.2$ GeV-fm and given m in GeV we get the rough estimate for the radius

$$R \approx \frac{0.1}{m}\frac{\mu}{\sqrt{2\omega}}\ \text{fm}. \tag{4.18}$$

Let us now turn to the analysis of the spectrum of resonant states.

5. Approximate solution of the radial equation by means of the $1/N$–method

As mentioned above, the spectrum of resonant levels in Eq. (4.16) could be obtained by requiring the wave function in $r > \mu/\sqrt{2\omega}$ to be an outcoming wave and looking for solutions in the complex μ-plane. This is, however, a complicated procedure. We will make use of a non- perturbative analytical method consisting in writing formally in D dimensions the Laplacian entering this equation and using $(D+2J)^{-1}$ as an expansion parameter (see, for example, [10] and references therein for applications in non-relativistic quantum mechanics and for the computation of energy eigenvalues from the Dirac equation in an external potential). We shall mention that using this method we get in first approximation a very localized wave function. It means that in fact we are neglecting the coupling to the disintegration channel and, consequently, we will obtain a real valued μ. The decay width may be obtained by computing the probability of tunneling through the barrier at $r < \mu/\sqrt{2\omega}$.

In the present paper, we will use a refined version of this method known as the shifted $1/N$–expansion. We start from the equation

$$\left\{-\frac{1}{N^2}\frac{d^2}{dx^2} + \frac{1}{4x^2}\left(1-\frac{a+1}{N}\right)\left(1-\frac{a+3}{N}\right) + \frac{b^2(\nu)}{4(1-x^2)}\right\}\psi = \frac{\nu^4}{4}\psi, \tag{5.1}$$

which may be taken as an extension of Eq. (4.16) to D dimensions, which coincides with it at the physical dimensions $D = 3$. The notations are as follows: $N = D + 2J + a$, where the magnitude a will be specified below; ν is related to μ as $\mu = \sqrt{N\omega}\,\nu$, and $r^2 = (\mu^2/2\omega)x^2$. Finally, $b(\nu)$ is defined in the following way:

$$b(\nu) = \frac{2\nu}{\sqrt{N_0\omega}} \pm \frac{1}{N_0}\sqrt{J(J+1)}, \tag{5.2}$$

where $N_0 = 3 + 2J + a$.

The solutions of Eq. (5.1) may be looked for as a power series in $1/N^{1/2}$, *i.e.*, $\nu = \nu_0 + \nu_{1/2}/N^{1/2} + \nu_1/N + \cdots$, $\psi = \psi_0 + \psi_{1/2}/N^{1/2} + \cdots$. We shall note, however, that $1/N$ plays in (5.1) the same role as $\hbar$ for the term $-d^2/dx^2$. Thus in the $N \to \infty$ limit we get a classical problem. In solving (5.1) we then follow the same steps as in a quasiclassic calculation of energy levels. First, we minimize the effective (classical) potential to find the equilibrium distance between particles and the minimal energy. Then small oscillations around the minimum are included, which lead to corrections to the ground-state energy and a first approximation to the excited states. After that, we include anharmonicities which lead to higher-order corrections.

Thus, in the leading approximation, $N \to \infty$, quantum fluctuations are suppressed and $\nu_0^4/4$ is determined as the minimum of the effective potential

$$\frac{\nu_0^4}{4} = \min U(x) = \min \left\{ \frac{1}{4x^2} + \frac{b^2(\nu_0)}{4(1-x^2)} \right\}. \tag{5.3}$$

This leads to transcendental equations for ν_0 and the position of the minimum, x_0, which may be solved explicitly. One obtains the following positive solutions for μ_0 at the physical dimension $D = 3$:

$$\mu_0^{(+)} = 1 + \sqrt{1 + \omega \left[N_0 + \sqrt{J(J+1)} \right]}, \quad \text{all values of } \omega,\ J; \tag{5.4}$$

$$\mu_0^{(-)} = 1 + \sqrt{1 + \omega \left[N_0 - \sqrt{J(J+1)} \right]}, \quad \omega < \frac{4N_0}{J(J+1)}; \tag{5.5}$$

and

$$\tilde{\mu}_0 = -1 + \sqrt{1 + \omega \left[N_0 + \sqrt{J(J+1)} \right]}, \quad \omega > \frac{4N_0}{J(J+1)}. \tag{5.6}$$

The square of the distance between a quark and an antiquark in this approximation is obtained as

$$x_0^2 = \frac{1}{\nu_0^2} = \frac{N_0 \omega}{\mu_0^2}, \tag{5.7}$$

or, in ordinary units

$$r_0^2 = \left(\frac{0.2 \text{ GeV-fm}}{m} \right)^2 \frac{N_0}{2}. \tag{5.8}$$

One shall note that r_0 defined in Eq. (5.8) may be taken as an estimate of the meson diameter only if it is of the same order of magnitude as $(0.2 \text{ GeV-fm}/m)\mu/\sqrt{2\omega}$, *i.e.*, if $x_0 \approx 1$.

The next-to-leading corrections to ν_0 are easily computed by writing $x = x_0 + y/N^{1/2}$ and considering the small (harmonic) oscillations around the equilibrium distance x_0. One can verify that $\nu_{1/2} = 0$. The wave function, ψ_0, and ν_1 are determined from the equation

$$\left\{ -\frac{d^2}{dy^2} + \tfrac{1}{2} U''(x_0) y^2 - \frac{2a+4}{4x_0^2} + \frac{b(\nu_0)\nu_1}{\sqrt{N_0\omega}\,(1-x_0^2)} \right\} \psi_0 = \nu_0^3 \nu_1 \psi_0, \tag{5.9}$$

leading to

$$\left(n+\tfrac{1}{2}\right)\lambda-\frac{2a+4}{4x_0^2}+\frac{b(\nu_0)\nu_1}{\sqrt{N_0\omega}\,(1-x_0^2)}=\nu_0^3\nu_1, \tag{5.10}$$

where λ is defined by

$$\lambda=\sqrt{2U''(x_0)}=\frac{2}{x_0^2(1-x_0^2)^{1/2}}. \tag{5.11}$$

The parameter a so far has not been determined. It is fixed by the requirement that the next-to-leading corrections give no contribution to the energy. In other words, Eqs. (5.4)–(5.6) are required to be exact up to corrections of order $1/N^2$.

If we require $\nu_1=0$, then after some simple algebraic manipulations we are led to the following equation for a

$$\frac{2+a}{4\left(n+\frac{1}{2}\right)}\left[\left(\frac{2+a}{4\left(n+\frac{1}{2}\right)}\right)^2-1\right]^{-1/2}=\nu_0=\frac{\mu_0}{\sqrt{N_0\omega}}, \tag{5.12}$$

and the energy of the level, μ_{nJ}, with n radial quanta and angular momentum J is obtained up to terms of order $1/N^2$ from one of the Eqs. (5.4)–(5.6), with a determined from Eq. (5.12).

Once obtained the energy (ν_0), we may estimate the level width in the quasiclassical (in $1/N$) approximation [13]

$$\Gamma=m\Omega T. \tag{5.13}$$

In this latter formula the quark mass has been explicitly written to express Γ in GeV. The transmission coefficient, T, is given by

$$T=\exp\left[-2N\int_{x_0}^{1}dx\,\sqrt{U(x)-\nu_0^4/4}\right], \tag{5.14}$$

where $U(x)$ is the potential defined in Eq. (5.3). Finally, Ω is the frequency of periodic motion around the minimum of U. It differs from λ in a coefficient which can be easily found with the help of Eq. (5.10) and the relation between μ and ν. Indeed, the $1/N$ series for μ is written as

$$\mu=\sqrt{N\omega}+\sqrt{\frac{\omega}{N}}\left[\nu_0^3-\frac{b(\nu_0)}{\sqrt{N_0\omega}\,(1-x_0^2)}\right]^{-1}\left\{\left(n+\tfrac{1}{2}\right)\lambda-\frac{2a+4}{4x_0^2}\right\}+\cdots \tag{5.15}$$

leading to

$$\Omega=\sqrt{\frac{\omega}{N}}\left[\nu_0^3-\frac{b(\nu_0)}{\sqrt{N_0\omega}\,(1-x_0^2)}\right]^{-1}\lambda. \tag{5.16}$$

After a bit of algebra we obtain

$$\Omega = \frac{2\mu_0}{\mu_0 - 1}\frac{\omega}{\sqrt{\mu_0^2 - N_0\omega}}. \tag{5.17}$$

Putting together (5.13), (5.14) and (5.17), we get

$$\Gamma = \frac{2\mu_0 m}{\mu_0 - 1}\frac{\omega}{\sqrt{\mu_0^2 - N_0\omega}}\exp\left[-2N\int_{x_0}^{1} dx\,\sqrt{U(x) - \nu_0^4/4}\right]. \tag{5.18}$$

Let us note that for Eq. (5.18) to hold we shall obtain $\Gamma \ll 1$. The integral in (5.18) may be explicitly evaluated out to give

$$\Gamma = \frac{2\mu_0 m}{\mu_0 - 1}\frac{\omega}{\sqrt{\mu_0^2 - N_0\omega}}\exp\left\{-\left[\frac{\mu_0}{\omega}\sqrt{\mu_0^2 - N_0\omega} - N_0\ln\frac{\mu_0 + \sqrt{\mu_0^2 - N_0\omega}}{\sqrt{N_0\omega}}\right]\right\}. \tag{5.19}$$

We shall discuss below the meaning of the widths of the levels computed from (5.17). An alternative estimation of the width of states with $J = 0$ may be obtained by computing the transmission probability of the effective Coulomb barrier at $r = \mu/\sqrt{2\omega}$. This has been done in Ref. [12].

6. Comparison with real mesons

Let us briefly describe the properties of mesons which follow from Eqs. (5.4)–(5.7), (5.12) and (5.19), and compare them with the properties of real mesons. Before doing this we wish to stress that we are dealing with a one parameter (the frequency ω) model theory, and so we are concerned only with the part of the wave function dependent on the coordinates and ordinary spins. No mention is then made of the dominant color interaction, as was done in the work of Godfrey and Isgur in Ref. [4], because their model is more complex and has many parameters. Furthermore we shall mention some of their results [4], only because the experimental information is frequently too small for meaningful comparisons.

The first point to notice is that in our model there is no mixing between quarks. A flavour is defined by a value, m_q, of the quark mass, (in GeV) and a value of the frequency, ω_q. This may be taken as an approximate description of $b\bar{b}, c\bar{c}$ and light isovector mesons in states with $J = L$.

So, we will consider the analogues of these mesons in our model. Let us start with the $b\bar{b}$. There is almost no uncertainty in the value of mass one may assign to the b quark. We take it from Ref. [4], $m_b \approx 4.977$ GeV. The frequency may be chosen to fit the value of a physical magnitude, for example, the energy gap to the first excited state in the subsector with $J = 0$. This gap is expected to be 0.58 GeV [4]. So, from

$$\Delta E(n, J) = m_b\left\{\mu_{nJ}(\omega_b) - \mu_{00}(\omega_b)\right\}, \tag{6.1}$$

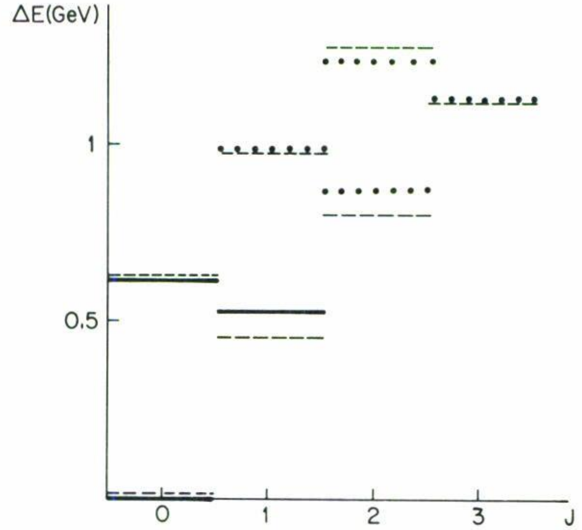

FIGURE 2. Excitation energies of $c\bar{c}$ mesons. The bold lines are experimentally observed levels, dotted lines correspond to calculations by Godfrey and Isgur [4], while our model levels (the $\mu^{(+)}$ branch) are represented by dashed lines.

and particularizing to $n = 1$, $J = 0$, one obtains $\omega_b = 0.062$. It means that the $b\bar{b}$ system is contained in the frequency region $\omega \ll 1$ (perturbative). In this region, a may be looked for as a power series in ω. One obtains

$$a = 4n + \mathcal{O}(\omega), \tag{6.2}$$

where $\mathcal{O}(\omega)$ means a correction or order ω, and the mass spectrum coincides with the perturbative result found in Refs. [5,6]

$$\mu^{(\pm)} = 2 + \frac{\omega}{2}\left[3 + 2J + 4n \pm \sqrt{J(J+1)}\right] + \mathcal{O}(\omega^2). \tag{6.3}$$

Energy differences computed from (6.1) and the $\mu^{(+)}$ branch of (6.3) reproduce qualitatively the first two expected [4] (and partially observed [14]) Regge trajectories in bottonium in the sector with $J = L$, *i.e.*, those corresponding to $n = 0$ and $n = 1$. The splitting of the two levels corresponding to a definite J is not, however, given by the difference $\mu^{(+)} - \mu^{(-)}$. In our model, this is a very strong splitting caused by a strong $L - S$ coupling, while in bottonium the splitting is supposed to be insignificant [4].

The experimental data available for $c\bar{c}$ mesons in states with $J = 0$ are the following. Three lines with their corresponding widths have been reported [4]: $\eta_c(2980)$, $\Gamma = 10$ MeV; $\eta_c(3950)$, $\Gamma = 8$ MeV; $\chi_{c1}(3510)$, $\Gamma = 1.3$ MeV. On the other hand, one can get an idea of the expected radii of these mesons from the measured strong interaction radius of the J/ψ, $R^2 = 0.04$ fm^2 [15]. One shall note, however, that the J/ψ is not a state with $J = L$.

In Fig. 2, excitation energies computed from Eqs. (5.4) and (5.12) are compared with the observed [14] (expected [4]) values for $c\bar{c}$ states with $J = L$. The mass $m_c = 1.628$ GeV has been taken from Ref. [4], while the frequency ω_c has again been chosen to fit the energy gap in the subsector with $J = 0$ (0.61 GeV). We obtained $\omega_c = 0.235$.

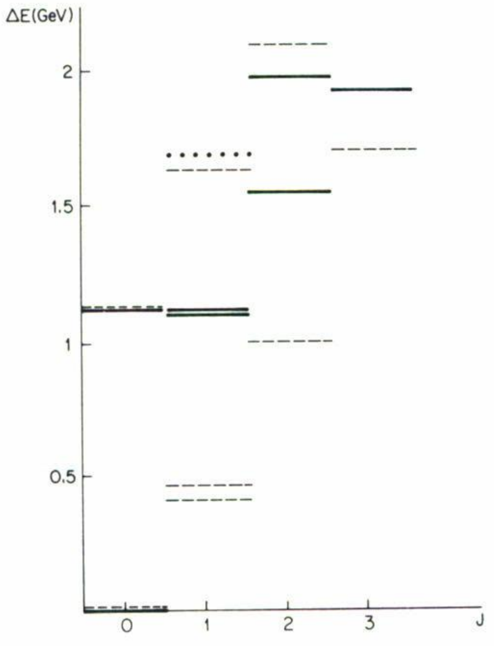

FIGURE 3. The same as Fig. 2 for isovector mesons, but in this case dashed lines represent levels calculated from the $\mu^{(-)}$ branch, except that the two terms of the $n=0$, $J=1$ doublet are drawn.

The radius of the $\eta_c(2980)$ computed from Eq. (4.18) is shown to be $R_{\eta_c} = 0.21$ fm, a very reasonable value. One shall note that for charmonium (and also for bottonium), the x_0 defined in Eq. (5.7) is a very small magnitude as compared with the diameter of the region where the meson lives until it decays, *i.e.*, $x_0 \ll 1$, and thus this value can not be used to estimate the meson diameter.

The experimental excitation energies of $J = L$ isovector resonances are represented in Fig. 3 [14]. Missing states (dotted lines) are taken from Ref. [4]. Dashed lines are the results of our calculations.

In the present case, the uncertainty in the mass of the u quark is very high, and we take it as a free parameter together with the frequency ω_u to fit two experimentally observed magnitudes: the energy gap (1.162 GeV) in the subsector with $J = 0$, and the pion radius $R_\pi = 0.64$ fm. We estimated the radius as half the r_0 given in Eq. (5.8). One obtains

$$m_u = 0.242 \text{ GeV}, \tag{6.4}$$

$$\omega_u = 7.48. \tag{6.5}$$

As may be seen, light mesons are contained in the highly non perturbative region $\omega \gg 1$. In this frequency range some interesting phenomena take place. For example, the branches $\mu^{(+)}$ and $\mu^{(-)}$ change their relative position, *i.e.*, $\mu^{(-)}$ is higher in energy than $\mu^{(+)}$. Something analogous to this is experimentally observed at least in $J = 1$ states, in which $J^{PC} = 1^{+-}$ states are lower in energy than 1^{++} contrary to what is expected in charmonium: $1^{+-} > 1^{++}$ [4]. So, in Fig. 3 we drew the $\mu^{(-)}$ branch except for the doublet in $J = 1, n = 0$ for which both branches are represented.

Concerning the radii of light mesons, we shall make a remark. We choose m_u to fit R_π, but Eq. (5.8) contains more information than a simple number. From Eq. (5.8) we obtain for the radius

$$R_{q\bar{q}}^2 = \frac{0.01}{m_q^2} \left(\frac{3+2J}{2} + \frac{a}{2} \right) \text{ fm}^2. \tag{6.6}$$

Equation (6.6) reproduces qualitatively the known properties of strong interaction radii of light mesons [15]. the main contribution to $R_{q\bar{q}}^2$, *i.e.*, the term $\frac{3}{2} + J$, is independent of the interaction potential (of ω). In non relativistic potential models, an analogous term is usually interpreted as a relativistic smearing of quark coordinates. A soft dependence on ω comes from a and corresponds in potential models to the radius computed from the wave function.

The law (5.19) does not give the experimentally observed widths of $b\bar{b}$, $c\bar{c}$ and light isovector mesons, but it leads to a qualitatively correct dependence of Γ on the quark flavour. Indeed, according to Eq. (5.19) we obtain that Γ should increase as ω is increased, becoming even greater than μ for $u\bar{u}$ mesons.

7. Concluding remarks

The main result of the present paper is the qualitative picture of mesons following from Eq. (4.16): mesons are resonant states mainly confined to a region of squared radius $(0.01/m^2)(\mu^2/2\omega)$ fm^2.

We compared the properties of our $b\bar{b}$, $c\bar{c}$ and $u\bar{u}$ model mesons with the properties of real mesons. To achieve this goal, we determined the quark parameter, ω_q, as to fit the values of an observable magnitude, for example the energy gap in the subsector $J = 0$. This is, of course, a rough procedure because we are working in a zeroth order approximation, without realistic and hyperfine and other forces.

We obtained reasonable values for the radii of the studied mesons. For light mesons, we obtained an expression for the radius which reproduces what is known about strong interaction radii of hadrons [15].

Excitation energies of states with $J = L$ were calculated by using mass formulae obtained by means of the $1/N$-expansion. The agreement with experiment is only qualitative due to the fact that the forces acting in the model are not realistic. In particular, the splitting of the doublets in subsectors with $J > 0$ is excessive. However, some qualitative properties are correctly reproduced. For example, the fact that the branches $\mu^{(+)}$ and $\mu^{(-)}$ change their relative position as we go from $\omega \ll 1$ (heavy quarkonia) to $\omega \gg 1$ (light mesons) seems to have its analogue in the spectra of real mesons.

We shall note that we found no conditions under which the branch $\mu^{(-)}$ may be replaced by $\tilde{\mu}$. Even in light mesons the inequality $\omega > 4N_0/J(J+1)$ is not fulfilled.

We estimated the level width by computing the quasiclassical (in $1/N$) transmission probability of the barrier at $x = 1$. This leads to qualitatively correct predictions, for example that heavy mesons have smaller widths (according to the law $\Gamma \approx 2m\omega \exp(-4/\omega)$, where ω decreases as m is increased). But the values we obtain for Γ are far from the real ones. This is due to the major role played by the vacuum in the decay of a meson, which is not taken into account in the two-particle equation (4.16). In particular, we obtain that every meson is unstable (against to its disintegration into q and $\bar{q}$), while in reality the meson may be stabilized because of the lack of final states for its decay to proceed.

We see that further developments of the present work could go along two directions. The first is to carry out the zeroth-order analysis for $J = L \pm 1$ states, and the second, to include realistic forces. Both seem to be very promising.

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APPENDIX

We shall indicate in this Appendix how to evaluate the matrix elements of $\mathbf{w}\cdot\mathbf{S}$, $\mathbf{w}=\mathbf{r}$ or $\mathbf{p}$, appearing on the right hand side of (4.8).

From Racah's analysis [16] we have that

$$\big\langle (L',S')Jm\big|\mathbf{w}\cdot\mathbf{S}\big|(L,S)Jm\big\rangle = (-1)^{L'+S-J}W(LL'SS';1J)\big\langle L'\big\|\mathbf{w}\big\|L\big\rangle\big\langle S'\big\|\mathbf{S}\big\|S\big\rangle, \quad (A.1)$$

where W is a Racah coefficient and $\langle L'\|\mathbf{w}\|L\rangle$, $\langle S'\|\mathbf{S}\|S\rangle$ are the reduced matrix elements of the vector operators $\mathbf{w}$, $\mathbf{S}$ respectively.

It is well known [16] that

$$\big\langle S'\big\|\mathbf{S}\big\|S\big\rangle = \delta_{S'S}[S(S+1)]^{1/2}, \quad (A.2)$$

$$\big\langle L'\big\|\mathbf{r}\big\|L\big\rangle = \delta_{L'L+1}r\sqrt{\frac{L+1}{2L+3}} - \delta_{L'L-1}r\sqrt{\frac{L}{2L-1}}. \quad (A.3)$$

On the other hand, in another publication [17] it was found that

$$\big\langle L'\big\|\mathbf{p}\big\|L\big\rangle = \delta_{L'L+1}\sqrt{\frac{L+1}{2L+3}}(-i)\left(\frac{d}{dr}-\frac{L}{r}\right) - \delta_{L'L-1}\sqrt{\frac{L}{2L-1}}(-i)\left(\frac{d}{dr}+\frac{L+1}{r}\right). \quad (A.4)$$

Combining all of these results we get the following matrix elements for $\mathbf{r}\cdot\mathbf{S}$ and $\mathbf{p}\cdot\mathbf{S}$:

$$\big\langle (J,1)Jm\big|\mathbf{p}\cdot\mathbf{S}\big|(J+1,1)Jm\big\rangle = i\left(\frac{J}{2J+1}\right)^{1/2}\left(\frac{d}{dr}+\frac{J+2}{r}\right), \tag{A.5a}$$

$$\big\langle (J,1)Jm\big|\mathbf{r}\cdot\mathbf{S}\big|(J+1,1)Jm\big\rangle = -\left(\frac{J}{2J+1}\right)^{1/2} r, \tag{A.5b}$$

$$\big\langle (J,1)Jm\big|\mathbf{p}\cdot\mathbf{S}\big|(J-1,1)Jm\big\rangle = i\left(\frac{J+1}{2J+1}\right)^{1/2}\left(\frac{d}{dr}-\frac{J-1}{r}\right), \tag{A.5c}$$

$$\big\langle (J,1)Jm\big|\mathbf{r}\cdot\mathbf{S}\big|(J-1,1)Jm\big\rangle = -\left(\frac{J+1}{2J+1}\right)^{1/2} r, \tag{A.5d}$$

$$\big\langle (J+1,1)Jm\big|\mathbf{p}\cdot\mathbf{S}\big|(J,1)Jm\big\rangle = i\left(\frac{J}{2J+1}\right)^{1/2}\left(\frac{d}{dr}-\frac{J}{r}\right), \tag{A.5e}$$

$$\big\langle (J+1,1)Jm\big|\mathbf{r}\cdot\mathbf{S}\big|(J,1)Jm\big\rangle = -\left(\frac{J}{2J+1}\right)^{1/2} r, \tag{A.5f}$$

$$\big\langle (J-1,1)Jm\big|\mathbf{p}\cdot\mathbf{S}\big|(J,1)Jm\big\rangle = i\left(\frac{J+1}{2J+1}\right)^{1/2}\left(\frac{d}{dr}+\frac{J+1}{r}\right), \tag{A.5g}$$

$$\big\langle (J-1,1)Jm\big|\mathbf{r}\cdot\mathbf{S}\big|(J,1)Jm\big\rangle = -\left(\frac{J+1}{2J+1}\right)^{1/2} r. \tag{A.5h}$$

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