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Solitary waves in ideal Hall-MHD

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Abstract. We show the existence of soliton solutions in the frame of the Hall-MHD description. The system under study consists in a nondissipative incompressible plasma column confined into a cylindrical perfectly conducting vessel. The plasma is separated from the wall by a vacuum region. The Korteweg-de Vries equation is obtained by performing an expansion in power series, up to second order, of a small positive parameter of every physical variable and governing equations after stretching some variables and assuming axisymmetry. The soliton solution is shown explicitly and its stability conditions are briefly discussed. Likewise, the relevance of this result for astrophysical plasmas is also commented.

Resumen. Se muestra que existen soluciones tipo solitón en el marco de la descripción Hall-MHD. Aquí se estudia el caso de una columna cilíndrica de plasma incomprensible y no disipativo que está confinada dentro de un depósito cilíndrico perfectamente conductor. Entre el plasma y el depósito media una región de vacío. La ecuación de Korteweg-de Vries se obtiene efectuando un desarrollo en serie de potencias, hasta segundo orden, en un parámetro positivo pequeño de las variables físicas y las ecuaciones dinámicas previo reescalamiento de algunas de las variables, suponiendo axisimetría. La solución tipo onda solitaria es obtenida explícitamente y se discuten brevemente sus condiciones de estabilidad. Asimismo, la relevancia de este resultado para plasmas astrofísicos es también comentada.

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1. Introduction

The study of nonlinear plasma waves is one of the most relevant and interesting problems from both physical and mathematical points of view. The case of solitary waves is a beautiful example. Regarding the propagation of weakly nonlinear, long wavelength waves, there are many experimental observations ranging from the hydrodynamics to mechanical and electric models; of course, they include plasma physics, solid state physics, magnetic systems, nonlinear optics, and so on [1].

The nonlinear effects play a very important role in the plasma theory, in fact they are present in both laboratory [2] and astrophysical plasmas. Broadly speaking, they are linked to turbulent dynamics [3–5] and solar plasmas [6]; also they are particularly interesting in relation with the study of Jupiter's Great Red Spot [7–10].

In the 70's the studies of solitary waves in nonconducting fluids constituted an important subject of mathematical physics [11] because they can provide useful information respect to the properties of nonlinear dispersive media [12–14]. Subsequently, solutions of soliton type were obtained for plasmas confined into cylindrical devices in the frame of the magnetohydrodynamic (MHD) model [15–18].

The aim of this paper is to study surface waves propagating at the interface of a plasma. The plasma column is contained into a cylindrical perfectly conducting vessel and is separated from the cylindrical wall by a vacuum region. Unlike the usual treatments we use the ideal magnetohydrodynamics (MHD) with a Ohm's law modified by including the Hall term. This ideal Hall-MHD description is relevant to astrophysical and laboratory plasmas because of the Hall mechanism for preferential acceleration of ions in the presence of an electron background [19] and other related phenomena [20,21]. Regarding the dispersion relations in the framework of the Hall-MHD description for the linear case, it is known that the presence of the Hall-current yields a dispersive behavior of the corresponding waves [22,23]. Here we will show that the total (thermodynamic plus magnetic) pressure, to first-order in ϵ (a small parameter that measures the importance of the dispersive terms), satisfies the Korteweg-de Vries (KdV) equation which has a soliton-type solution.

An indicative feature of the possible existence of solitary waves, which provided the motivation for the present work, arises when one looks at the limiting value, in the long wavelength approximation, of the dispersion relations resulting from the linear analysis for both z-pinch and slab configurations in the frame of the nondissipative MHD with and without Hall effect in the Ohm's law. In order to clarify this point, let us examine in some detail the corresponding dispersion relations [23–25]. Firstly, we have

$$\frac{\omega^2}{\omega_{\text{A}}^2} = -b_{\text{i}}^2 + \frac{I'_m(kr_0)}{kr_0 I_m(kr_0)}\left[1 + \frac{(m+b_{\text{o}}kr_0)^2}{kr_0}\frac{K'_m(kR_0)I_m(kr_0) - I'_m(kR_0)K_m(kr_0)}{K'_m(kR_0)I'_m(kr_0) - I'_m(kR_0)K'_m(kr_0)}\right], \quad (1)$$

for a cylindrical plasma. Here ω is the oscillation frequency of the linear perturbations; ω_{A} is the Alfvén frequency; and b_{i} and b_{o} are the normalized (to the equilibrium magnetic field B_0) values of the magnetic field inside and outside the plasma column, respectively; R_0 is the radius of the perfectly conducting cylindrical vessel and r_0 is the initial radius of the plasma column; k and m denote, respectively, the axial and azimuthal wave numbers; I_m and K_m are the modified Bessel functions. The prime stands for the first derivative with respect to the corresponding argument. The dispersion relation (1) is valid for incompressible plasmas with homogeneous equilibrium in the frame of both ideal MHD and ideal Hall-MHD; it has already been elsewhere [23].

Secondly, for a nondissipative plasma slab the following dispersion relations [22] are obtained:

$$\frac{\omega^2}{k^2} = \begin{cases} v_{\text{S}}^2, \\ \left[(1+\tfrac{1}{2}\lambda_{\text{H}}^2\rho_0 k^2) \pm \lambda_{\text{H}}\rho_0^{1/2}k(1+\tfrac{1}{4}\lambda_{\text{H}}^2\rho_0 k^2)^{1/2}\right]v_{\text{A}}^2, \end{cases} \quad (2)$$

$$\frac{\omega^2}{k^2} = v_{\text{S}}^2 + v_{\text{A}}^2, \quad (3)$$

where v_{S} the sound speed, v_{A} the Alfvén speed, and $\lambda_{\text{H}} = C_{\text{H}}/a\rho_0$, with C_{H} denoting the Hall scaling parameter, a is the ion charge-to-mass ratio, and ρ_0 is the constant equilibrium mass density. Expression (2) is valid for parallel (respect to the equilibrium magnetic field) modes, while (3) is valid for perpendicular modes. It may be noted that the Hall-current only affects the parallel modes, at least in the case of uniform plasmas. Equations (2) and (3) were obtained using the ideal Hall-MHD description. It is worth to mention that for a slab plasma, when one uses the ideal MHD description, a dispersion relation is obtained of magneto-acoustic type.

It is clear from Eqs. (1)–(3) that the dispersive term is basically due to a couple of effects. On one side there is the finite frequency effect, on the other there are the geometric effects. The finite frequency effect means that the frequency of the linear wave is not vanishingly small compared with the ion cyclotron frequency, therefore it arises from the Hall term in the generalized Ohm's law. The balancing of the dispersive terms by nonlinearities can, at least in principle, lead to soliton-type solutions of the nonlinear equation.

As mentioned above there is another way to introduce dispersion, through geometric effects. That is the case for a cylindrical plasma which is surrounded by a vacuum when described by the ideal MHD model, a surface wave can propagate at the interface plasma-vacuum. Thus the dispersive term, in this case, is connected to the finite width of the cylinder.

Expressions (1) and (2) exhibit similar behavior as we discuss below but it is due to different reasons. Specifically, Eq. (1) exhibits a dispersive term due to a geometric effect, whereas Eq. (2) owes its dispersive term to a dynamic effect, the presence of the Hall effect. In the present paper we combine both of them: a perfectly conducting cylinder containing a plasma with Hall current and separated from the plasma column by a vacuum region. On the other hand, it is known [17] that for waves that are symmetric about the axis of the cylinder, the nonlinear wave that arises is described by the Benjamin-Ono or the KdV equation.

The precedent dispersion relations, Eqs. (1)–(2), in the long wavelength approximation, reduce to

$$c(k) = \alpha v_{\mathrm{A}} - \beta k^2, \tag{4}$$

where $c(k)$ is the phase speed, and α and β are constants. For the cylindrical geometry, relation (4) is valid when $m = 0$. In the particular case of the plasma slab $\alpha = 1$ and β is a constant which depends on the parameter λ_{H}. Equation (4) is similar to the familiar dispersion relation for shallow water in the linearized case [26–28]. With the substitutions $\omega \to i\partial/\partial t$, $k \to -i\partial/\partial z$ (*i.e.*, by using the Fourier's theorem), Eq. (4) yields

$$\frac{\partial \psi}{\partial t} + \alpha v_{\mathrm{A}} \frac{\partial \psi}{\partial z} + \beta \frac{\partial^3 \psi}{\partial z^3} = 0, \tag{5}$$

where ψ is the corresponding amplitude. Equation (5) can be obtained from the integro-differential equation due to Whitham [29],

$$\frac{\partial \psi}{\partial t} + \mu \psi \frac{\partial \psi}{\partial z} + \int_{-\infty}^{\infty} \tilde{c}(z - x) \frac{\partial \psi(x,t)}{\partial x}\, dx = 0, \tag{6}$$

where $\tilde{c}$ denotes the Fourier transform of the phase speed $c(k)$ given by Eq. (4). In the case under consideration $\mu = 0$; that is, it corresponds to the linear case. Thus we expect the governing equations at second order in ϵ yield the KdV equation with a soliton like solution.

This paper is organized as follows. In Sect. 2 we present the basic equations of the ideal Hall-MHD model for incompressible plasmas and suitable boundary conditions assuming a homogeneous equilibrium state. An ordering for some involved variables is proposed in Sect. 3 and all physical variables are expanded in powers of ϵ. Next the governing equations, to first-order in ϵ, are obtained and the related eigenvalue problem is established. Sect. 4 deals with the governing equations to second-order which, with the help of the first-order solutions, yield the KdV equation and then the soliton solution is obtained. Finally, in Sect. 5 we give a brief summary and a discussion of the results.

2. Basic Equations

We consider an incompressible and nondissipative plasma which is contained inside a perfectly conducting cylindrical vessel of radius R_0. The plasma is separated from the wall by a vacuum region. Also it is assumed that a constant magnetic field (along the axis of the vessel) is present inside the axisymmetric and infinitely long plasma column, while the vacuum carries a twisted magnetic field. The initial plasma radius is r_0. We will use a cylindrical coordinates system (r, θ, z) with the z–axis coinciding with the axis of the cylindrical vessel. Since we are interested only in axisymmetric perturbations, we set $\partial/\partial\theta = 0$.

Because the nondissipative Hall-MHD equations and the conditions under which they are an adequate physical model were discussed elsewhere [30] we simply assume that model as a starting point for this study.

The set of equations governing the motion of the plasma with the Hall current included in Ohm's law is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{\rho_0}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{7}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \lambda_{\text{H}} \nabla \times \big((\nabla \times \mathbf{B}) \times \mathbf{B}\big), \tag{8}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{10}$$

where $\mathbf{v}$, p, and $\mathbf{B}$ are the fluid velocity, pressure, and magnetic field, respectively. The remaining symbols were already defined.

The vacuum magnetic field $\mathbf{B}^{\text{V}}$ satisfies

$$\nabla \times \mathbf{B}^{\text{V}} = 0, \tag{11}$$

$$\nabla \cdot \mathbf{B}^{\text{V}} = 0. \tag{12}$$

If $\varphi(z,t)$ denotes the form of the free-boundary (*i.e.*, the plasma-vacuum interface), the deviations from the initial position r_0, say ξ, are given by

$$\xi = \varphi(z,t) - r_0. \tag{13}$$

As the plasma surface must move with the plasma

$$\frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi = v_r, \tag{14}$$

with $v_r = \mathbf{v} \cdot \hat{e}_r$ denoting the radial component of the velocity, and $\hat{e}_r$ being a unit vector along the radial direction.

The plasma under consideration must evolve in such a way that the total pressure and the normal components of the magnetic field be continuous across the free-surface. Likewise, the radial component of the vacuum magnetic field must vanish at the conducting wall. Hence the appropriate boundary conditions are:

i) At the free-boundary,

$$\pi = \tfrac{1}{2} \left|\mathbf{B}^{\mathrm{V}}\right|^2, \tag{15}$$

$$\mathbf{B} \cdot \hat{n} = 0, \quad \mathbf{B}^{\mathrm{V}} \cdot \hat{n} = 0, \tag{16}$$

where π is the total pressure and $\hat{n}$ is the unit vector along the outward normal direction.

ii) At the conducting wall, $r = R_0$,

$$\mathbf{B}^{\mathrm{V}} \cdot \hat{e}_r = 0. \tag{17}$$

For simplicity we assume an equilibrium characterized by a constant plasma pressure p_0 and a constant mass density ρ_0. The equilibrium fluid velocity is $\mathbf{v}_{\mathrm{e}} = v_0 \hat{e}_z$ with v_0 constant and $\hat{e}_z$ being the unit vector along the z-direction. On the other hand, as initial magnetic field we choose

$$\mathbf{B}_{\mathrm{e}} = B_0 b_i \hat{e}_z, \qquad 0 \le r \le r_0; \tag{18}$$

$$\mathbf{B}^{\mathrm{V}} = B_0 b_0 \hat{e}_z + B_0 \left(\frac{r_0}{r}\right) \hat{e}_\theta, \qquad r_0 < r \le R_0; \tag{19}$$

where B_0, b_0 and b_i are constants and $\hat{e}_\theta$ is the unit vector along the θ-direction.

3. First order analysis

As stressed in the Introduction, we need define new "stretched quantities" using a small parameter ϵ that measures the importance of the dispersive terms. That parameter will be also used to expand all physical variables in regular power series about their corresponding equilibrium values. It results from (2) that a suitable parameter for expand in power series is $\epsilon = \lambda_{\mathrm{H}}^2 \rho_0 k^2 = C_{\mathrm{H}}^2 (\omega_{\mathrm{A}}/\omega_{\mathrm{ci}})^2$, where ω_{ci} is the ion-cyclotron frequency. Hence to speak of ϵ

as small means we are using a long wavelength approximation in terms of a characteristic length defined by $\lambda_{\mathrm{H}}\sqrt{\rho_0}$. Note that the frequency at lowest order goes as $\epsilon^{3/2}$ [see Eq. (2)], so that for soliton-type solutions we require nonlinear terms to also be of order $\epsilon^{3/2}$. As usual, the ratio of the time scaling and the space scaling must go as ϵ; this is due, in particular, to the frequency shift. In view of these considerations, in order to carry out the nonlinear expansion of Eqs. (7)–(11), let us introduce the following transformations:

$$\frac{\partial}{\partial t} \rightarrow \epsilon^{3/2}\frac{\partial}{\partial t}, \tag{20}$$

$$\frac{\partial}{\partial z} \rightarrow \epsilon^{1/2}\frac{\partial}{\partial z}. \tag{21}$$

Additionally we choose for the radial components the following scaling:

$$v_r \rightarrow \epsilon^{1/2} v_r, \tag{22}$$

$$B_r \rightarrow \epsilon^{1/2} B_r, \tag{23}$$

$$B_r^{\mathrm{V}} \rightarrow \epsilon^{1/2} B_r^{\mathrm{V}}. \tag{24}$$

On the other hand, the physical quantities, say f, are expressed as power series in ϵ about the equilibrium state as

$$f = f_{\mathrm{e}} + \epsilon f_1 + \epsilon^2 f_2 + \cdots, \tag{25}$$

where f_{e} is the corresponding equilibrium value.

Furthermore, the governing equations are supplemented by imposing that the perturbations are localized (*i.e.*, the plasma tends to the equilibrium state as $|z| \rightarrow \infty$) and bounded at $r = 0$.

Applying (20)–(25) to the set of equations (7)–(12) we have to first order:

$$\frac{\partial \pi_1}{\partial r} = 0, \quad \pi_1 = p_1 + B_{\mathrm{e}} B_{z_1}, \tag{26}$$

$$\frac{\partial(\rho_0 v_0 v_{\theta_1} - B_{\mathrm{e}} B_{\theta_1})}{\partial z} = 0, \tag{27}$$

$$\frac{\partial(\rho_0 v_0 v_{z_1} + \pi_1)}{\partial z} = B_{\mathrm{e}}\frac{\partial B_{z_1}}{\partial z}, \tag{28}$$

$$\frac{\partial(B_{\mathrm{e}} v_{r_1} - v_0 B_{r_1})}{\partial z} + \lambda_{\mathrm{H}} B_{\mathrm{e}}\frac{\partial^2 B_{\theta_1}}{\partial z^2} = 0, \tag{29}$$

$$\frac{\partial(B_{\mathrm{e}} v_{\theta_1} - v_0 B_{\theta_1})}{\partial z} + \lambda_{\mathrm{H}} B_{\mathrm{e}}\frac{\partial^2 B_{z_1}}{\partial z \partial r} = 0, \tag{30}$$

$$\frac{\partial(B_{\rm e}v_{z_1}-v_0B_{z_1})}{\partial z}-\lambda_{\rm H}B_{\rm e}\left(\frac{\partial^2B_{\theta_1}}{\partial r\partial z}+\frac{1}{r}\frac{\partial B_{\theta_1}}{\partial r}\right)=0, \tag{31}$$

$$\frac{1}{r}\frac{\partial(rv_{r_1})}{\partial r}+\frac{\partial v_{z_1}}{\partial z}=0, \tag{32}$$

$$\frac{1}{r}\frac{\partial(rB_{r_1})}{\partial r}+\frac{\partial B_{z_1}}{\partial z}=0, \tag{33}$$

$$\frac{1}{r}\frac{\partial(rB^{\rm V}_{r_1})}{\partial r}+\frac{\partial B^{\rm V}_{z_1}}{\partial z}=0, \tag{34}$$

$$\frac{\partial(rB^{\rm V}_{\theta_1})}{\partial z},\frac{\partial(rB^{\rm V}_{\theta_1})}{\partial r}=0, \tag{35}$$

$$\frac{\partial B^{\rm V}_{z_1}}{\partial r}=0, \tag{36}$$

where the subscript 1 stands for first-order quantities, whereas r, θ, and z refer to the r–, θ–, and z–component, respectively. $B_{\rm e}$ is B_0b_i for short.

In a similar way, we obtain for the free-boundary, Eqs. (14)–(16),

$$v_{r_1}=v_0\frac{\partial\xi_1}{\partial z}, \tag{37}$$

$$\pi_1=B_0b_0B^{\rm V}_{z_1}-\frac{B_0^2}{r_0}\xi_1, \tag{38}$$

$$B_{r_1}=B_{\rm e}\frac{\partial\xi_1}{\partial z},\quad B^{\rm V}_{r_1}=B_0b_0\frac{\partial\xi_1}{\partial z}. \tag{39}$$

For the conducting wall we have

$$B^{\rm V}_{r_1}=0. \tag{40}$$

As it is obvious from Eq. (26), the total pressure is a function of z and t only, $\pi_1=\pi_1(z,t)$.

Substituting Eqs. (32)–(33) into Eq. (28) it is obtained, after integrating in r, an expression for B_{r_1}, namely

$$B_{r_1}=\frac{\rho_0v_0}{B_{\rm e}}v_{r_1}-\frac{r}{2B_{\rm e}}\frac{\partial\pi_1}{\partial z}. \tag{41}$$

On integrating Eq. (29) in z, making use of Eq. (41), leads to

$$\frac{\partial B_{\theta_1}}{\partial z}=\frac{\rho_0v_0^2-B_{\rm e}^2}{\lambda_{\rm H}B_{\rm e}^2}v_{r_1}-\frac{v_0r}{2\lambda_{\rm H}B_{\rm e}^2}\frac{\partial\pi_1}{\partial z}. \tag{42}$$

Combining Eq. (30) together with Eqs. (27), (33), (41) and (42), yields

$$\frac{1}{r}\frac{\partial^2(rv_{r_1})}{\partial r^2}-\frac{1}{r^2}\frac{\partial(rv_{r_1})}{\partial r}+\frac{(B_{\rm e}^2-\rho_0v_0^2)^2}{\rho_0^2\lambda_{\rm H}^2v_0^2B_{\rm e}^2}v_{r_1}=\frac{(\rho_0v_0^2-B_{\rm e}^2)}{2\lambda_{\rm H}^2B_{\rm e}^2\rho_0^2v_0}r\frac{\partial\pi_1}{\partial z}, \tag{43}$$

a differential equation relating the perturbed radial velocity and the total pressure. In obtaining Eq. (43) we have assumed that $\lambda_{\rm H}$, v_0, ρ_0, $B_{\rm e}$, and $v_0^2-B_{\rm e}^2/\rho_0$ are non-zero. We have also supposed that $\partial\pi_1/\partial z$ is not identically zero.

For the vacuum region, Eqs. (34)–(36), we obtain the following solution:

$$B_{r_1}^{\rm V}=\frac{R_0^2-r^2}{2r}\frac{\partial B_{z_1}^{\rm V}}{\partial z}, \tag{44}$$

$$B_{\theta_1}^{\rm V}=0, \tag{45}$$

where $B_{z_1}^{\rm V}$ is a function of z and t only.

From Eqs. (37), (39) and (41) one arrives at

$$\frac{\partial\xi_1}{\partial z}=\frac{r_0}{2(\rho_0v_0^2-B_{\rm e}^2)}\frac{\partial\pi_1}{\partial z}, \tag{46}$$

and by using Eqs. (39), (44) and (46) it is obtained

$$\frac{\partial B_{z_1}^{\rm V}}{\partial z}=\frac{B_0b_0r_0^2}{(R_0^2-r_0^2)(\rho_0v_0^2-B_{\rm e}^2)}\frac{\partial\pi_1}{\partial z}. \tag{47}$$

A combination of Eqs. (46) and (47) together with Eq. (38) yields the equation that relates the equilibrium speed, v_0, with the other equilibrium quantities, namely

$$v_0^2=\frac{B_{\rm e}^2}{\rho_0}-\frac{B_0^2}{2\rho_0}\left(1-\frac{2b_0^2r_0^2}{R_0^2-r_0^2}\right). \tag{48}$$

This relation imposes a restriction upon the admissible values for v_0 and, as we will discuss later, is related to the stability of the soliton solution.

Returning to Eq. (43), we might rewrite it as

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\phi(r)\right)+\frac{\left(B_{\rm e}^2-\rho_0v_0^2\right)^2}{\lambda_{\rm H}^2\rho_0^2v_0^2B_{\rm e}^2}\frac{\phi(r)}{r}=\frac{(\rho_0v_0^2-B_{\rm e}^2)}{2\lambda_{\rm H}^2\rho_0^2v_0B_{\rm e}^2}r, \tag{49}$$

casting the perturbed radial velocity as $rv_{r_1}(r,z,t)=\phi(r)\partial\pi_1(z,t)/\partial z$, where $\phi(r)$ is a function to be determined. The ordinary differential equation (49) together with (48) define an eigenvalue problem for $\phi(r)$. The suitable boundary conditions for determining $\phi(r)$ are

$$\phi(0)=0, \tag{50}$$

$$\phi(r_0) = \frac{r_0^2 v_0}{2(\rho_0 v_0^2 - B_e^2)}. \tag{51}$$

Equation (51) comes of using Eqs. (37) and (46).

The solution to Eq. (49) is easily obtained, it is

$$\phi(r) = \phi_0 r J_1(\varrho r) + \frac{v_0 r^2}{2(\rho_0 v_0^2 - B_e^2)}, \tag{52}$$

where ϕ_0 is a constant and

$$\varrho = \frac{B_e^2 - \rho_0 v_0^2}{\lambda_H \rho_0 v_0 B_e}. \tag{53}$$

Equation (52) satisfies condition (50) and also satisfies Eq. (51) provided that $\phi_0 = 0$ or $\phi_0 \neq 0$, with $\varrho r_0 = x_{n1}$; x_{n1} denotes the n^{th} root of J_1, with J_1 being the ordinary Bessel function of first order. Here the first root, $x_{11} = 0$, is neglected because we have assumed for previous calculations that $v_e^2 \neq B_e^2/\rho_0$. The case $v_e^2 = B_e^2/\rho_0$ will be discussed later. Therefore the solution to Eq. (49) is

$$\phi(r) = \phi_0 r J_1\left(x_{n1}\frac{r}{r_0}\right) + \frac{v_0 r^2}{2(\rho_0 v_0^2 - B_e^2)}, \tag{54}$$

with $x_{n1} \neq 0$ and arbitrary ϕ_0, or

$$\phi(r) = \frac{v_0 r^2}{2(\rho_0 v_0^2 - B_e^2)}, \tag{55}$$

with $\phi_0 = 0$.

As the values of v_0 depend on the particular equilibrium state through B_e, b_0, b_i, ρ_0, r_0, and R_0 [see Eq. (48)], the condition $\varrho r_0 = x_{n1} \neq 0$ sets a restriction on the possible values of λ_H in terms of the specific equilibrium state. However, when one takes $\phi_0 = 0$, Eq. (55) satisfies Eqs. (50) and (51) without any additional condition upon λ_H.

4. Second order analysis

In this section we present the governing equations (only those required for our purposes) to second order and shall obtain the KdV equation for surface waves that are related with the first order perturbed total pressure evaluated at the free-surface. The required second-order equations are, for the plasma region:

$$\frac{\partial \pi_2}{\partial r} = \frac{\partial (B_e B_{r_1} - \rho_0 v_0 v_{r_1})}{\partial z} + \frac{(\rho_0 v_{\theta_1}^2 - B_{\theta_1}^2)}{r}, \tag{56}$$

$$\rho_0 \frac{\partial v_{\theta_1}}{\partial t} = \frac{\partial (B_e B_{\theta_2} - \rho_0 v_0 v_{\theta_2})}{\partial z} - \rho_0 \mathbf{v}_1 \cdot \nabla_\perp v_{\theta_1} + \mathbf{B}_1 \cdot \nabla_\perp B_{\theta_1} + \frac{(B_{\theta_1} B_{r_1} - \rho_0 v_{\theta_1} v_{r_1})}{r}, \tag{57}$$

$$\rho_0 \frac{\partial v_{z_1}}{\partial t} = \frac{\partial (B_e B_{z_2} - \rho_0 v_0 v_{z_2})}{\partial z} - \rho_0 \mathbf{v}_1 \cdot \nabla_\perp v_{z_1} + \mathbf{B}_1 \cdot \nabla_\perp B_{z_1} - \frac{\partial \pi_2}{\partial z}, \tag{58}$$

$$\frac{1}{r}\frac{\partial (r v_{r_2})}{\partial r} + \frac{\partial v_{z_2}}{\partial z} = 0, \tag{59}$$

$$\frac{1}{r}\frac{\partial (r B_{r_2})}{\partial r} + \frac{\partial B_{z_2}}{\partial z} = 0, \tag{60}$$

$$\frac{1}{r}\frac{\partial (r B^{\vee}_{r_2})}{\partial r} + \frac{\partial B^{\vee}_{z_2}}{\partial z} = 0, \tag{61}$$

$$\frac{\partial (r B^{\vee}_{\theta_2})}{\partial z}, \quad \frac{\partial (r B^{\vee}_{\theta_2})}{\partial r} = 0, \tag{62}$$

$$\frac{\partial B^{\vee}_{z_2}}{\partial r} = \frac{\partial B^{\vee}_{r_1}}{\partial z}, \tag{63}$$

with $\nabla_\perp = \hat{e}_r \partial/\partial r + \hat{e}_z \partial/\partial z$ and $\pi_2 = p_2 + \frac{1}{2}(B^2_{\theta_1} + B^2_{z_1} + 2B_0 B_{z_2})$.

At the free-boundary we have

$$\frac{\partial \xi_1}{\partial t} + v_0 \frac{\partial \xi_2}{\partial z} + v_{r_1}\frac{\partial \xi_1}{\partial r} + v_{z_1}\frac{\partial \xi_1}{\partial z} - v_{r_2} = 0, \tag{64}$$

$$\pi_2 = B^{\vee}_{z_0}\xi_1 \frac{\partial B^{\vee}_{z_1}}{\partial r} + \frac{1}{2} B^{V^2}_{z_1} + B^{\vee}_{z_0} B^{\vee}_{z_2} + \frac{3B_0^2}{2r_0^2}\xi_1^2 - \frac{B_0^2}{r_0}\xi_2, \tag{65}$$

$$\xi_1 \frac{\partial B_{r_1}}{\partial r} + B_{r_2} - B_e \frac{\partial \xi_2}{\partial z} - B_{z_1}\frac{\partial \xi_1}{\partial z} = 0, \tag{66}$$

$$\xi_1 \frac{\partial B^{\vee}_{r_1}}{\partial r} + B^{\vee}_{r_2} - B^{\vee}_{z_0}\frac{\partial \xi_2}{\partial z} - B^{\vee}_{z_1}\frac{\partial \xi_1}{\partial z} = 0, \tag{67}$$

whereas at the conducting wall, $r = R_0$, we have

$$B^{\vee}_{r_2} = 0. \tag{68}$$

Integrating Eq. (56) in r yields

$$\pi_2 = \int_0^r P(r', z, t)\, dr' - Q(z, t), \tag{69}$$

where $Q(z,t)$ is a function to be determined and

$$P = \frac{\partial(B_e B_{r_1} - \rho_0 v_0 v_{r_1})}{\partial z} + \frac{1}{r}(\rho_0 v_{\theta_1}^2 - B_{\theta_1}^2). \tag{70}$$

By using Eqs. (58), (60) and (69), it is obtained

$$B_{r_2} = \frac{\rho_0}{rB_e}\int_0^r r' R(r',z,t)\,dr' - \frac{\rho_0}{B_e r}\int_0^r \int_0^{r'} r' \frac{\partial P(r'',z,t)}{\partial z}\,dr''\,dr' + \frac{\rho_0 r}{2B_e}\frac{\partial Q(z,t)}{\partial z}, \tag{71}$$

where

$$R(r',z,t) = \frac{1}{\rho_0}\mathbf{B}_1 \cdot \nabla_\perp B_{z_1} - \frac{\partial v_{z_1}}{\partial t} - v_0 \frac{\partial v_{z_2}}{\partial z} - \mathbf{v}_1 \cdot \nabla_\perp v_{z_1}. \tag{72}$$

On the other hand, the solutions for the vacuum region are

$$\frac{\partial B_{z_2}^{\mathrm{V}}}{\partial z} = \left[\frac{R_0^2}{2}\ln\left(\frac{R_0}{r}\right) - \frac{R_0^2 - r^2}{4}\right]\frac{\partial^3 B_{z_1}^{\mathrm{V}}}{\partial z^3} + \frac{\partial S(z,t)}{\partial z}, \tag{73}$$

$$B_{r_2}^{\mathrm{V}} = -\left[\frac{R_0^2 r}{4}\ln\left(\frac{R_0}{r}\right) + \frac{R_0^4 - r^4}{16r}\right]\frac{\partial^3 B_{z_1}^{\mathrm{V}}}{\partial z^3} + \frac{R_0^2 - r^2}{2r}\frac{\partial S(z,t)}{\partial z}, \tag{74}$$

where $S(z,t)$ is a function to be determined.

From Eqs. (67) and (74) it is obtained

$$\frac{R_0^2 - r_0^2}{2r_0}\frac{\partial S}{\partial z} = B_{z_0}^{\mathrm{V}}\frac{\partial \xi_2}{\partial z} + B_{z_1}^{\mathrm{V}}\frac{\partial \xi_1}{\partial z} - \xi_1 \frac{\partial B_{r_1}^{\mathrm{V}}}{\partial r} + \left[\frac{R_0^2 r_0}{4}\ln\left(\frac{R_0}{r_0}\right) + \frac{R_0^4 - r_0^4}{16r_0}\right]\frac{\partial^3 B_{z_1}^{\mathrm{V}}}{\partial z^3}, \tag{75}$$

at $r = r_0$.

By using Eqs. (66) and (71) we have

$$\frac{\partial Q}{\partial z} = \frac{2}{r_0^2}\int_0^{r_0}\int_0^{r'} r' \frac{\partial P(r'',z,t)}{\partial z}\,dr''\,dr' - \frac{2\rho_0}{r_0^2}\int_0^{r_0} r' R(r',z,t)\,dr' - \frac{2B_e \xi_1}{r_0}\frac{\partial B_{r_1}}{\partial r} + \frac{2B_e^2}{r_0}\frac{\partial \xi_2}{\partial z} + \frac{2B_e B_{z_1}}{r_0}\frac{\partial \xi_1}{\partial z}, \tag{76}$$

evaluating at $r = r_0$.

Now, differentiating π_2, as given by Eq. (65), with respect to z and using Eq. (69) at $r = r_0$ and (76), we obtain an expression for π_1 that involves only perturbed quantities of first order except for v_{z_2}, $B^{\rm V}_{z_2}$, and the terms containing ξ_2. v_{z_2} can be expressed in terms of first order quantities by using Eqs. (59) and (64). Likewise, we express $B^{\rm V}_{z_2}$ in terms of first order quantities by utilizing Eqs. (73) and (75). The terms involving ξ_2 disappear by virtue of Eq. (48). Hence the remaining equation involves only physical variables of first order which can be expressed in terms of $\pi_1(z,t)$ using the results of the preceding section. After some long but straightforward algebraic work, one gets

$$M_1 \frac{\partial \pi_1}{\partial t} + M_2 \pi_1 \frac{\partial \pi_1}{\partial z} + M_3 \frac{\partial^3 \pi_1}{\partial z^3} = 0, \tag{77}$$

where

$$M_1 = -\frac{2\rho_0 v_0}{\rho_0 v_0^2 - B_{\rm e}^2}, \tag{78}$$

$$\begin{aligned} M_2 = {} & \frac{3B_0^2}{4\left(\rho_0 v_0^2 - B_{\rm e}^2\right)^2} + \frac{B_0^2 b_0^2 r_0^2 (R_0^2 + 5r_0^2)}{2(R_0^2 - r_0^2)^2 (\rho_0 v_0^2 - B_{\rm e}^2)^2} \\ & - \frac{2\rho_0 v_0 r_0 \phi'(r_0) - 2\rho_0 v_0 \phi(r_0) - r_0^2}{2r_0^2(\rho_0 v_0^2 - B_{\rm e}^2)} + \frac{1}{\rho_0 v_0^2 - B_{\rm e}^2} \\ & + \frac{\rho_0 v_0^2 - B_{\rm e}^2}{2\lambda_{\rm H}^2 \rho_0 v_0^2 B_{\rm e}^4} \int_0^{r_0} \frac{1}{r} \left[2(\rho_0 v_0^2 - B_{\rm e}^2) \frac{\phi(r)}{r} - v_0 r \right]^2 dr \\ & - \frac{2}{r_0^2 B_{\rm e}^2} \int_0^{r_0} r \left[1 - \rho_0 v_0 \frac{\phi'(r)}{r} \right]^2 dr \\ & - \frac{\rho_0 v_0^2 - B_{\rm e}^2}{\lambda_{\rm H}^2 \rho_0 v_0^2 B_{\rm e}^2 r_0^2} \int_0^{r_0} \int_0^{r} \frac{r}{\sigma} \left[2(\rho_0 v_0^2 - B_{\rm e}^2) \frac{\phi(\sigma)}{\sigma} - v_0 \sigma \right]^2 d\sigma\, dr \\ & + \frac{2\rho_0}{r_0^2} \int_0^{r_0} \left[\frac{\phi'^2(r)}{r} - \phi(r) \frac{d}{dr} \left(\frac{\phi'(r)}{r} \right) \right] dr, \end{aligned} \tag{79}$$

$$\begin{aligned} M_3 = {} & \frac{r_0^2}{8} + \frac{B_0^2 b_0^2 r_0^2}{2(R_0^2 - r_0^2)(\rho_0 v_0^2 - B_{\rm e}^2)} \left[R_0^2 \ln\left(\frac{R_0}{r_0}\right) - \frac{R_0^2 - r_0^2}{2} \right] \\ & + \frac{B_0^2 b_0^2 r_0^2}{(R_0^2 - r_0^2)^2 (\rho_0 v_0^2 - B_{\rm e}^2)} \left[\frac{R_0^2 r_0^2}{2} \ln\left(\frac{R_0}{r_0}\right) + \frac{R_0^4 - r_0^4}{8} \right]. \end{aligned} \tag{80}$$

It is clear from Eq. (78) that $M_1 \neq 0$ for any non-zero value of v_0. Now, dividing Eq. (77) by M_1 and making the following change of variables:

$$\pi_1 = \left(\frac{M_1}{M_2}\right) \left(\frac{M_3}{M_1}\right)^{1/3} \psi(z,t), \tag{81}$$

$$z = \left(\frac{M_3}{M_1}\right)^{1/3} \eta, \tag{82}$$

the KdV equation is obtained,

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial \eta} + \frac{\partial^3 \psi}{\partial \eta^3} = 0. \tag{83}$$

(By the way, notice that M_3/M_1 plays in dispersive media the analogous role that the Reynolds number in viscous media.) As we have considered the evolution of localized initial disturbances, $|\pi_1(z,0)| \to 0$ as $|z| \to \infty$ and $\pi_1(z,0) = \pi_0(z)$, then the same is valid for ψ [see Eq. (81)].

If we move to a frame of reference moving at speed u in the η direction, a suitable change of variables is $\zeta = \eta - ut$, then Eq. (83) has the well known stationary solution [31,32] of soliton-type given by

$$\psi(\zeta) = 3u \operatorname{sech}^2\left(\tfrac{1}{2}\sqrt{u}\,\zeta\right), \tag{84}$$

assuming that $uM_1/M_3 > 0$.

5. DISCUSSION AND CONCLUDING REMARKS

In this paper we have shown the existence of solitary waves at the plasma-vacuum interface of a plasma column confined into a perfectly conducting vessel. These waves are in fact related to a tendency of the plasma column to sustain the balance of the total pressure at the free-surface against a localized disturbance, and result from the balance between nonlinearity and dispersion. The linear previously known [22–25] dispersion relations were the starting point for this work. From them we were able to obtain suitable ordering in terms of a positive small parameter related with the Hall term which is the cause of the dispersive behavior. It was also shown that the linear dispersion relations reduce, in the long wavelength limit, to those resembling shallow water waves. This suggested the presence of solitary waves with the ordering previously obtained. Indeed, an expansion in power series leads to the Korteweg-de Vries equation for which a soliton is a solution. That is, in the regime we have analyzed, the disturbance propagates along the axis of the cylinder like an undeformable pulse (or solitary wave) and its amplitude is proportional to the first order total pressure at the plasma-vacuum surface.

Here we have considered only MHD waves with long wavelength. In the long wavelength limit, the charge separation between electrons and ions can be ignored, this allowed us to assume that the plasma is quasi-neutral.

It may be noted that the expansion given by Eqs. (20)–(24) is a good choice for the present analysis in the sense that the leading order terms, respectively, can be the largest among various ways of expanding. The treatment here used is related with a method of reduction based on a singular perturbation expansion due to Taniuti and Wei [33].

Next, we consider the particular case when $v_0^2 = B_e^2/\rho_0$, that is, when the flow velocity is equal to the Alfvén speed. In this case instead of Eq. (43) we get

$$\left\{\frac{d}{dr}\left[\frac{\sqrt{\rho_0}}{r}\frac{d\phi}{dr} - \frac{r^2}{2B_e}\right]\right\}\left\{\frac{\partial \pi_1}{\partial z}\right\} = 0.$$

This expression leads to either $\partial\pi_1/\partial z = 0$ which is irrelevant because it implies that $\pi_1 = \pi_1(t)$ at most, and therefore there is no localized perturbation, so that there does not exist a soliton, or, with $\partial\pi_1/\partial z$ undetermined but non-zero, we hold

$$\frac{d}{dr}\left[\frac{\sqrt{\rho_0}}{r}\frac{d\phi}{dr} - \frac{r^2}{2B_e}\right] = 0$$

and might recover the solitary solution. This twofold implication is not surprising; it has a simple explanation on physical backgrounds. The nondissipative MHD description of an axisymmetric plasma possesses, in general, three singular points [34,35] corresponding, respectively, to magnetoacoustic waves, and slow and fast MHD waves. Here, by singular points we mean that such values correspond to values of the Alfvén Mach number Q at which the governing system of equations changes from *elliptic* to *hyperbolic* and viceversa. Thus, ellipticity holds over the intervals $0 \leq Q^2 < \beta$, $Q_s^2 < Q^2 < 1$, and $1 < Q^2 < Q_f^2$. Here Q_s and Q_f denote, respectively, the dimensionless slow and fast waves in MHD, and $\beta = v_S^2/(v_A^2 + v_S^2)$. Consequently, in such regions there are neither hyperbolic nor dispersive wave equations. On the other hand, the point $v_0 = v_A$ seems to be a point of transition but this is not the case; in fact, a close examination reveals that it corresponds to an undetermined situation in the following sense. If $\mathcal{D}(Q)$ denotes the determinant characterizing the type of equation, then $\mathcal{D}(Q) < 0$ defines an elliptic system, whereas $\mathcal{D}(Q) > 0$ defines a hyperbolic system, and $\mathcal{D}(Q) = 0$ remains undetermined. In the above mentioned case, when $Q = 1$ (viz., $v_0 = v_A$), we have $\mathcal{D}(1) = 0$, the undetermined case, which gives rise to the twofold implication.

It is important to mention that the soliton solutions for the parallel case have been used as a basis for the description of MHD turbulence in the solar wind by several authors [36–38]. But, as was pointed out by Campos and Isaeva [39], the Hall effect on Alfvén waves in the solar wind is significant beyond the $1AU$. The solar wind is an abundant source of finite amplitude hydromagnetic turbulence and so it can be regarded as a natural plasma laboratory where nonlinear theories of finite amplitude waves can be tested. On the other hand, large-amplitude hydromagnetic waves are present with a variety of waveforms in association with interplanetary shocks [40] and in the environment of comets. Also it should be noted that MHD fluctuations have been observed in the Earth's magnetosphere (specifically, at the magnetotail) with a period comparable to the local proton gyroperiod (about $5 \sim 12$ s), and it has also been observed that such fluctuations have large amplitude. Therefore, the approximation and the model here considered are applicable. In addition, as Roberts [41] has pointed out, the solar magnetic flux tubes can support solitary waves. Consequently, the present treatment may be relevant to such phenomena.

One last point to be discussed is the stability of the obtained solution. The corresponding stability condition, of course, is related with the sign of v_0, Eq. (48). As mentioned in the Introduction, when the nondissipative plasma column is incompressible and the equilibrium current density is zero [23–25] the dispersion relation is the same for both the MHD and the HMHD model [Eq. (1)], thus one could expect that the stability criteria be the same. In fact, when the plasma completely fills the cylindrical vessel, the criteria for linear and nonlinear stability are analogous in both models [19,42]. Moreover, as we have shown, the soliton solution persists in both models. Thus, it is highly plausible that the stability criterion, in the present case, be that obtained by Tayler [43] for the corresponding situation in the MHD description, $b_i^2 > \frac{1}{2} - \frac{b_0^2 r_0^2}{R_0^2 - r_0^2}$ for axisymmetric perturbations (*viz.*, $m = 0$). It is clear from Eq. (48) that in the present case such condition is fulfilled trivially taking into account that v_0 is a real number. Regarding the ordering of the physical variables, the way it is done here is usual; however a detailed discussion of this point and its relation with the stability conditions deserves a further study.

In this paper we restrict our study to the case of incompressible plasmas but apparently the compressibility effects only modify the value of the coefficients in the nonlinear terms but not the dispersive term, as a consequence the resulting behavior would be similar to that discussed here.

In concluding this section, it should be mentioned that to examine the effect of removing the condition of axisymmetric perturbations, upon the existence of soliton solutions, is beyond the scope of the present study, but it is undoubtedly an interesting question.

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