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Cooper pairing with a one-dimensional δ-potential

V.C. AGUILERA-NAVARRO* AND M. DE LLANO
Physics Department, North Dakota State University
Fargo, ND 58105 USA

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ABSTRACT. Measurements of quantized magnetic flux establish that pairing is empirically present in all known superconductors, whether the old low-temperature elemental ones or the newer high-temperature cuprates. The pairing phenomenon is explained and derived, analytically and graphically, in a simple one-dimensional model with an attractive δ-potential. The model displays a smooth continuous evolution from the loosely-bound, strongly-overlapping Cooper pair extreme reminiscent of low-temperature, to the well-separated, tightly-bound composite bosons extreme more characteristic of high-temperature superconductivity. This latter property makes it useful in addressing *exotic* superconductivity where strong suggestions of a link with Bose condensation emerge from recent experiments. The interaction model does not exhibit all the physical properties of actual electron-electron (or hole-hole) pairing forces in superconductors. However, it lends itself to a simple, pedagogically useful solution at the Cooper problem level in terms of combined analytical, graphical and numerical techniques which are standard.

RESUMEN. La acumulación por pares (apareamiento) en un gas de electrones (o huecos) parece ser un hecho experimental en todos los superconductores, ya sean los convencionales de baja temperatura crítica o los cupratos recientes con altas temperaturas críticas. Deducimos este fenómeno, analítica y gráficamente, a partir de un modelo didáctico unidimensional sencillo que consiste en una interacción atractiva tipo delta de Dirac, resuelto a la manera de Cooper. Aparece una evolución suave y continua entre un régimen de pares débilmente ligados y fuertemente trasplantados entre sí (que recuerda el caso de la superconductividad de bajas temperaturas críticas), y el régimen de bosones compuestos fuertemente amarrados y bien separados entre sí que caracteriza más a la superconductividad moderna de altas temperaturas.

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1. INTRODUCTION

The Cooper fermion-pair problem [1] consists in solving the Schrödinger equation (in the momentum representation) for two fermions interacting via a potential $V(r)$, and which cannot scatter by phonon exchange into single-fermion states already occupied by the $N-2$ background fermions, where $N \simeq 10^{23}$. The two fermions will maximize their mutual attractive interaction if they have opposite spins and zero center-of-mass motion. Thus, if $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2 = 0$ so that $\mathbf{k}_1 = -\mathbf{k}_2 \equiv \mathbf{k}$, and the intrinsic space pair

*On leave of absence from Instituto de Física Teórica, UNESP, São Paulo, 01405 SP, Brazil, with a grant from FAPESP, Brazil.

wave function is $\phi(\mathbf{r}) = \Sigma_{\mathbf{k}} C_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$, the coefficients $C_{\mathbf{k}}$ then satisfy the algebraic equations

$$2\epsilon_k^0 C_{\mathbf{k}} + \sum_{\mathbf{k}'} C_{\mathbf{k}'} V_{\mathbf{k}',\mathbf{k}} = EC_{\mathbf{k}}, \tag{1}$$

$$C_{\mathbf{k}} \equiv 0 \quad (k < k_{\mathrm{F}}), \tag{2}$$

where $\epsilon_k^0 \equiv \hbar^2 k^2/2m$, E is the pair energy eigenvalue and, if L^D is the D-dimensional system volume, then

$$V_{\mathbf{k}',\mathbf{k}} \equiv L^{-D} \int d^D r\, e^{-i\mathbf{k}'\cdot\mathbf{r}} V(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

For the *net* electron-electron interaction, Cooper employed what is now called the "BCS model interaction": $V_{\mathbf{k}',\mathbf{k}} = -V$ if $E_{\mathrm{F}} < \epsilon_{\mathbf{k}}^0,\ \epsilon_{\mathbf{k}'}^0 < E_{\mathrm{F}} + \hbar\omega_D$, and $= 0$ otherwise, where $\hbar\omega_D$ is the maximum energy a phonon can have, E_{F} is the Fermi energy (which is fixed by the carrier density),and $V > 0$. This simplification immediately leads to the simple eigenvalue equation

$$1 = V \sum_k{}' \frac{1}{2\epsilon_k^0 - E}, \tag{4}$$

where the prime on the summation sign means the k–sum is restricted such that $E_{\mathrm{F}} < \epsilon_k^0 < E_{\mathrm{F}} + \hbar\omega_D$. The sum in (4) can be converted to an energy integral if the density of states $g(\varepsilon) \equiv (L/2\pi)^D\, d^D k/d\varepsilon$ is introduced, so that (4) becomes

$$1 = V \int_{E_{\mathrm{F}}}^{E_{\mathrm{F}}+\hbar\omega_D} d\varepsilon\, \frac{g(\varepsilon)}{2\varepsilon - E} \simeq V g(E_{\mathrm{F}}) \int_{E_{\mathrm{F}}}^{E_{\mathrm{F}}+\hbar\omega_D} \frac{d\varepsilon}{2\varepsilon - E}, \tag{5}$$

where the last step follows in $3D$ from the empirical physical fact that in metals $\hbar\omega_D \ll E_{\mathrm{F}}$, since the former is $\sim 10^2$ K while the latter is $\sim 10^4$ K to 10^5 K. Note that this reduces the problem to a $2D$ one, where $g(\varepsilon)$ is a constant [2] independent of ε. The remaining integral is elementary and gives

$$E = 2E_{\mathrm{F}} - \frac{2\hbar\omega_D}{e^{1/\lambda} - 1} \xrightarrow[\lambda\to 0]{} 2E_{\mathrm{F}} - 2\hbar\omega_D e^{-1/\lambda} \qquad (3D), \tag{6}$$

where $\lambda \equiv g(E_{\mathrm{F}})V/2$.

This shows a *lowering* in energy of the pair, relative to the interactionless case, and constitutes the well-known "Cooper pairing", which is the fundamental ingredient of all superconductivity, whether low- or high-temperature.

2. Attractive δ-potential model in $1D$

To illustrate and better exhibit this fundamental nature of Cooper pairing, we wish to solve the same problem in $1D$ for the interaction model $V(r) = -v_0\delta(x)$, with $v_0 > 0$

and $x \equiv x_1 - x_2$. This means that $\hbar\omega = \infty$, and implies an unphysical property of the attractive pairing force in that it allows for arbitrarily large momenta transfers. But it is precisely this shortcoming in the model interaction, as we shall see, that

$$1 = \frac{v_0}{L} \sum_k{}' \frac{1}{2\epsilon_k^0 - E}, \tag{7}$$

with the prime now meaning only that $E_F < \epsilon_k^0 < \infty$.

If this restriction were removed, (7) would become for $E < 0$

$$1 = \frac{v_0}{L} \frac{L}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\hbar^2 k^2/m + |E|} = \frac{v_0}{2\hbar} \sqrt{m/|E|}, \tag{8}$$

which on squaring gives the exact two-body Schrödinger result $E = -mv_0^2/4\hbar^2$ for the eigenvalue E, as expected. This result is familiar from elementary quantum mechanics [3].

Going back to the restriction $E_F < \epsilon_k^0 < \infty$ in the corresponding $1D$ Cooper problem, instead of (5) we now have, exactly,

$$1 = \frac{v_0}{L} C_1 \int_{E_F}^{\infty} d\varepsilon \, \frac{\varepsilon^{-1/2}}{2\varepsilon - E}, \tag{9}$$

where the $1D$ density of states [2] $g(\varepsilon) = C_1 \varepsilon^{-1/2}$, with $C_1 = \sqrt{m/2}\, L/\pi\hbar$, has been inserted. Introducing the dimensionless quantity $\epsilon \equiv E/2E_F$, the integral in (9) yields

$$\frac{1}{2\sqrt{E_F}} \frac{1}{\sqrt{\epsilon}} \ln\left(\frac{1+\sqrt{\epsilon}}{1-\sqrt{\epsilon}}\right), \qquad \text{if } \epsilon > 0; \tag{10}$$

$$\frac{1}{\sqrt{E_F}} \frac{1}{\sqrt{|\epsilon|}} \left(\frac{\pi}{2} - \arctan \frac{1}{\sqrt{|\epsilon|}}\right), \qquad \text{if } \epsilon < 0. \tag{11}$$

The Fermi energy $E_F = \hbar^2 k_F^2/2m$ is related to the $1D$ number density $\rho = N/L = 2k_F/\pi$, namely $E_F = \hbar^2\pi^2\rho^2/8m$. Defining the dimensionless (coupling) quantity $\lambda \equiv mv_0/\hbar^2\rho$ the prefactor in (9) is then just $2\lambda\sqrt{E_F}/\pi^2$, so that Eqs. (9), (10) and (11) lead to two distinct implicit transcendental equations for ϵ, namely,

$$\frac{\pi^2\sqrt{\epsilon}}{\lambda} = \ln\left(\frac{1+\sqrt{\epsilon}}{1-\sqrt{\epsilon}}\right), \qquad \text{if } \epsilon > 0; \tag{12}$$

$$\frac{\pi^2\sqrt{|\epsilon|}}{\lambda} = \pi - 2\arctan \frac{1}{\sqrt{|\epsilon|}} \qquad \text{if } \epsilon < 0. \tag{13}$$

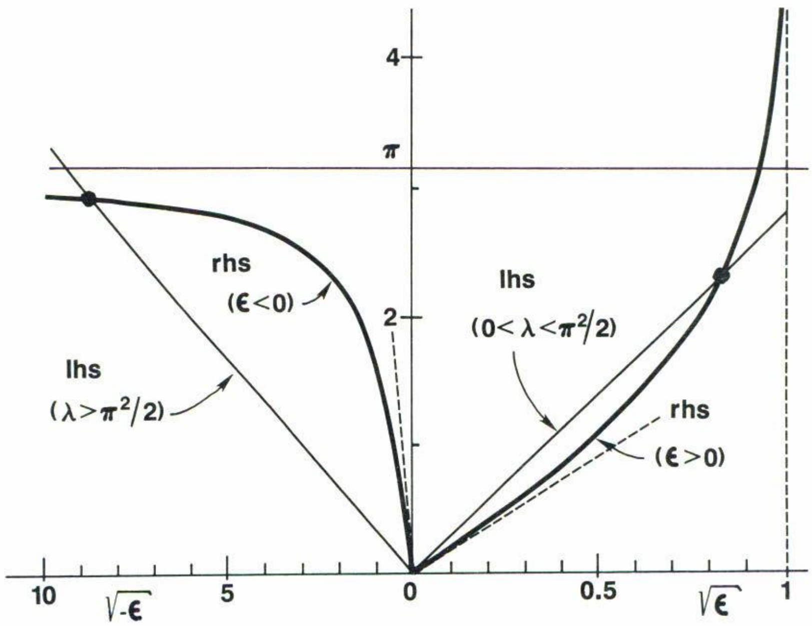

FIGURE 1. Graphical solution of Eq. (12) (right half) for the typical value of $\lambda = 3.5$, as explained in text, and of Eq. (13) (left half) for the typical value of $\lambda = 30$. Note that scales on the horizontal axis are different for right and left halves.

Expanding both rhs members we see that they both behave like $2\sqrt{|\epsilon|} + O(|\epsilon|^{3/2})$ for small $|\epsilon|$, but (12) is concave up in ϵ while (13) is concave down in $|\epsilon|$ (see Fig. 1). Consequently, there will be non-trivial ($|\epsilon| > 0$) solutions whenever

$$0 < \lambda < \pi^2/2, \qquad \text{for } \epsilon > 0; \tag{14}$$

$$\pi^2/2 < \lambda, \qquad \text{for } \epsilon < 0, \tag{15}$$

with the value $\lambda = \pi^2/2$ corresponding to $\epsilon = 0$. This concludes the *analytical* aspects of the problem. To proceed, we must apply either *graphical* or *numerical* methods.

Figure 1 depicts the graphical solutions of Eqs. (12) and (13). The curves in the right panel refer to the rhs (thick curve) and lhs (thin curve) of (12) for the typical value of $\lambda = 3.5$. Those on the left refer to the rhs (thick curve) and lhs (thin curve) of (13). The dashed straight lines are the asymptotes $2\sqrt{|\epsilon|}$ common to both rhs members of (12) and (13) for small $|\epsilon|$. The large dots designate the rhs and lhs intersections whereby the *solution* ϵ can be found in either case $\epsilon > 0$ or $\epsilon < 0$. These solutions are numerically summarized in Table I, and graphically displayed in Fig. 2 as a function of $1/\lambda \equiv \hbar^2\rho/mv_0$. This variable has a simple physical meaning: since the pair wave function in vacuum is $\exp(-mv_0|x|/2\hbar^2)$ [3], $1/\lambda$ is proportional to the pair diameter in units of the average interparticle spacing $L/N = 1/\rho$.

Consider the two extremes of $\lambda \to 0^+$ and $\lambda \to \infty$, which are, respectively, the weakly-bound, strongly-overlapping Cooper pair limit and the tightly-bound (point boson) "dimer" limit. Let $\Delta \equiv 2E_F - E$ be the (positive) binding energy of the weakly-bound

TABLE I. Cooper pairing eigenvalues $\epsilon \equiv E/2E_F$ for different inverse coupling $1/\lambda \equiv \hbar^2\rho/mv_0$ in the $1D$ attractive δ–potential model, obtained from the graphical solution shown in Fig. 1.

$1/\lambda$	ϵ	$1/\lambda$	ϵ
0	$-\infty$	0.090	−7.5738
0.010	−972.4000	0.100	−5.6443
0.020	−232.6500	0.203	0
0.030	−98.6600	0.300	0.7243
0.040	−52.7790	0.400	0.9119
0.050	−32.0040	0.500	0.9694
0.060	−20.9679	0.600	0.9890
0.070	−14.4625	1.000	0.9998
0.080	−10.3365		

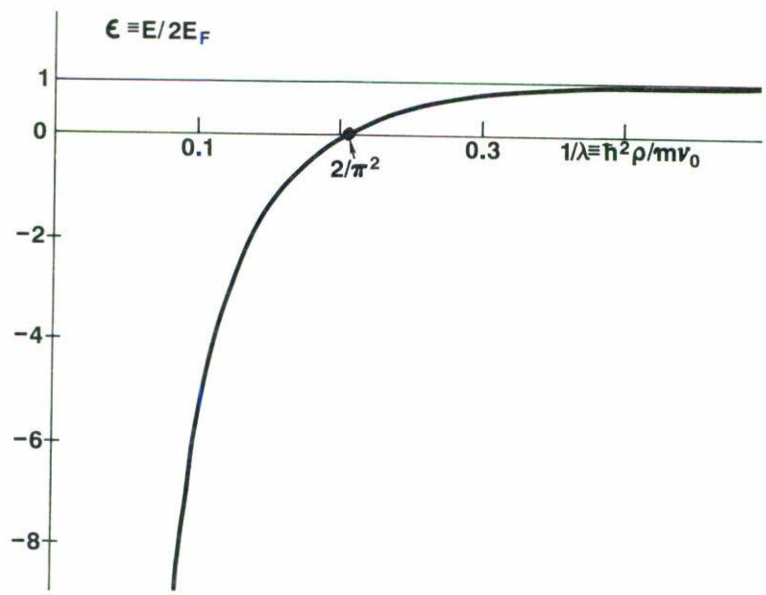

FIGURE 2. Evolution of the pair eigenvalue energy E from $2E_F$ to zero and down to negative values as the coupling v_0 of the attractive pairwise δ-potential is increased from zero.

Cooper pair. Defining $\delta \equiv \Delta/2E_F$, Eq. (12) can be expanded for $1-\epsilon \equiv \delta \to 0^+$ and yields $\pi^2/\lambda \simeq -\ln(\delta/4)$, or

$$\delta \underset{\lambda\to 0}{\simeq} 4e^{-\pi^2/\lambda}, \tag{16}$$

meaning that the Cooper pair binding energy for weak coupling can be expressed as

$$\begin{aligned} \Delta &\simeq 8E_F e^{-\pi^2\hbar^2\rho/mv_0} \\ &\neq \Delta_0 + \Delta_1 v_0 + \Delta_2 v_0^2 + \cdots, \end{aligned} \tag{17}$$

i.e., it *cannot* be expanded as a power series in λ, or v_0. The same type essential singularity in coupling Eq. (16) also appears in the $3D$ Cooper pair problem [1], in the $3D$ many-electron BCS theory gap (order) parameter and, finally, in the BCS superconductive transition temperature $T_c \simeq 1.13\Theta_D e^{-1/2\lambda}$, where $\Theta_D = \hbar\omega_D/k_B T_F$, with k_B the Boltzmann constant and λ defined as in Sect. 1 [4]. On the other hand, in the present section equivalent to finite v_0 but ρ(or $E_F) \to 0$, and so it should correspond to treating the two fermions in a *vacuum*. In this case, (13) becomes, since $|\epsilon| \equiv E/2E_F \to \infty$,

$$\frac{\pi^2\sqrt{|\epsilon|}}{\lambda} = \pi - \frac{2}{\sqrt{|\epsilon|}} + \cdots \tag{18}$$

or

$$\sqrt{|\epsilon|} \simeq \frac{\lambda}{\pi}, \tag{19}$$

which upon squaring again gives $E = -mv_0^2/4\hbar^2$, the binding energy of two particles in a vacuum, as it should.

Leggett [5] has asked the question "What happens to a Cooper pair as density is decreased?". He answers the question partially by showing that as density decreases the basic equation in BCS theory, the gap equation, becomes the two-particles Schrödinger equation in momentum space with *twice* the chemical p model, which can be shown to possess the same dynamics as the $3D$ electron gas "jellium" model [6], as straightforward: the Cooper pair becomes a point boson dimer (called a "diatomic molecule" by Leggett) reminiscent of "bipolarons". These objects play an increasing important role [7] in modern theories of high-temperature superconductivity where empirical "coherence lengths" (pair diameters) —typically 3 to 4 *order of magnitude larger* than the lattice spacing in *low*-temperature superconductors— can be comparable and even smaller than the average lattice spacing in the new cuprate superconductors. This latter class of materials, together with organic, Chevrel-phase, heavy-fermion and bismuthate superconductors, are known as "*exotic*" —to distinguish them from *conventional*, low-temperature, elemental superconductors. Exotic superconductors can be either 2D-like or 3D-like. Transition temperatures T_c for exotic superconductor range over three orders of magnitude (from tenths to over a hundred kelvins) but have recently been found experimentally [8] to scale like the corresponding Bose-Einstein condensation critical temperatures. This suggests the existence for $T > T_c$ of "pre-formed" boson charge carriers which are sufficiently well defined, distinct and non-overlapping-unlike the low-temperature, weakly-bound Cooper pairs of conventional BCS superconductors existing only when $T \leq T_c$ —which somehow Bose condense into the superconducting phase.

3. CONCLUSION

The Cooper problem of two fermions attracting pairwise via a δ–function potential in $1D$, and submersed in a background sea of $N-2$ inert fermions obeying the Pauli principle, can be solved exactly in analytical-graphical-numerical form. The relevant parameter is just the ratio of the pair size to the average interparticle spacing. As this ratio decreases from infinity (weakly-quasi-bound, strongly-overlapping Cooper pairs) to zero (tightly-bound well-separated composite boson dimers), the energy eigenvalue decreases smoothly from $2E_F$ to zero and finally to negative values characteristic of "real" bound pairs. When the ratio actually reaches the limit of zero (infinite dimer separation) one recovers the familiar bound state energy of the attractive δ-function potential. The model, in spite of being only $1D$, also highlights some recent views of both low- and high-temperature superconductivity in $2D$ and $3D$.

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