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Eigenfunctions of the curl operator via the method of adjoint operators

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ABSTRACT. The divergenceless eigenfunctions of the curl operator in spherical, circular cylindrical, parabolic cylindrical and elliptic cylindrical coordinates are obtained by means of the method of adjoint operators. In each case, the eigenfunctions of the curl operator are expressed in terms of a single scalar potential that satisfies the Helmholtz equation.

RESUMEN. Se obtienen las eigenfunciones sin divergencia del operador rotacional en coordenadas esféricas, cilíndricas circulares, cilíndricas parabólicas y cilíndricas elípticas por medio del método de operadores adjuntos. En cada caso, las eigenfunciones del operador rotacional se expresan en términos de un solo potencial escalar que satisface la ecuación de Helmholtz.

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1. INTRODUCTION

The eigenfunctions of the curl operator appear in various areas of theoretical physics. For instance, the source-free Maxwell equations in empty space can be written as $\nabla \cdot \mathbf{F} = 0$, $\nabla \times \mathbf{F} = \frac{i}{c}\dot{\mathbf{F}}$, where $\mathbf{F} \equiv \mathbf{E} + i\mathbf{B}$; therefore, assuming a time dependence of the form $e^{-i\omega t}$, the complex vector field $\mathbf{F}$ is a divergenceless eigenfunction of the curl operator. If a magnetic field $\mathbf{B}$ in a plasma is an eigenfunction of the curl operator, then the magnetic force density vanishes; hence, $\mathbf{B}$ is called a force-free field (see *e.g.*, Refs. [1–4]). The eigenfunctions of the curl operator are also useful in the expansion of vector fields in electromagnetism, fluid dynamics and acoustics (see, *e.g.*, Refs. [5–7] and the references cited therein).

The divergenceless eigenfunctions of the curl operator in circular cylindrical and spherical coordinates can be expressed in terms of partial derivatives of scalar potentials that satisfy the Helmholtz equation [1,7–9]. The fact that the most general eigenfunction of the curl operator with vanishing divergence is determined by a single scalar potential has been demonstrated in Ref. [9], where the eigenvalue equation is solved by separation of variables in spherical and cylindrical coordinates, making use of the spin-weighted harmonics. In this paper, a simple derivation of the expressions for the eigenfunctions of the curl operator in terms of scalar potentials in spherical, circular cylindrical, parabolic cylindrical and elliptic cylindrical coordinates is given making use of the method of adjoint operators, which allows the reduction of systems of homogeneous linear partial differential equations to simpler equations [10,11].

2. Debye potentials for the eigenfunctions of the curl operator

The eigenvalue equation

$$\nabla \times \mathbf{u} = \lambda \, \mathbf{u}, \tag{1}$$

can be written in the form

$$\mathcal{E}(\mathbf{u}) = 0, \tag{2}$$

where $\mathcal{E}$ is the linear partial differential operator that maps vector fields into vector fields given by

$$\mathcal{E}(\mathbf{u}) \equiv \nabla \times \mathbf{u} - \lambda \, \mathbf{u}. \tag{3}$$

If the adjoint of a linear operator $\mathcal{A}$ that maps n-index tensor fields into m-index tensor fields is defined as that linear operator $\mathcal{A}^\dagger$ that maps m-index tensor fields into n-index tensor fields such that

$$\int g^{\alpha\beta\cdots}[\mathcal{A}(f_{\mu\nu\cdots})]_{\alpha\beta\cdots} \, dv = \int [\mathcal{A}^\dagger(g^{\alpha\beta\cdots})]^{\mu\nu\cdots} f_{\mu\nu\cdots} \, dv, \tag{4}$$

for any pair of square-integrable n-index and m-index tensor fields $f_{\mu\nu\cdots}$ and $g^{\alpha\beta\cdots}$, respectively, then one finds that[1]

$$(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger \mathcal{A}^\dagger, \qquad (\mathcal{A} + \mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger, \qquad (\mathcal{A}^\dagger)^\dagger = \mathcal{A} \tag{5}$$

and

$$\text{curl}^\dagger = \text{curl}, \qquad \text{grad}^\dagger = -\text{div}, \qquad \text{div}^\dagger = -\text{grad}, \tag{6}$$

therefore, $\mathcal{E}$ and the laplacian operator are self-adjoint (*i.e.*, $\mathcal{E}^\dagger = \mathcal{E}$ and $(\nabla^2)^\dagger = \nabla^2$).

If there exist linear operators $\mathcal{O}$, $\mathcal{T}$, $\mathcal{S}$ such that

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}, \tag{7}$$

then, using Eqs. (5) and the fact that $\mathcal{E}^\dagger = \mathcal{E}$, Eq. (7) yields

$$\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger. \tag{8}$$

Hence, if ψ satisfies the condition

$$\mathcal{O}^\dagger(\psi) = 0, \tag{9}$$

[1] The definition of the adjoint of an operator given here is slightly different from that used in Refs. [10,11]. The definition used here guarantees the uniqueness of the adjoint operator, which is essential to obtain Eqs. (5) and (6).

from Eq. (8) it follows that $\mathbf{u} = \mathcal{S}^\dagger(\psi)$ satisfies Eq. (2), which means that $\mathbf{u}$ is an eigenfunction of the curl operator.

In order to find operators $\mathcal{O}$, $\mathcal{T}$, $\mathcal{S}$, satisfying Eq. (7) it is convenient to introduce the vector field

$$\mathbf{K} \equiv \mathcal{E}(\mathbf{u}) = \nabla \times \mathbf{u} - \lambda\, \mathbf{u}. \tag{10}$$

Then, $\mathbf{K} = 0$ if and only if $\mathbf{u}$ is an eigenfunction of the curl operator. Taking the curl of Eq. (10) using the identity $\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$, one gets

$$\nabla \times \mathbf{K} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} - \lambda\, \nabla \times \mathbf{u}, \tag{11}$$

and taking now the curl of Eq. (11) one finds that

$$\nabla \times \nabla \times \mathbf{K} = -\nabla^2\, \nabla \times \mathbf{u} - \lambda\, \nabla\, (\nabla \cdot \mathbf{u}) + \lambda\, \nabla^2 \mathbf{u}. \tag{12}$$

Therefore, from Eqs. (11) and (12) one obtains the identity

$$\lambda\, \nabla \times \mathbf{K} + \nabla \times \nabla \times \mathbf{K} = -(\nabla^2 + \lambda^2)\, \nabla \times \mathbf{u}, \tag{13}$$

which implies that

$$\mathbf{r} \cdot (\lambda\, \nabla \times \mathbf{K} + \nabla \times \nabla \times \mathbf{K}) = -(\nabla^2 + \lambda^2)(\mathbf{r} \cdot \nabla \times \mathbf{u}) \tag{14}$$

and, similarly,

$$\hat{e}_z \cdot (\lambda\, \nabla \times \mathbf{K} + \nabla \times \nabla \times \mathbf{K}) = -(\nabla^2 + \lambda^2)(\hat{e}_z \cdot \nabla \times \mathbf{u}). \tag{15}$$

Thus, if $\mathbf{u}$ is an eigenfunction of the curl operator (*i.e.*, $\mathbf{K} = 0$), then the scalars $\mathbf{r} \cdot \nabla \times \mathbf{u} = (\mathbf{r} \times \nabla) \cdot \mathbf{u}$ and $\hat{e}_z \cdot \nabla \times \mathbf{u}$ satisfy the Helmholtz equation. As shown below, Eqs. (14) and (15) allow us to find the eigenfunctions of the curl operator adapted to spherical and cylindrical coordinates, respectively.

2.1. Spherical coordinates

Since $\mathbf{K} = \mathcal{E}(\mathbf{u})$, Eq. (14) is equivalent to an operator identity of the form (7) with

$$\begin{aligned} \mathcal{S}(\mathbf{K}) &\equiv -\mathbf{r} \cdot (\lambda\, \nabla \times \mathbf{K} + \nabla \times \nabla \times \mathbf{K}) \\ &= -\lambda\, (\mathbf{r} \times \nabla) \cdot \mathbf{K} - (\mathbf{r} \times \nabla) \cdot \nabla \times \mathbf{K}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathcal{T}(\mathbf{u}) &\equiv \mathbf{r} \cdot \nabla \times \mathbf{u}, \\ \mathcal{O}(\phi) &\equiv (\nabla^2 + \lambda^2)\phi. \end{aligned} \tag{17}$$

Therefore,

$$\mathbf{u} = \mathcal{S}^\dagger(\psi) = \lambda\, \mathbf{r} \times \nabla\psi + \nabla \times (\mathbf{r} \times \nabla\psi) \tag{18}$$

is a solution of Eq. (2), provided that $\mathcal{O}^\dagger(\psi) = 0$, *i.e.*,

$$(\nabla^2 + \lambda^2)\psi = 0. \tag{19}$$

As is very well known, the Helmholtz equation (19) admits separable solutions in spherical coordinates of the form

$$\psi(r, \theta, \varphi) = (A j_j(\lambda r) + B n_j(\lambda r))\, Y_{jm}(\theta, \varphi), \tag{20}$$

where A, B are arbitrary constants, j_l, n_l are spherical Bessel functions and the Y_{lm} are spherical harmonics. Substituting Eq. (20) into Eq. (18) one obtains an eigenfunction of the curl operator that is also an eigenfunction of the square of the total angular momentum J^2 and of the z-component of the total angular momentum J_z, with eigenvalues $j(j+1)$ and m, respectively [12]. Since a vector field has spin 1, j must be greater than, or equal to, 1. In fact, if one substitutes Eq. (20) with $j = 0$ into Eq. (18), one obtains $\mathbf{u} = 0$. (However, there exist eigenfunctions of curl that are eigenfunctions of J^2 with eigenvalue 0; they correspond to $\lambda = 0$ [9].)

The first term in the right-hand side of Eq. (18) can be written in the equivalent form $-\lambda\, \nabla \times (\mathbf{r}\psi)$, which shows that the eigenfunctions of the curl operator given by Eq. (18) have vanishing divergence. In fact, solving directly Eq. (1) in spherical coordinates, it turns out that all the divergenceless eigenfunctions of the curl operator can be written in the form (18) [9]. On the other hand, taking the divergence of Eq. (1) one finds that $0 = \lambda\, \nabla \cdot \mathbf{u}$, therefore the only eigenfunctions of the curl operator that can have a nonvanishing divergence are those with eigenvalue $\lambda = 0$. When $\lambda = 0$, Eq. (1) reduces to $\nabla \times \mathbf{u} = 0$, which implies that locally $\mathbf{u}$ is the gradient of some function, $\mathbf{u} = \nabla\phi$.

2.2. Cylindrical coordinates

Equation (15) is equivalent to an identity of the form (7) with

$$\begin{aligned}\mathcal{S}(\mathbf{K}) &\equiv -\hat{e}_z \cdot (\lambda\, \nabla \times \mathbf{K} + \nabla \times \nabla \times \mathbf{K}) \\ &= -\lambda\, (\hat{e}_z \times \nabla) \cdot \mathbf{K} - (\hat{e}_z \times \nabla) \cdot \nabla \times \mathbf{K},\end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathcal{T}(\mathbf{u}) &\equiv \hat{e}_z \cdot \nabla \times \mathbf{u}, \\ \mathcal{O}(\phi) &\equiv (\nabla^2 + \lambda^2)\phi.\end{aligned} \tag{22}$$

Therefore,

$$\mathbf{u} = \mathcal{S}^\dagger(\psi) = \lambda\, \hat{e}_z \times \nabla\psi + \nabla \times (\hat{e}_z \times \nabla\psi) \tag{23}$$

is an eigenfunction of the curl operator with eigenvalue λ provided that $\mathcal{O}^\dagger(\psi) = 0$, *i.e.*,

$$(\nabla^2 + \lambda^2)\psi = 0 \tag{24}$$

(*cf.* Eqs. (18) and (19)). Rewriting Eq. (23) as $\mathbf{u} = \nabla \times (-\lambda\, \hat{e}_z \psi + \hat{e}_z \times \nabla\psi)$, one finds that the eigenfunctions (23) have vanishing divergence.

The expression (23) is adapted to cylindrical coordinates (u, v, z), where

$$u = u(x, y), \qquad v = v(x, y) \tag{25}$$

and (x, y, z) are cartesian coordinates, since the operator $\hat{e}_z \times \text{grad}$ appearing in Eq. (23) involves partial derivatives with respect to u and v only. Looking for separable solutions of the Helmholtz equation (24) of the form

$$\psi(u, v, z) = \Phi(u, v)e^{ikz}, \tag{26}$$

one finds that $\Phi(u, v)$ satisfies the two-dimensional Helmholtz equation

$$(\Delta_2 + \alpha^2)\Phi = 0, \tag{27}$$

where Δ_2 is the Laplace operator on the plane and

$$\alpha^2 \equiv \lambda^2 - k^2. \tag{28}$$

If (u, v, z) are orthogonal coordinates, Eq. (23) amounts to

$$\mathbf{u} = -\left(\frac{\lambda}{h_2}\frac{\partial\psi}{\partial v} + \frac{1}{h_1}\frac{\partial^2\psi}{\partial z \partial u}\right)\hat{e}_1 + \left(\frac{\lambda}{h_1}\frac{\partial\psi}{\partial u} - \frac{1}{h_2}\frac{\partial^2\psi}{\partial z \partial v}\right)\hat{e}_2 + \Delta_2\psi\, \hat{e}_z, \tag{29}$$

where h_1, h_2 are the scale factors corresponding to the coordinates u and v, and $\hat{e}_1$ and $\hat{e}_2$ are unit vectors in the u–direction and the v–direction, respectively (we are assuming that $\{\hat{e}_1, \hat{e}_2, \hat{e}_z\}$ is a right-handed basis).

Equation (27) admits separable solutions in cartesian, polar, parabolic and elliptic coordinates, which are orthogonal (see, *e.g.*, Ref. [13]). In polar coordinates (ρ, φ), Eq. (27) has separable solutions of the form

$$\Phi(\rho, \varphi) = (AJ_m(\alpha\rho) + BN_m(\alpha\rho))e^{im\varphi}, \tag{30}$$

where A, B are arbitrary constants, J_m, N_m are Bessel functions and m is an integer. The eigenfunctions of curl given by Eqs. (23), (26) and (30) (called Chandrasekhar-Kendall eigenfunctions) are also eigenfunctions of the square of the linear momentum in the x–y plane, $P_1^2 + P_2^2$, and of the z–component of the total angular momentum, J_z, with eigenvalues α^2 and m, respectively. These eigenfunctions form a basis for the divergenceless vector fields [7]. By solving directly Eq. (1) in circular cylindrical coordinates, one finds that all the divergenceless eigenfunctions of the curl operator are given by Eqs. (23)–(24) [9].

In parabolic coordinates (u, v), which are defined by

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv, \tag{31}$$

the two-dimensional Helmholtz equation (27) admits separable solutions $\Phi(u, v) = U(u)V(v)$, where the separate functions U and V satisfy

$$\frac{d^2U}{du^2} + (-\alpha a + \alpha^2 u^2)U = 0, \qquad \frac{d^2V}{dv^2} + (\alpha a + \alpha^2 v^2)V = 0, \tag{32}$$

and a is a separation constant. Therefore, U and V can be expressed in terms of the parabolic cylinder functions (also called Weber functions) (see, *e.g.*, Refs. [8,13,14]). Similarly, in elliptic coordinates (ξ, η), which are given by

$$x = d \cosh\xi \cos\eta, \qquad y = d \sinh\xi \sin\eta, \tag{33}$$

where d is a constant scale factor, Eq. (27) admits separable solutions $\Phi(\xi, \eta) = U(\xi)V(\eta)$, where

$$-\frac{d^2U}{d\xi^2} + (b - d^2\alpha^2 \cosh^2\xi)U = 0, \qquad \frac{d^2V}{d\eta^2} + (b - d^2\alpha^2 \cos^2\eta)V = 0, \tag{34}$$

and b is a separation constant. The solutions of Eqs. (34) are linear combinations of Mathieu functions (see, *e.g.*, Refs. [8,13]).

3. Concluding remarks

One of the advantages of the procedure used here to obtain Eqs. (18) and (23) is that the coordinates are introduced only at the end, which simplifies the derivation of these expressions. It should be noticed that by substituting any solution of the scalar Helmholtz equation into Eq. (18) or (23), one gets an eigenfunction of the curl operator (in fact, one can also consider singular solutions of the scalar Helmholtz equation which generate well behaved vector fields [9]). However, the separable solutions of the form (20) and (26) are adapted to Eqs. (18) and (23), respectively, in the sense that they yield relatively simple expressions when written in terms of the basis induced by the corresponding coordinates (see, *e.g.*, Eq. (29)).

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