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Spinors in three dimensions. II

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ABSTRACT. A spinor calculus applicable to three-dimensional riemannian spaces of any signature is given. It is shown that any orthogonal transformation in three dimensions can be expressed in terms of 2×2 matrices and that a spinor defines a triad of vectors.

RESUMEN. Se presenta un cálculo espinorial aplicable a espacios riemannianos de dimensión tres de cualquier signatura. Se muestra que cualquier transformación ortogonal en tres dimensiones se puede expresar en términos de matrices 2×2 y que un espinor define una tríada de vectores.

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1. INTRODUCTION

The spinor formalism is a very powerful tool in general relativity and, in some applications, it is almost an indispensable tool. For a three-dimensional riemannian manifold with a positive definite metric, a spinor calculus can also be developed [1], which possesses the advantages of that employed in the four-dimensional space-time of general relativity.

Spinors can be defined in riemannian spaces of any dimension and with any signature (see, *e.g.*, Refs. [2,3]); however, for a given dimension, the group of spin transformations and the properties of the spinor equivalents of tensors depend on the signature of the metric. The aim of this paper is to extend the results of Ref. [1], developing a 2-spinor calculus for three-dimensional riemannian manifolds with any signature.

In Sect. 2 the relationship between spinor components and tensor components is established and in the specific case of signature $(++-)$, two alternative relationships are given (*cf.* also Refs. [3,4]). In Sect. 3 it is explicitly shown that the orthogonal transformations in three dimensions can be expressed in terms of 2×2 matrices with unit determinant, no matter what the signature of the metric. In Sects. 4 and 5 the Levi-Civita connection and the curvature are expressed in spinor form and in Sect. 6 the concept of spin-weight and the effect of the prime operation, which interchanges the basis spinors, are given. In Sect. 7 it is shown that a spinor defines an orthogonal basis. Lower-case Latin indices a, b, $\ldots$, run from 1 to 3 and capital Latin indices A, B, $\ldots$, run from 1 to 2.

2. CORRESPONDENCE BETWEEN TENSORS AND SPINORS

The components of a spinor, or a spinor field, in a three-dimensional space have the form $\psi^{AB\cdots}_{CD\cdots}$, where each of the indices A, B, C, $\ldots$, can take two values (*e.g.*, 1 and 2). Under a change of spinor basis, the components of any spinor transform according to

$$\psi'^{AB\ldots}_{CD\ldots} = (U^{-1})^A{}_R (U^{-1})^B{}_S \cdots U^T{}_C U^V{}_D \cdots \psi^{RS\ldots}_{TV\ldots}, \tag{1}$$

where $(U^A{}_B)$ is a non-singular matrix (the superscript labels rows and the subscript labels columns) and $((U^{-1})^A{}_B)$ is its inverse. Here and henceforth whenever an index occurs twice in a term, once up, once down, it is to be summed. The spinor indices are raised and lowered by means of

$$(\varepsilon_{AB}) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv (\varepsilon^{AB}), \tag{2}$$

according to the convention

$$\psi_A = \varepsilon_{AB}\psi^B, \qquad \psi^B = \varepsilon^{AB}\psi_A, \tag{3}$$

($i.e.$, $\psi_1 = \psi^2$, $\psi_2 = -\psi^1$) which implies that $\psi_A\phi^A = -\psi^A\phi_A$ and $\varepsilon^A{}_B = \delta^A_B$.

The compatibility of Eqs. (1) and (3) requires that the matrices $(U^A{}_B)$ appearing in Eq. (1) have unit determinant. Indeed, from Eq. (1) one has, for instance, $\psi'_A = U^B{}_A\psi_B$ which, in view of Eq. (3), amounts to $\varepsilon_{AC}\psi'^C = U^B{}_A\varepsilon_{BD}\psi^D$ and using again Eq. (1), $\varepsilon_{AC}(U^{-1})^C{}_D\psi^D = U^B{}_A\varepsilon_{BD}\psi^D$. The validity of this last equation for any ψ^D is equivalent to

$$\varepsilon_{AC} = U^B{}_A U^D{}_C \varepsilon_{BD}, \tag{4}$$

which amounts to the condition $\det(U^A{}_B) = 1$. (This means that ε_{AB} is invariant under the spin transformations (1).) Equation (4) can also be expressed as

$$(U^{-1})^A{}_B = -\varepsilon_{BC}U^C{}_D\varepsilon^{DA} = -U_B{}^A, \tag{5}$$

where we have made use of Eqs. (3).

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthogonal basis such that the inner product of $\mathbf{e}_a$ with itself is plus or minus one. Let

$$g_{ab} \equiv \mathbf{e}_a \cdot \mathbf{e}_b, \tag{6}$$

then (g_{ab}) is a diagonal matrix whose diagonal entries are $+1$ or -1. Thus, the metric is positive definite if $(g_{ab}) = \mathrm{diag}(1,1,1)$; the metric is indefinite if one of the diagonal entries of (g_{ab}) is different from the other two.

Proposition. Let (g_{ab}) be a diagonal matrix whose diagonal entries are $+1$ or -1, then there exist scalars σ_{aAB} such that

$$\sigma_{aAB} = \sigma_{aBA}, \tag{7}$$

$$\sigma_{aAB}\sigma_b{}^{AB} = -2g_{ab}. \tag{8}$$

Proof. Let us introduce the matrices

$$(s_{1AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (s_{2AB}) \equiv \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad (s_{3AB}) \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{9}$$

which are obtained by multiplying ε by each of the Pauli matrices, then it is easy to see that the matrices $(\sigma_{aAB}) = (s_{aAB})$ satisfy Eqs. (7) and (8) with $g_{ab} = \delta_{ab}$ (see, *e.g.*, Ref. [1]). Therefore, the matrices $(\sigma_{aAB}) = \lambda_a (s_{aAB})$ (no summation over a) satisfy Eqs. (7) and (8) with $(g_{ab}) = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)$. Thus, by choosing the λ_a in such a way that λ_a^2 is $+1$ or -1 one obtains matrices satisfying Eqs. (7-8) for any signature.

It may be noticed that the matrices (σ_{aAB}) are not uniquely determined by Eqs. (7) and (8). In fact, expressing conditions (7) and (8) in the form

$$\mathrm{tr}\ \sigma_a = 0, \tag{10}$$

$$\mathrm{tr}\ \sigma_a \sigma_b = 2 g_{ab}, \tag{11}$$

where σ_a is the matrix with entries

$$(\sigma_a)^A{}_B \equiv \varepsilon^{CA} \sigma_{aCB} \tag{12}$$

and tr denotes the trace, it is easy to see that given a set of matrices σ_a satisfying Eqs. (10-11), the matrices $\tilde{\sigma}_a = U^{-1} \sigma_a U$ also satisfy Eqs. (10-11) for any non-singular matrix U.

From Eqs. (7-8) it follows that

$$\sigma_{aAB} \sigma^a{}_{CD} = -(\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{BC} \varepsilon_{AD}). \tag{13}$$

(The indices a, b, ..., are lowered and raised by means of (g_{ab}) and its inverse (g^{ab}), *e.g.*, $\sigma^a{}_{CD} = g^{ab} \sigma_{bCD}$.) Therefore, if $t_{ab...c}$ are the components of an n-index three-dimensional tensor relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the components of its spinor equivalent are defined by

$$t_{ABCD...EF} \equiv \left(\tfrac{1}{\sqrt{2}} \sigma^a{}_{AB}\right) \left(\tfrac{1}{\sqrt{2}} \sigma^b{}_{CD}\right) \cdots \left(\tfrac{1}{\sqrt{2}} \sigma^c{}_{EF}\right) t_{ab...c}, \tag{14}$$

then the tensor components are given in terms of the spinor components by

$$t_{ab...c} = \left(-\tfrac{1}{\sqrt{2}} \sigma_a{}^{AB}\right) \left(-\tfrac{1}{\sqrt{2}} \sigma_b{}^{CD}\right) \cdots \left(-\tfrac{1}{\sqrt{2}} \sigma_c{}^{EF}\right) t_{ABCD...EF}, \tag{15}$$

and from Eqs. (13) and (15) it follows that

$$t_{...a...}\, s^{...a...} = -t_{...AB...}\, s^{...AB...}. \tag{16}$$

3. The spin-groups

If the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is replaced by a second orthonormal basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ such that

$$\mathbf{e}'_a \cdot \mathbf{e}'_b = \mathbf{e}_a \cdot \mathbf{e}_b, \tag{17}$$

then the components of an n-index tensor with respect to the new basis, $t'_{ab\ldots c}$, are given by

$$t'_{ab\ldots c} = L^d{}_a L^e{}_b \cdots L^f{}_c t_{de\ldots f}, \tag{18}$$

where, owing to Eqs. (6) and (17), $(L^a{}_b)$ is a real matrix such that

$$g_{ab} = L^c{}_a L^d{}_b g_{cd}. \tag{19}$$

The matrices $(L^a{}_b)$ satisfying Eq. (19) form the group $\mathrm{O}(p,q)$, where p and q are the numbers of positive and negative eigenvalues of the matrix (g_{ab}) or vice versa. Equation (19) implies that $\det(L^a{}_b) = \pm 1$. The matrices with unit determinant that satisfy Eq. (19) form the subgroup $\mathrm{SO}(p,q)$ of $\mathrm{O}(p,q)$.

The spinor equivalent of Eqs. (18-19) are

$$t'_{ABCD\ldots} = (-1)^n L^{RS}{}_{AB} L^{TV}{}_{CD} \cdots t_{RSTV\ldots} \tag{20}$$

[see Eq. (16)] and

$$\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC} = L^{RS}{}_{AB} L^{TV}{}_{CD} \left(\varepsilon_{RT}\varepsilon_{SV} + \varepsilon_{RV}\varepsilon_{ST}\right) \tag{21}$$

where

$$L^{AB}{}_{CD} \equiv \tfrac{1}{2}\sigma_a{}^{AB}\sigma^b{}_{CD} L^a{}_b. \tag{22}$$

Hence, using Eqs. (19), (13) and Eq. (6) of Ref. [1],

$$\begin{aligned}
\varepsilon_{AC} L^{AB}{}_{11} L^{CD}{}_{11} &= \tfrac{1}{4}\sigma_a{}^{AB}\sigma^b{}_{11} L^a{}_b \sigma_{cA}{}^D \sigma^d{}_{11} L^c{}_d \\
&= \tfrac{1}{8}(\sigma_a{}^{AB}\sigma_{cA}{}^D + \sigma_c{}^{AB}\sigma_{aA}{}^D) L^a{}_b \sigma^b{}_{11} L^c{}_d \sigma^d{}_{11} \\
&= -\tfrac{1}{4} g_{ac}\varepsilon^{BD} L^a{}_b \sigma^b{}_{11} L^c{}_d \sigma^d{}_{11} \\
&= -\tfrac{1}{4} g_{bd}\varepsilon^{BD}\sigma^b{}_{11}\sigma^d{}_{11} = 0,
\end{aligned}$$

this implies that

$$L^{AB}{}_{11} = \alpha^A \alpha^B, \tag{23}$$

for some α^A. In a similar manner, one finds that

$$L^{AB}{}_{22} = \beta^A \beta^B, \tag{24}$$

for some β^A. From Eq. (21) we have $\varepsilon_{AB}\varepsilon_{CD}L^{AC}{}_{11}L^{BD}{}_{22} = 1$, which, owing to Eqs. (23-24), yields

$$(\alpha^A\beta_A)^2 = 1. \tag{25}$$

Then, using Eq. (25) and the fact that $\alpha^A\beta^B - \alpha^B\beta^A = (\alpha^C\beta_C)\varepsilon^{AB}$, from Eqs. (21) and (23-24) one finds that

$$L^{AB}{}_{12} = \alpha^{(A}\beta^{B)}, \tag{26}$$

where the parenthesis denotes symmetrization on the indices enclosed.

If $\alpha^A\beta_A = 1$, we define $U^A{}_1 \equiv \alpha^A$, $U^A{}_2 \equiv \beta^A$; then, Eqs. (23-24) and (26) are equivalent to

$$L^{AB}{}_{CD} = U^{(A}{}_C U^{B)}{}_D \tag{27}$$

with

$$\det(U^A{}_B) = 1, \tag{28}$$

while if $\alpha^A\beta_A = -1$, we make $U^A{}_1 \equiv i\alpha^A$, $U^A{}_2 \equiv i\beta^A$ and from Eqs. (23-24) and (26) we get

$$L^{AB}{}_{CD} = -U^{(A}{}_C U^{B)}{}_D, \tag{29}$$

where $(U^A{}_B)$ again satisfies Eq. (28). Thus from Eqs. (22), (27) and (29) it follows that any matrix $(L^a{}_b)$ belonging to $\mathrm{O}(p,q)$ can be expressed in the form

$$L^a{}_b = \pm\tfrac{1}{2}\sigma^a{}_{AB}\sigma_b{}^{CD}U^A{}_C U^B{}_D, \tag{30}$$

where $(U^A{}_B)$ has unit determinant. The determinant of the matrix $(L^a{}_b)$ given by Eq. (30) is equal to $+1$ or -1 according to whether one takes the negative or the positive sign in the right-hand side, respectively.

Substituting Eq. (29) into Eq. (20) and comparing with Eq. (1) we see that the spin transformations (1) correspond to orthogonal transformations with unit determinant, *i.e.*, elements of $\mathrm{SO}(p,q)$. Thus, the orthogonal transformation corresponding to the spin transformation (1) is given by

$$L^a{}_b = -\tfrac{1}{2}\sigma^a{}_{AB}\sigma_b{}^{CD}U^A{}_C U^B{}_D. \tag{31}$$

It may be noted that the two matrices $(U^a{}_B)$ and $-(U^A{}_B)$ give rise to the same orthogonal matrix $(L^a{}_b)$. Using Eqs. (5) and (12), one finds that Eq. (31) can also be written as $L^a{}_b = \frac{1}{2}\,\mathrm{tr}\,\sigma^a U\sigma_b U^{-1}$, where $U = (U^A{}_B)$.

Making use of Eqs. (13), (4) and (8), it is easy to see that if $(U^A{}_B)$ is any complex matrix with unit determinant then the $L^a{}_b$ given by Eq. (30) satisfy Eq. (19); however, the $L^a{}_b$ will be complex, in general. As shown below, the conditions that $(U^A{}_B)$ has to satisfy in order for $(L^a{}_b)$ to be real depend on the choice of the connection symbols σ_{aAB}.

3.1. Positive definite metric

As pointed out in the preceding section, the matrices $(\sigma_{aAB}) = (s_{aAB})$ given by Eqs. (9) satisfy conditions (7-8) with $g_{ab} = \delta_{ab}$. From Eqs. (9) and (3) it follows that, under complex conjugation,

$$\overline{\sigma_{a11}} = -\sigma_{a22} = -\sigma_a{}^{11}, \qquad \overline{\sigma_{a12}} = \sigma_{a12} = -\sigma_a{}^{12}, \qquad \overline{\sigma_{a22}} = -\sigma_{a11} = -\sigma_a{}^{22},$$

or equivalently

$$\overline{\sigma_{aAB}} = -\sigma_a{}^{AB}. \tag{32}$$

Therefore, using Eqs. (14) and (32) we find that the spinor equivalent of an n-index tensor $t_{ab...c}$ satisfy

$$\overline{t_{ABCD...EF}} = (-1)^n t^{ABCD...EF} \tag{33}$$

if and only if the tensor components $t_{ab...c}$ are real.

The admissible spin transformations must preserve condition (33); this means that, for instance, if t_a is real, $\overline{t'_{AB}} = -t'^{AB}$, which, according to Eq. (1), amounts to $\overline{U^R{}_A U^S{}_B t_{RS}} = -(U^{-1})^A{}_R (U^{-1})^B{}_S t^{RS}$ and, making use of Eq. (33), $\overline{U^R{}_A U^S{}_B} t^{RS} = (U^{-1})^A{}_R (U^{-1})^B{}_S t^{RS}$. Hence, $\overline{U^R{}_A} = \pm (U^{-1})^A{}_R$ and using the condition $\det(U^A{}_B) = 1$ one gets

$$\overline{U^R{}_A} = (U^{-1})^A{}_R, \tag{34}$$

which means that $(U^A{}_B)$ is unitary; therefore, $(U^A{}_B)$ belongs to SU(2). Using Eqs. (5) and (32), one readily sees that if $(U^A{}_B)$ satisfies Eq. (34), then the orthogonal matrix $(L^a{}_b)$ given by Eq. (31) is real. Thus, Eq. (31) gives the well-known two-to-one mapping (in fact, homomorphism) of SU(2) onto SO(3) (an alternative derivation is given in Ref. [8]).

3.2. Indefinite metric

The matrices $(\sigma_{1AB}) = (s_{1AB})$, $(\sigma_{2AB}) = -i(s_{2AB})$, $(\sigma_{3AB}) = (s_{3AB})$, given explicitly by

$$(\sigma_{1AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma_{2AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (\sigma_{3AB}) \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{35}$$

satisfy Eqs. (7-8) with

$$(g_{ab}) = \mathrm{diag}(1, -1, 1). \tag{36}$$

Since in this case the σ_{aAB} are all real, from Eq. (14) we see that the spinor components of a tensor are real if and only if the tensor is real. Thus, the admissible spin transformations (1) correspond to real or pure imaginary matrices; in the first case $(U^A{}_B)$ belongs to SL(2,R). Taking into account Eqs. (35), it is clear that if $(U^A{}_B)$ is real or pure imaginary then the matrix $(L^a{}_b)$ given by Eq. (31) is real. An explicit calculation shows that if $(U^A{}_B) \in$ SL(2,R) then $L^2{}_2 > 0$ and that $L^2{}_2 < 0$ if $(U^A{}_B)$ is pure imaginary. Equation (31) establishes in this case a two-to-one homomorphism of SL(2,R) onto $\mathrm{SO}^\uparrow(2,1)$ — the connected component of the identity in SO(2,1).

Alternatively, if we choose $(\sigma_{1AB}) = (s_{1AB})$, $(\sigma_{2AB}) = (s_{2AB})$, $(\sigma_{3AB}) = -i(s_{3AB})$, *i.e.*,

$$(\sigma_{1AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma_{2AB}) \equiv \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad (\sigma_{3AB}) \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{37}$$

then Eqs. (7-8) are satisfied with

$$(g_{ab}) = \mathrm{diag}(1, 1, -1). \tag{38}$$

Now we have

$$\overline{\sigma_{a11}} = -\sigma_{a22}, \qquad \overline{\sigma_{a12}} = -\sigma_{a12}, \tag{39}$$

which can be expressed as

$$\overline{\sigma_{aAB}} = -\eta_{AR}\eta_{BS}\sigma_a{}^{RS}, \tag{40}$$

where

$$(\eta_{AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{41}$$

With the choice (37), the spinor equivalent of an n-index tensor $t_{ab...c}$ satisfies the conditions

$$\overline{t_{AB...EF}} = (-1)^n \eta_{AR}\eta_{BS} \cdots \eta_{EW}\eta_{FX} t^{RS...WX} \tag{42}$$

if and only if the tensor components $t_{ab...c}$ are real.

Proceeding as in the previous subsection, one finds that the spin transformations $(U^A{}_B)$ that preserve the reality conditions (42) must obey the condition

$$\eta_{AB}\overline{U^B{}_C} = \pm\eta_{CB}(U^{-1})^B{}_A \tag{43}$$

or, equivalently,

$$U^\dagger \eta U = \pm\eta, \tag{44}$$

where $U \equiv (U^A{}_B)$ and $\eta \equiv (\eta_{AB})$. The matrices that satisfy Eq. (44) with the positive sign form the group SU(1,1) (which is isomorphic to SL(2,R)).

Making use of Eqs. (5) and (40) one finds that if $(U^A{}_B)$ satisfies Eq. (43), then the matrix $(L^a{}_b)$ given by Eq. (31) is real. Furthermore, $L^3{}_3$ is positive if U satisfies Eq. (44) with the positive sign and $L^3{}_3$ is negative if U satisfies Eq. (44) with the negative sign. Therefore, with the σ_{aAB} given by Eqs. (37), Eq. (31) defines a two-to-one correspondence between SU(1,1) and $\mathrm{SO}^\uparrow(2,1)$, which is a group homomorphism.

4. Covariant differentiation

Let ∂_a denote the directional derivative with respect to $\mathbf{e}_a$ and let ∇_a denote the covariant derivative with respect to ∂_a. The components of the Levi-Civita connection relative to the basis $\{\partial_1, \partial_2, \partial_3\}$ are the real-valued functions $\Gamma^c{}_{ba}$ defined by

$$\nabla_a \partial_b = \Gamma^c{}_{ba} \partial_c, \tag{45}$$

which are determined by

$$[\partial_a, \partial_b] = (\Gamma^c{}_{ba} - \Gamma^c{}_{ab})\, \partial_c. \tag{46}$$

The functions

$$\Gamma_{abc} \equiv g_{ad} \Gamma^d{}_{bc}$$

are anti-symmetric in the first pair of indices,

$$\Gamma_{abc} = -\Gamma_{bac}, \tag{47}$$

therefore the spinor equivalent of Γ_{abc}, Γ_{ABCDEF}, can be written as

$$\Gamma_{ABCDEF} = -\Gamma_{ACEF}\varepsilon_{BD} - \Gamma_{BDEF}\varepsilon_{AC} \tag{48}$$

(see, *e.g.*, Ref. [1]), where

$$\Gamma_{ABCD} = -\tfrac{1}{2}\varepsilon^{RS}\Gamma_{RASBCD} = -\tfrac{1}{2}\Gamma^R{}_{ARBCD}. \tag{49}$$

The components Γ_{ABCD} are symmetric in the first and second pairs of indices

$$\Gamma_{ABCD} = \Gamma_{BACD} = \Gamma_{ABDC}, \tag{50}$$

and from Eqs. (33) and (42), using the fact that ε^{RS} is real and that $\det(\eta_{AB}) = -1$, one finds that

$$\overline{\Gamma_{ABCD}} = \begin{cases} -\Gamma^{ABCD} & \text{if } (g_{ab}) = \mathrm{diag}(1,1,1), \\ \Gamma_{ABCD} & \text{if } (g_{ab}) = \mathrm{diag}(1,-1,1), \\ \eta_{AR}\eta_{BS}\eta_{CT}\eta_{DV}\Gamma^{RSTV} & \text{if } (g_{ab}) = \mathrm{diag}(1,1,-1). \end{cases} \tag{51}$$

Denoting by ∂_{AB} the differential operators

$$\partial_{AB} \equiv \tfrac{1}{\sqrt{2}}\sigma^a{}_{AB}\partial_a, \tag{52}$$

and making use of Eqs. (16) and (48) one finds that the spinor equivalent of Eq. (45) is

$$\nabla_{AB}\partial_{CD} = \Gamma^R{}_{CAB}\partial_{RD} + \Gamma^R{}_{DAB}\partial_{CR}, \tag{53}$$

where ∇_{AB} denotes the covariant derivative with respect to ∂_{AB}. The components of the covariant derivative of a spinor field $\psi^{CD\dots}_{FG\dots}$ with respect to ∂_{AB} are given by

$$\begin{aligned}\nabla_{AB}\psi^{CD\dots}_{FG\dots} &= \partial_{AB}\psi^{CD\dots}_{FG\dots} + \Gamma^{C}{}_{RAB}\psi^{RD\dots}_{FG\dots} + \Gamma^{D}{}_{RAB}\psi^{CR\dots}_{FG\dots} \\ &\quad + \cdots - \Gamma^{R}{}_{FAB}\psi^{CD\dots}_{RG\dots} - \Gamma^{R}{}_{GAB}\psi^{CD\dots}_{FR\dots} - \cdots . \end{aligned} \tag{54}$$

The symmetry of Γ_{ABCD} in the first pair of indices [Eq. (50)] implies that the covariant derivatives of ε_{AB} vanish and, therefore, the covariant derivative commutes with the raising and lowering of spinor indices. From Eqs. (53) or (54), making use of Eq. (5), it follows that under the spin transformation (1) the components Γ_{ABCD} transform according to

$$\Gamma'_{ABCD} = U^{T}{}_{C}U^{V}{}_{D}(U^{R}{}_{A}U^{S}{}_{B}\Gamma_{RSTV} + U^{M}{}_{A}\,\partial_{TV}U_{MB}). \tag{55}$$

5. Curvature

The curvature tensor of a three-dimensional manifold can be expressed in the form

$$R_{abcd} = -g\;\varepsilon_{abe}\varepsilon_{cdf}G^{ef}, \tag{56}$$

where $g \equiv \det(g_{ab})$, G_{ab} is a symmetric tensor and $\varepsilon_{123} \equiv 1$. Using the fact that

$$g^{ac}\varepsilon_{abe}\varepsilon_{cdf} = g^{-1}(g_{bd}g_{ef} - g_{bf}g_{ed}), \tag{57}$$

from Eq. (56) we find that G_{ab} is related to the Ricci tensor $R_{ab} \equiv R^{c}{}_{acb}$ through

$$G_{ab} = R_{ab} - \tfrac{1}{2}R\,g_{ab}, \tag{58}$$

where $R \equiv R^{a}{}_{a}$ is the scalar curvature. Therefore, if Φ_{ab} denotes the trace-free part of the Ricci tensor ($\Phi_{ab} \equiv R_{ab} - \tfrac{1}{3}R\,g_{ab}$), Eq. (58) gives

$$G_{ab} = \Phi_{ab} - \tfrac{1}{6}R\,g_{ab}. \tag{59}$$

The spinor equivalent of the alternating symbol ε_{abc} is given by

$$\varepsilon_{ABCDEF} = \frac{i}{2\sqrt{2g}}(\varepsilon_{AC}\varepsilon_{BE}\varepsilon_{DF} + \varepsilon_{AC}\varepsilon_{BF}\varepsilon_{DE} + \varepsilon_{BD}\varepsilon_{AE}\varepsilon_{CF} + \varepsilon_{BD}\varepsilon_{AF}\varepsilon_{CE}) \tag{60}$$

(see, *e.g.*, Ref. [1], Eq. (16)), hence the spinor equivalent of Eq. (56) is

$$\begin{aligned}R_{ABCDEFHI} &= \tfrac{1}{2}(\varepsilon_{AC}\varepsilon_{EH}G_{BDFI} + \varepsilon_{AC}\varepsilon_{FI}G_{BDEH} \\ &\quad + \varepsilon_{BD}\varepsilon_{EH}G_{ACFI} + \varepsilon_{BD}\varepsilon_{FI}G_{ACEH}),\end{aligned} \tag{61}$$

where G_{ABCD} are the spinor components of G_{ab}, and from Eqs. (13) and (59) we have

$$G_{ABCD} = \Phi_{ABCD} + \tfrac{R}{12}(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}), \tag{62}$$

where Φ_{ABCD} are the spinor components of Φ_{ab}, which are totally symmetric.

The spinor equivalent of the equation

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, t_c = -R^d{}_{cab} t_d, \tag{63}$$

which defines the curvature tensor, can be written in the form

$$(\varepsilon_{AC} \Box_{BD} + \varepsilon_{BD} \Box_{AC})\, t_{EF} = R^{HI}{}_{EFABCD} t_{HI} \tag{64}$$

(see, *e.g.*, Ref. [1]), where

$$\Box_{AB} \equiv \nabla^R{}_{(A} \nabla_{B)R}. \tag{65}$$

Therefore, from Eqs. (61), (62) and (64) it follows that

$$\Box_{AB} \psi_C = -\tfrac{1}{2} \Phi_{ABCD} \psi^D - \tfrac{R}{24} (\varepsilon_{AC} \psi_B + \varepsilon_{BC} \psi_A). \tag{66}$$

By expanding the left-hand side of Eq. (66) one obtains

$$\begin{aligned} -\tfrac{1}{2} \Phi_{ABCD} - \tfrac{R}{24} (\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BD}) = \partial^R{}_{(A} \Gamma_{|DC|B)R} - \Gamma^S{}_R{}^R{}_{(A} \Gamma_{|DC|B)S} \\ - \Gamma^S{}_{(A}{}^R{}_{B)} \Gamma_{DCSR} - \Gamma^S{}_C{}^R{}_{(A} \Gamma_{|DS|B)R}, \end{aligned} \tag{67}$$

where the indices between bars are excluded from the symmetrization.

Since Φ_{ab} is real, from Eqs. (33) and (42) we find that

$$\overline{\Phi_{ABCD}} = \begin{cases} \Phi^{ABCD} & \text{if } (g_{ab}) = \text{diag}(1,1,1), \\ \Phi_{ABCD} & \text{if } (g_{ab}) = \text{diag}(1,-1,1), \\ \eta_{AR} \eta_{BS} \eta_{CT} \eta_{DV} \Phi^{RSTV} & \text{if } (g_{ab}) = \text{diag}(1,1,-1),. \end{cases} \tag{68}$$

Owing to Eq. (56), in a three-dimensional manifold, the Bianchi identities are equivalent to the contracted Bianchi identities $\nabla^a G_{ab} = 0$, which amount to

$$\nabla^{AB} \Phi_{ABCD} + \tfrac{1}{6} \partial_{CD} R = 0. \tag{69}$$

6. Spin-weight and prime operation

The concept of spin-weight and the prime operation employed in the 2-spinor calculus of general relativity (see, *e.g.*, Refs. [5,6]), can also be defined in a three-dimensional riemannian manifold. Since the case with a positive definite metric has been considered in Ref. [1], in this section we restrict ourselves to the case where the metric has signature $(++-)$, with the σ_{aAB} given by Eq. (37). Nevertheless, except for Eqs. (73) and (77-78), all the formulae of this section are also valid in the case of positive definite metric, with the σ_{aAB} given by Eq. (9).

A quantity ξ has spin-weight s if under the spin transformation defined by

$$(U^A{}_B) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \tag{70}$$

(which belongs to SU(1,1) and corresponds to a rotation through an angle θ about $\mathbf{e}_3$), it transforms according to

$$\xi \mapsto e^{is\theta}\xi. \tag{71}$$

Making use of Eqs. (1) and (70-71) one finds that each component $\psi_{AB\ldots D}$ of a spinor has spin-weight equal to one half of the difference between the number of indices A, B, $\ldots$, D taking the value 1 and those taking the value 2. The $2n+1$ independent components of a totally symmetric $2n$-index spinor can be labeled by their spin-weight

$$\psi_s \equiv \psi_{\underbrace{1\ldots1}_{n+s}\underbrace{2\ldots2}_{n-s}}, \qquad (s = 0, \pm1, \ldots, \pm n). \tag{72}$$

From Eq. (71) it is clear that if ξ has spin-weight s, then $\overline{\xi}$ has spin-weight $-s$. The spin-weighted components of a real trace-free totally symmetric n-index tensor $t_{ab\ldots c}$, defined by

$$t_s \equiv t_{\underbrace{1\ldots1}_{n+s}\underbrace{2\ldots2}_{n-s}},$$

where $t_{AB\ldots D}$ are the spinor components of $t_{ab\ldots c}$, which are totally symmetric, satisfy the relations

$$\overline{t_s} = (-1)^n t_{-s}, \tag{73}$$

where we have made use of Eq. (42) (*cf.* Ref. [1], Eq. (37)).

The spinor components of the connection Γ_{11AB} and Γ_{22AB} have a well-defined spin-weight; in fact, from Eq. (55) one finds that under the spin transformation (70)

$$\begin{aligned} &\Gamma_{1111} \mapsto e^{2i\theta}\Gamma_{1111}, && \Gamma_{1112} \mapsto e^{i\theta}\Gamma_{1112}, && \Gamma_{1122} \mapsto \Gamma_{1122}, \\ &\Gamma_{2222} \mapsto e^{-2i\theta}\Gamma_{2222}, && \Gamma_{2212} \mapsto e^{-i\theta}\Gamma_{2212}, && \Gamma_{2211} \mapsto \Gamma_{2211}, \\ &\Gamma_{1211} \mapsto e^{i\theta}(\Gamma_{1211} - \tfrac{i}{2}\partial_{11}\theta), && \Gamma_{1212} \mapsto \Gamma_{1212} - \tfrac{i}{2}\partial_{12}\theta, && \Gamma_{1222} \mapsto e^{-i\theta}(\Gamma_{1222} - \tfrac{i}{2}\partial_{22}\theta), \end{aligned} \tag{74}$$

therefore, if ξ has spin-weight s, then $(\partial_{11}+2s\Gamma_{1211})\xi$, $(\partial_{12}+2s\Gamma_{1212})\xi$ and $(\partial_{22}+2s\Gamma_{1222})\xi$ have spin-weight $s+1$, s and $s-1$, respectively.

The unimodular matrix

$$(U^A{}_B) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{75}$$

which satisfies Eq. (44) with the negative sign and represents a rotation through 180° about $\mathbf{e}_1$, defines a spin transformation called prime operation. Under this transformation, the components of a totally symmetric $2n$-index spinor defined by Eq. (72), transform as

$$\psi'_s = i^{2n}\psi_{-s}. \tag{76}$$

Using Eqs. (1), (39), (55) and (75) one finds that if $t_{AB...D}$ are the spinor components of a *real* n-index tensor then

$$t'_{AB...D} = \overline{t_{AB...D}}, \tag{77}$$

and

$$\partial'_{AB} = \overline{\partial_{AB}}, \qquad \Gamma'_{ABCD} = \overline{\Gamma_{ABCD}}. \tag{78}$$

7. The triad defined by a spinor

In the spinor calculus employed in general relativity, two linearly independent spinors define a tetrad of vectors. In the case of three-dimensional spaces, a single spinor determines an orthogonal basis. When the metric is positive definite, this relationship is well known and allows the representation of a spinor by means of an ax or a flag (see, *e.g.*, Refs. [7,8,3]).

In the case of signature $(++-)$ we shall make use of the σ_{aAB} given by Eq. (37). The mate of an arbitrary spinor ψ^A will be defined by

$$\hat{\psi}_A \equiv \eta_{AB}\overline{\psi^B}, \tag{79}$$

(*i.e.*, $\hat{\psi}^1 = \overline{\psi^2}$, $\hat{\psi}^2 = \overline{\psi^1}$). Using Eqs. (1), (79) and the complex conjugate of Eq. (43) one finds that under a spin transformation belonging to SU(1,1), $\hat{\psi}_A$ transforms according to

$$\hat{\psi}'_A = \eta_{AB}\overline{\psi'^B} = \eta_{AB}\overline{(U^{-1})^B{}_C\psi^C} = \eta_{CB}U^B{}_A\overline{\psi^C} = U^B{}_A\hat{\psi}_B,$$

which shows that $\hat{\psi}_A$ transforms as a spinor under SU(1,1) transformations. Therefore, the components

$$R_a \equiv i\sigma_{aAB}\hat{\psi}^A\psi^B \tag{80}$$

transform under $SO^{\uparrow}(2,1)$ as the components of a vector. The components R_a are given explicitly by

$$(R_a) = \left(-i\left(\overline{\psi^1}\psi^2 - \overline{\psi^2}\psi^1\right), -\overline{\psi^1}\psi^2 - \overline{\psi^2}\psi^1, -|\psi^1|^2 - |\psi^2|^2\right)$$

which shows that R_a is real and $R^3 = -R_3 \geq 0$. Using Eqs. (13) and (80) one finds that

$$R_aR^a = -(\psi^A\hat{\psi}_A)^2. \tag{81}$$

(Note that $\psi^A \hat{\psi}_A = |\psi^1|^2 - |\psi^2|^2$ is invariant under the spin transformations.)

Introducing now

$$M_a \equiv \sigma_{aAB} \psi^A \psi^B, \tag{82}$$

using Eqs. (13), (80) and (82) one readily obtains $R_a M^a = 0$, $M_a M^a = 0$ which means that the real and imaginary parts of M_a are orthogonal to R_a and to each other and that they have the same magnitude. In fact, a straightforward computation yields

$$(\text{Re } M_a)(\text{Re } M^a) = (\text{Im } M_a)(\text{Im } M^a) = (\psi^A \hat{\psi}_A)^2. \tag{83}$$

(Note that from Eqs. (40), (79) and (82) it follows that $\overline{M_a} = -\sigma_{aAB} \hat{\psi}^A \hat{\psi}^B$.) Thus, if ψ^A satifies the condition $\psi^A \hat{\psi}_A = 1$, then $\{\text{Re } \mathbf{M}, \text{Im } \mathbf{M}, \mathbf{R}\}$ is an orthonormal basis with the same orientation as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The spinors ψ^A and $-\psi^A$ give rise to the same triad. (If $\psi^A \hat{\psi}_A = 0$, then $\hat{\psi}^A$ is proportional to ψ^A and from Eqs. (80) and (82) it follows that R_a and M_a are proportional.)

When the metric has signature $(+++)$, we take the σ_{aAB} given by Eq. (9) and define the mate of a spinor ψ^A as

$$\hat{\psi}_A \equiv \overline{\psi^A} \tag{84}$$

(*i.e.*, $\hat{\psi}^1 = -\overline{\psi^2}$, $\hat{\psi}^2 = \overline{\psi^1}$). As a consequence of Eq. (34), $\hat{\psi}^A$ transforms according to Eq. (1). In the present case

$$R_a \equiv -\sigma_{aAB} \hat{\psi}^A \psi^B \tag{85}$$

are real and transform as the components of a vector under SO(3) and they satisfy $R_a R^a = (\hat{\psi}^A \psi_A)^2$. On the other hand,

$$M_a \equiv \sigma_{aAB} \psi^A \psi^B \tag{86}$$

are the components of a complex vector such that $R_a M^a = 0$, $M_a M^a = 0$ and $(\text{Re } M_a) \times (\text{Re } M^a) = (\text{Im } M_a)(\text{Im } M^a) = (\hat{\psi}^A \psi_A)^2$. (In the present case we also have $\overline{M_a} = -\sigma_{aAB} \hat{\psi}^A \hat{\psi}^B$.) Therefore, if ψ^A is such that $\psi^A \hat{\psi}_A = 1$ (*i.e.*, $|\psi^1|^2 + |\psi^2|^2 = 1$), then $\{\text{Re } \mathbf{M}, \text{Im } \mathbf{M}, \mathbf{R}\}$ is an orthonormal basis with the same orientation as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Also in this case the spinors ψ^A and $-\psi^A$ define the same triad.

8. Concluding remarks

The results of this paper, together with those of Ref. [1], provide a spinor calculus for three-dimensional manifolds with any signature; the signature of the metric enters through the choice of the connection symbols σ_{aAB}. In the case where the metric is indefinite, the symbols σ_{aAB} given by Eq. (37) seem to be more convenient that those given by Eq. (35), since the use of complex quantities reduces the number of independent equations.

REFERENCES

1. G.F. Torres del Castillo, *Rev. Mex. Fís.* **38** (1992) 863.
2. R. Penrose and W. Rindler, *Spinors and space-time*, Vol. 2, Cambridge University Press, Cambridge (1986); Appendix.
3. G.F. Torres del Castillo, *Rev. Mex. Fís.* **35** (1989) 123. (In Spanish.)
4. J.F. Plebański and G.F. Torres del Castillo, *Spinors, tetrads and forms*, (in preparation).
5. R.P. Geroch, A. Held and R. Penrose, *J. Math. Phys.* **14** (1973) 874.
6. R. Penrose and W. Rindler, *Spinors and space-time*, Vol. 1, Cambridge University Press, Cambridge (1984).
7. W.T. Payne, *Am. J. Phys.* **20** (1952) 253.
8. G.F. Torres del Castillo, *Rev. Mex. Fís.* **40** (1994) 119.