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A Hamiltonian structure for the Euler equations

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Abstract. A Hamiltonian structure for the Euler equations for an ideal compressible fluid is given and it is shown that the corresponding Poisson bracket is degenerate. The case of an ideal incompressible fluid is also discussed.

Resumen. Se da una estructura hamiltoniana para las ecuaciones de Euler para un fluido ideal compresible y se muestra que el paréntesis de Poisson correspondiente es degenerado. Se trata también el caso de un fluido ideal incompresible.

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1. Introduction

As is very well known, the Lagrangian and Hamiltonian formalisms employed in the treatment of mechanical systems with a finite number of degrees of freedom can be extended to the treatment of continuous media and fields. The Hamiltonian description of continuous systems is usually obtained starting from the Lagrangian formulation, imitating the steps followed in the case of the systems of point particles (see, *e.g.*, Refs. [1,2]); however, in several cases of interest it is impossible to apply this canonical procedure in a straightforward manner in order to find a Hamiltonian description, since the momentum densities are not independent of the field variables (see, *e.g.*, Ref. [3]).

Nevertheless, it is possible to give a Hamiltonian formulation for a given continuous system, without making reference to the Lagrangian formulation, if its evolution equations can be written in the form

$$\dot{\phi}_\alpha = D_{\alpha\beta}\frac{\delta H}{\delta\phi_\beta}, \tag{1}$$

where the field variables ϕ_α represent the state of the system, H is some functional of the ϕ_α, $\delta H/\delta\phi_\alpha$ is the functional derivative of H with respect to ϕ_α, and the $D_{\alpha\beta}$ are differential operators that must satisfy certain conditions that allow the definition of a

Poisson bracket between functionals of the ϕ_α (see, *e.g.*, Refs. [4,3]). Here and henceforth a dot denotes partial differentiation with respect to the time and there is summation over repeated indices.

In this paper we give a Hamiltonian structure for the Euler equations for an ideal, compressible, isentropic fluid. The Euler equations and the equation of continuity are written in the Hamiltonian form (1), without having to introduce constraints or auxiliary quantities (see also Refs. [5-7]). As is known, the Hamiltonian formalism allows one to make the transition to quantum field theory, by replacing Poisson brackets by commutators (a discussion about the application of quantum hydrodynamics to superfluidity is given in Ref. [8]). In Sect. 2 we obtain a Hamiltonian structure for the Euler equations assuming that the pressure depends on the density only and we show that the corresponding Poisson bracket is degenerate. In Sect. 3, following Ref. [4], we consider the case of an ideal incompressible fluid, using the components of the vorticity as field variables and we show that the corresponding Hamiltonian structure is, essentially, a reduction of that obtained in Sect. 2.

2. Hamiltonian formulation for the Euler equations

The Euler equations for an inviscid fluid are

$$\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{B}, \tag{2}$$

where $\mathbf{u}$ is the velocity field of the fluid, ρ is the density, p is the pressure, and $\mathbf{B}$ is the external force per unit volume. The velocity and the density of the fluid are related through the equation of continuity

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{3}$$

Making use of the vorticity

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}, \tag{4}$$

the Euler equations can be written as

$$\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u} + \nabla(\tfrac{1}{2}u^2) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{B}, \tag{5}$$

and taking the curl of this last equation one obtains

$$\dot{\boldsymbol{\omega}} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \left(\frac{1}{\rho}\mathbf{B}\right). \tag{6}$$

In what follows we shall restrict ourselves to those cases where the external force per unit mass, $\mathbf{B}/\rho$, is the gradient of a function $-\Omega$

$$\frac{1}{\rho}\mathbf{B} = -\nabla\Omega \tag{7}$$

and the density and the pressure of the fluid are such that

$$\nabla\rho \times \nabla p = 0; \tag{8}$$

then Eq. (6) reduces to

$$\dot{\boldsymbol{\omega}} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}). \tag{9}$$

Equation (8) means that ρ is constant or that p is a function of ρ only.

If the pressure is a function of the density only, the term $\frac{1}{\rho}\nabla p$ appearing in the right-hand side of Eq. (5) can be written as the gradient of a certain function. In fact, by introducing the intrinsic energy of the fluid per unit mass $E(\rho)$, defined by

$$E \equiv \int \frac{p}{\rho^2}\, d\rho \tag{10}$$

(see, *e.g.*, Ref. [9]), one finds that

$$\frac{1}{\rho}\nabla p = \nabla\left(\frac{d}{d\rho}(\rho E)\right). \tag{11}$$

Hence, under the present assumptions, using Eqs. (7) and (11), the Euler equations take the form

$$\dot{\mathbf{u}} = -\boldsymbol{\omega} \times \mathbf{u} - \nabla\left(\frac{1}{2}u^2 + \frac{d}{d\rho}(\rho E) + \Omega\right) \tag{12}$$

which amount to

$$\begin{aligned} \dot{u}_i &= -\epsilon_{ijk}\omega_j u_k - \partial_i\left(\frac{1}{2}u^2 + \frac{d}{d\rho}(\rho E) + \Omega\right) \\ &= \frac{1}{\rho}\omega_j\epsilon_{jik}\frac{\delta H}{\delta u_k} - \partial_i\frac{\delta H}{\delta\rho} \end{aligned} \tag{13}$$

(Latin indices $i,j,\ldots,$ range and sum over 1,2,3) where

$$H \equiv \int \left[\tfrac{1}{2}\rho u^2 + \rho E(\rho) + \rho\Omega\right] dv, \tag{14}$$

and $\partial_i \equiv \partial/\partial x_i$. On the other hand, Eq. (3) can be written as

$$\dot{\rho} = -\partial_i(\rho u_i) = -\partial_i\frac{\delta H}{\delta u_i}. \tag{15}$$

Thus, taking $(\phi_1, \phi_2, \phi_3, \phi_4) \equiv (u_1, u_2, u_3, \rho)$, Eqs. (13) and (15) can be expressed in the Hamiltonian form

$$\dot{\phi}_\alpha = D_{\alpha\beta} \frac{\delta H}{\delta \phi_\beta}, \tag{16}$$

where $\alpha, \beta, \ldots$, run from 1 to 4, with the Hamiltonian functional H defined by Eq. (14) and

$$\begin{aligned} D_{ik} &= \frac{1}{\rho} \omega_j \epsilon_{jik} = \frac{1}{\rho} (\partial_i u_k - \partial_k u_i), \\ D_{i4} &= D_{4i} = -\partial_i, \qquad D_{44} = 0. \end{aligned} \tag{17}$$

It is easy to see that the Poisson bracket

$$\begin{aligned} \{F, G\} &\equiv \int \frac{\delta F}{\delta \phi_\alpha} D_{\alpha\beta} \frac{\delta G}{\delta \phi_\beta} \, dv \\ &= \int \left[\frac{1}{\rho} \omega_j \epsilon_{jik} \frac{\delta F}{\delta u_i} \frac{\delta G}{\delta u_k} - \frac{\delta F}{\delta u_i} \partial_i \frac{\delta G}{\delta \rho} - \frac{\delta F}{\delta \rho} \partial_i \frac{\delta G}{\delta u_i} \right] dv \end{aligned} \tag{18}$$

(*cf.* Ref. [10]) is antisymmetric for functionals satisfying $n_i(\delta F/\delta u_i) = 0$ at the boundary of the fluid, where $\mathbf{n}$ is normal to the boundary. A straightforward computation, using the methods given in Ref. [11], shows that this Poisson bracket satisfies the Jacobi identity.

Using the fact that

$$\phi_\alpha(\mathbf{r}', t) = \int \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \phi_\beta(\mathbf{r}, t) \, dv$$

it follows that

$$\frac{\delta \phi_\alpha(\mathbf{r}', t)}{\delta \phi_\beta(\mathbf{r}, t)} = \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}'),$$

therefore, from Eq. (18), one gets

$$\{\phi_\alpha(\mathbf{r}, t), \phi_\beta(\mathbf{r}', t)\} = D_{\alpha\beta}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \tag{19}$$

(compare Ref. [1], p. 567). Thus, Eqs. (17) yield

$$\begin{aligned} \{u_i(\mathbf{r}, t), u_k(\mathbf{r}', t)\} &= \frac{1}{\rho} \omega_j \epsilon_{jik} \delta(\mathbf{r} - \mathbf{r}') \\ &= \frac{1}{\rho} (\partial_i u_k - \partial_k u_i) \delta(\mathbf{r} - \mathbf{r}'), \\ \{u_i(\mathbf{r}, t), \rho(\mathbf{r}', t)\} &= -\partial_i \delta(\mathbf{r} - \mathbf{r}'), \\ \{\rho(\mathbf{r}, t), \rho(\mathbf{r}', t)\} &= 0 \end{aligned} \tag{20}$$

(*cf.* Ref. [8], Sect. 5; note that the relation corresponding to the first of Eqs. (20) appears in Ref. [8] with the opposite sign). Similarly, one finds that [Eq.(4)]

$$\frac{\delta\omega_i(\mathbf{r}',t)}{\delta u_j(\mathbf{r},t)} = \epsilon_{ijk}\partial_k\delta(\mathbf{r}-\mathbf{r}'), \qquad \frac{\delta\omega_i(\mathbf{r}',t)}{\delta\rho(\mathbf{r},t)} = 0,$$

therefore

$$\begin{aligned}\{\omega_i(\mathbf{r},t),\omega_k(\mathbf{r}',t)\} &= \epsilon_{ijk}\partial_m\left(\frac{1}{\rho}\omega_m\partial_j\delta(\mathbf{r}-\mathbf{r}')\right)\\ &\quad - \epsilon_{mjk}\partial_m\left(\frac{1}{\rho}\omega_i\partial_j\delta(\mathbf{r}-\mathbf{r}')\right).\end{aligned} \tag{21}$$

The Poisson bracket (18) is degenerate in the sense that there exist nontrivial functionals $C[\phi_\alpha]$ such that $\{F,C\}=0$ for all F or, equivalently,

$$D_{\alpha\beta}\frac{\delta C}{\delta\phi_\beta} = 0. \tag{22}$$

In fact, denoting $a_i \equiv \delta C/\delta u_i$, $b \equiv \delta C/\delta\rho$, and using Eqs. (17) one finds that Eq. (22) amounts to

$$\frac{1}{\rho}\omega_j\epsilon_{jik}a_k - \partial_i b = 0, \tag{23}$$

and

$$-\partial_i a_i = 0. \tag{24}$$

Contracting Eq. (23) with ω_i we get $\omega_i\partial_i b = 0$, which means that b is a constant along $\boldsymbol{\omega}$; therefore, since the direction of $\boldsymbol{\omega}$ is arbitrary, we conclude that b must be a constant. Then, Eq. (23) yields $\boldsymbol{\omega}\times\mathbf{a}=0$, which implies that $\mathbf{a}=\lambda\boldsymbol{\omega}$ for some scalar λ. Using now the fact that $\nabla\cdot\boldsymbol{\omega}=0$ [Eq. (4)] and that, according to Eq. (24), $\nabla\cdot\mathbf{a}=0$, it follows that λ must be a constant. Thus,

$$C = \frac{\lambda}{2}\int \mathbf{u}\cdot\boldsymbol{\omega}\,dv + b\int\rho\,dv, \tag{25}$$

where λ and b are arbitrary constants. Clearly, the second integral corresponds to the total mass of the fluid. (The nontrivial functionals that satisfy Eq. (22) are called distinguished functionals or Casimir functionals.) It may be noticed that $C_1 \equiv \int \mathbf{u}\cdot\boldsymbol{\omega}\,dv$ satisfies the condition $n_i\delta C_1/\delta u_i = 0$ at the boundary if all components u_i vanish at the boundary. (An illuminating discussion about the invariant denoted here as C_1 and some examples of flows for which $C_1 \neq 0$ can be found in Ref. [12].)

The functionals corresponding to the cartesian components of the linear momentum P_k, must be the generators of translations along the coordinate axes, in the same way as $-H$ is the generator of translations in time; therefore (*cf.* Eq. (16))

$$-\partial_k \phi_\alpha = D_{\alpha\beta} \frac{\delta P_k}{\delta \phi_\beta} \tag{26}$$

or, equivalently,

$$-\partial_k u_i = \frac{1}{\rho} \omega_m \epsilon_{mij} \frac{\delta P_k}{\delta u_j} - \partial_i \frac{\delta P_k}{\delta \rho}, \tag{27}$$

and

$$-\partial_k \rho = -\partial_i \frac{\delta P_k}{\delta u_i}. \tag{28}$$

Equation (28) is satisfied with $\delta P_k / \delta u_i = \rho \delta_{ik}$, which, when substituted into Eq. (27), gives

$$\begin{aligned} -\partial_k u_i &= \omega_m \epsilon_{mik} - \partial_i \frac{\delta P_k}{\delta \rho} \\ &= \partial_i u_k - \partial_k u_i - \partial_i \frac{\delta P_k}{\delta \rho}. \end{aligned}$$

This last equation becomes an identity assuming that $\delta P_k / \delta \rho = u_k$. Therefore, the functionals

$$P_k = \int \rho u_k \, dv \tag{29}$$

satisfy Eq. (26) and coincide with the usual expressions for the components of the linear momentum of the fluid. In view of Eq. (22), the functionals P_k are defined by Eq. (26) up to the addition of a functional of the form (25). It may be remarked that the expression of P_k is determined once the Hamiltonian structure, defined by the operators $D_{\alpha\beta}$, is given, without using again the evolution equations. The functionals P_k need not be conserved; their conservation depends on the explicit expression of the potential energy per unit mass, Ω, and on the boundaries.

In a similar manner, by defining the functionals L_k, corresponding to the components of the angular momentum, as the generators of rotations about the coordinate axes (see, *e.g.*, Ref. [3], Eq. (54)) one finds that

$$L_k = \int \rho \epsilon_{kij} x_i u_j \, dv, \tag{30}$$

modulo functionals of the form (25).

3. Hamiltonian structure for ideal incompressible fluids

In the case of an inviscid incompressible fluid, assuming that Eq. (7) holds, it is convenient to use the components of the vorticity as field variables (see, *e.g.*, Ref. [4]). From Eq. (3) we have

$$\nabla \cdot \mathbf{u} = 0 \tag{31}$$

and therefore Eq. (9) can be written as

$$\dot{\boldsymbol{\omega}} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega}. \tag{32}$$

Since in the present case $\mathbf{u}$ is divergenceless, it can be expressed as the curl of a vector field $\boldsymbol{\psi}$ [4]

$$\mathbf{u} = \nabla \times \boldsymbol{\psi}. \tag{33}$$

In order to write Eq. (32) in the Hamiltonian form (16), one has to find the functional derivatives $\delta H/\delta\omega_i$. By assuming that the Hamiltonian functional is given by

$$H = \int \tfrac{1}{2}\rho u^2 \, dv \tag{34}$$

(*cf.* Eq. (14)), then

$$\begin{aligned} \frac{d}{d\varepsilon} H[\mathbf{u} + \varepsilon\, \delta\mathbf{u}]\bigg|_{\varepsilon=0} &= \int \rho \mathbf{u} \cdot \delta\mathbf{u}\, dv = \int \rho(\nabla \times \boldsymbol{\psi}) \cdot \delta\mathbf{u}\, dv \\ &= \int \rho\boldsymbol{\psi} \cdot \nabla \times \delta\mathbf{u}\, dv = \int \rho\boldsymbol{\psi} \cdot \delta\boldsymbol{\omega}\, dv, \end{aligned}$$

where we have assumed that $\delta\mathbf{u}$ vanishes at the boundary. Thus $\delta H/\delta\omega_i = \rho\psi_i$ and from Eqs. (32–33) one obtains

$$\begin{aligned} \dot{\omega}_i &= \omega_m \partial_m \epsilon_{ijk} \partial_j \frac{1}{\rho}\frac{\delta H}{\delta\omega_k} - (\partial_m\omega_i)\epsilon_{mjk}\partial_j \frac{1}{\rho}\frac{\delta H}{\delta\omega_k} \\ &= \tilde{D}_{ik}\frac{\delta H}{\delta\omega_k}, \end{aligned}$$

which are of the form (16), with $\phi_i = \omega_i$ and

$$\tilde{D}_{ik} \equiv \frac{1}{\rho}(\epsilon_{ijk}\omega_m\partial_m\partial_j - \epsilon_{mjk}(\partial_m\omega_i)\partial_j). \tag{35}$$

The Poisson bracket

$$\{F, G\} \equiv \int \frac{\delta F}{\delta\omega_i}\tilde{D}_{ij}\frac{\delta G}{\delta\omega_j}\, dv \tag{36}$$

is antisymmetric and satisfies the Jacobi identity provided one imposes suitable boundary conditions (*e.g.*, the vanishing of the functional derivatives $\delta F/\delta\omega_i$ at the boundary). From Eqs. (35–36) we obtain the basic relations (*cf.* Eq. (19))

$$\{\omega_i(\mathbf{r},t),\omega_k(\mathbf{r}',t)\} = \frac{1}{\rho}(\epsilon_{ijk}\omega_m\partial_m\partial_j - \epsilon_{mjk}(\partial_m\omega_i)\partial_j)\delta(\mathbf{r}-\mathbf{r}') \tag{37}$$

which, under the present assumptions ($\rho =$ const.), coincide with Eq. (21), thus showing that the Hamiltonian structure defined by Eq. (35) is, in a sense, a reduction of that given by Eqs. (17).

Looking for nontrivial functionals such that $\tilde{D}_{ik}\delta C/\delta\omega_k = 0$, one finds [4,13]

$$C = \frac{\lambda}{2}\int \mathbf{u}\cdot\boldsymbol{\omega}\,dv, \tag{38}$$

where λ is an arbitrary constant, which is consistent with the results of the preceding section since, in the present case, the second integral in Eq. (25) is a constant. The expressions for the functionals corresponding to the components of the linear and angular momenta can be obtained as in Sect. 2. For example, from Eqs. (26) and (35) we have

$$-\partial_k\omega_i = \frac{1}{\rho}(\epsilon_{ijn}\omega_m\partial_m\partial_j - \epsilon_{mjn}(\partial_m\omega_i)\partial_j)\frac{\delta P_k}{\delta\omega_n},$$

which are satisfied if $\frac{1}{\rho}\epsilon_{mjn}\partial_j(\delta P_k/\delta\omega_n) = \delta_{mk}$. Hence, using Eq. (4) and integrating by parts, one finds

$$\begin{aligned} P_k &= \int \omega_n \frac{\delta P_k}{\delta\omega_n}\,dv = \int(\epsilon_{njm}\partial_j u_m)\frac{\delta P_k}{\delta\omega_n}\,dv \\ &= -\int u_m\epsilon_{njm}\partial_j\left(\frac{\delta P_k}{\delta\omega_n}\right)dv = \int \rho u_m\delta_{mk}\,dv = \int \rho u_k\,dv, \end{aligned}$$

which coincides with Eq. (29). In a similar manner one finds that the components of the angular momentum are given by Eq. (30) up to the addition of a functional of the form (38). Alternatively, expressions (29-30) can be obtained looking for the conserved quantities associated with the translational and rotational invariance of the Hamiltonian functional [13].

4. Concluding remarks

The example considered here illustrates the advantages of the Hamiltonian formulation based on Eqs. (16) (compare, *e.g.*, Ref. [8] and the references cited therein). It is interesting to notice that the equation of continuity has to be taken into account in order to write the Euler equations in the Hamiltonian form (16). Another feature of this system is that, by contrast with other continuous systems for which the Hamiltonians corresponding to different choices of the field variables are not related by a simple change of variables (consider, *e.g.*, the case of sound waves [1,2]), the Hamiltonian and the linear and angular momenta have essentially the same form whether one uses the velocity and the density as field variables or the vorticity. In spite of this fact, the boundary conditions required to have a Hamiltonian structure in each case are different.

It should be remarked that, in order to write the evolution equations of a given continuous system in the Hamiltonian form (1), it is necessary to choose appropriately the field variables ϕ_α (see also Ref. [10]), which may be a difficult task.

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