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Approximation on a finite set of points through Kravchuk functions

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ABSTRACT. In a harmonic oscillator environment, such as Fourier optics in a multimodal parabolic index-profile fiber, data on a finite set of discrete observation points can be used to reconstruct the sampled wavefunction through partial wave synthesis of harmonic oscillator eigenfunctions. This procedure is generally far from optimal because a nondiagonal matrix must be inverted. Here it is shown that Kravchuk orthogonal functions (those obtained from Kravchuk polynomials by multiplication with the square root of the weight function) not only simplify the inversion algorithm for the coefficients, but also have a well-defined analytical structure *inside* the measurement interval. They can be regarded as the best set of approximants because, as the number of sampling points increases, these expansions become the standard oscillator expansion.

RESUMEN. Es un ambiente de oscilador armónico, tal como la óptica de Fourier en una fibra multimodal de perfil de índice de refracción parabólico, los datos sobre un conjunto finito de puntos de observación pueden ser usados para reconstruir la función de onda que se está muestreando, desarrollándola en eigenfunciones de oscilador armónico. Este procedimiento es generalmente lejano al óptimo porque es necesario invertir matrices no diagonales. Aquí mostramos que las funciones ortogonales de Kravchuk (aquellas obtenidas de polinomios de Kravchuk multiplicados por la raíz de su función de peso) no sólo simplifican el algoritmo de inversión para obtener sus coeficientes, sino que también proveen una estructura analítica bien definida *dentro* del intervalo de medición. Pueden considerarse como el mejor conjunto aproximante porque, conforme crece el número de puntos muestreados, este desarrollo se vuelve el desarrollo estándar en funciones de oscilador armónico.

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1. INTRODUCTION

Harmonic oscillator wavefunctions are widely used as a convenient basis for the expansion of square-integrable functions. When the "function" is the result of a finite number of measurements on a line segment, it is common to continue to use this set of functions even though the interpolation between measured points is thereby fixed by mathematics rather

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than physics, and its extrapolation to the real line outside the interval of measurement will bear the Gaussian decrease factor of this set.

In this article we discuss the approximation of a function given by its values on a finite number of points by means of the Kravchuk functions, which are the discrete analogues of harmonic oscillator functions [1,2]. We find that the use of Kravchuk functions not only simplifies the inversion algorithm for obtaining the linear combination coefficients, but also has a well-defined analytical structure *inside* the measurement interval. Lastly, when the number of sampling points increases, the Kravchuk function expansion becomes the standard harmonic oscillator expansion.

The physical problem to which we direct this development is the following: consider a plane, multimode optical waveguide whose refractive index profile is parabolic. Optical fibers present such a profile in the paraxial approximation; a scalar wavefield travelling along the fiber behaves as a quantum harmonic oscillator wavefunction in time. That is, we expect the harmonic oscillator wavefunctions to be a "good" physical basis of functions to describe mode propagation, as well as mixing by imperfections, accidental or designed. A finite line array of point sensors across the fiber yields a finite number of complex field values (amplitude and phase), whose mode content we want to determine —efficiently. Clearly, it is hardly optimal to use the standard "continuous" harmonic oscillator wavefunctions because they are not orthogonal over the set of sensors, and because they allow the presence of more modes than observation points. Rather, we require a finite system of functions, orthogonal on the same finite number of points, having a definite and close relation with the oscillator system, such that in the limit of a continuous measurement device become the standard harmonic oscillator wavefunctions.

In Sect. 2 we review the standard harmonic oscillator expansion series and its discrete version over a set of sampling points. In Sect. 3 we describe the properties of the Kravchuk functions; by analogy with the *Hermite functions* [3], the *Kravchuk functions* are the product of the Kravchuk polynomials times the square root of the weight function. Sect. 4 is devoted to the construction of the finite approximation to a square-integrable function in terms of the symmetric Kravchuk functions. In Sect. 5 we discuss discrete position and momentum functions, that correspond to localized states. Finally, we give in the Appendix the three-term recurrence relations for the symmetric Kravchuk polynomials $k_n^{(1/2)}(x+\frac{1}{2}N, N)$ and their explicit forms for n up to 10.

2. HARMONIC OSCILLATOR EXPANSIONS OVER A LATTICE OF SAMPLING POINTS

The harmonic oscillator eigenfuctions are

$$\psi_n(\xi) = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n!}} H_n(\xi)\, e^{-\xi^2/2}, \qquad (n = 0, 1, 2, \ldots), \tag{2.1}$$

where $H_n(\xi)$ are the Hermite polynomials and

$$\xi = \sqrt{m\omega/\hbar}\, x \tag{2.2}$$

is a dimensionless variable expressed in terms of the oscillator mass m, the oscillation frequency ω, the reduced Planck constant $\hbar$, and the position coordinate x. These functions are orthonormal under the $\mathcal{L}^2()$ inner product, integration over the full real line:

$$(\psi_m, \psi_n)_{\mathbb{R}} = \int_{-\infty}^{\infty} d\xi\, \psi_m(\xi)\psi_n(\xi) = \delta_{m,n} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } m \neq n. \end{cases} \tag{2.3}$$

Thus, an arbitrary function $f(\xi) \in \mathcal{L}^2(\mathbb{R})$ can be approximated in the norm by the expansion

$$f(\xi) = \sum_{n=0}^{\infty} c_n\, \psi_n(\xi), \tag{2.4}$$

where the expansion coefficients $\{c_n\}_{n=0}^{\infty}$ are determined by performing the integrals in

$$c_n = (\psi_n, f)_{\mathbb{R}} = \int_{-\infty}^{\infty} d\xi\, \psi_n(\xi) f(\xi). \tag{2.5}$$

When the $N+1$ values $\{f(\xi_j)\}_{j=0}^{N}$ of function $f(\xi)$ are known only at a set of $N+1$ discrete "sampling" points $\xi_0, \xi_1 \ldots \xi_N$, that are equidistant by $h = \xi_{j+1} - \xi_j$, and form an array that is symmetric around the origin,

$$\xi_0 = -\tfrac{1}{2}Nh, \ldots, \xi_j = \left(-\tfrac{1}{2}N + j\right)h, \ldots, \xi_N = \tfrac{1}{2}Nh, \tag{2.6}$$

then we can build an expansion that uses the first $N+1$ harmonic oscillator eigenfunctions (2.1) in the following way:

$$f(\xi_j) = \sum_{n=0}^{N} c_n^{(N)} \psi_n(\xi_j) \qquad (j = 0, 1, \ldots, N). \tag{2.7}$$

The task to determine the $N+1$ coefficients $\{c^{(N)}\}_{n=0}^{N}$ is simplified when we write these equations in matrix form

$$\mathbf{f} = \mathbf{\Psi}^{(N)} \mathbf{c}^{(N)}, \tag{2.8}$$

abbreviating by $(N+1)$-component vectors the observed values at the points and the coefficients,

$$\mathbf{f} = \begin{pmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_N) \end{pmatrix}, \qquad \mathbf{c}^{(N)} = \begin{pmatrix} c_0^{(N)} \\ c_1^{(N)} \\ \vdots \\ c_N^{(N)} \end{pmatrix}, \tag{2.9}$$

and the $(N+1)\times(N+1)$ transformation matrix between the two,

$$\mathbf{\Psi}^{(N)} = \begin{pmatrix} \psi_0(\xi_0) & \psi_0(\xi_1) & \dots & \psi_0(\xi_N) \\ \psi_1(\xi_0) & \psi_1(\xi_1) & \dots & \psi_1(\xi_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(\xi_0) & \psi_N(\xi_1) & \dots & \psi_N(\xi_N) \end{pmatrix}, \tag{2.10}$$

that depends only on the separation $h > 0$ and the number N. The traditional method to solve for the coefficients of the expansion (2.7) needs the inversion of the matrix (2.10). Therefore, a more efficient algorithm should sidetrack this matrix inversion.

3. Kravchuk functions as difference analogs of the oscillator eigenfunctions

Kravchuk polynomials $k_n^{(p)}(x,N)$ of degree $0 \le n \le N$, in the variable $x \in [0,N]$, and of the parameter $0 < p < 1$, are related to the binomial distribution of probability theory [4,5]. They satisfy the three-term recurrence relation

$$[x-n-p(N-2n)]k_n^{(p)}(x,N) = (n+1)k_{n+1}^{(p)}(x,N)+p(1-p)(N-n+1)k_{n-1}^{(p)}(x,N), \tag{3.1}$$

and are related to the Gauss hypergeometric function through $k_n^{(p)}(x,N) = (-1)^n \times C_N^n p^n F(-n,-x,-N;p^{-1})$. For each $N = 1,2,\dots$, the $N+1$ Kravchuk polynomials $\{k_n^{(p)}(x,N)\}_{n=0}^{N}$ are an orthogonal set with respect to a discrete weight function with finite support, namely

$$\sum_{j=0}^{N} \varrho(j)\, k_m^{(p)}(j,N)\, k_n^{(p)}(j,N) = d_n^2 \delta_{m,n}, \tag{3.2}$$

with $d_n^2 = C_N^n[p(1-p)]^n$, where $C_N^n = \binom{N}{n} = N!/\Gamma(n+1)\,\Gamma(N-n+1)$ is the binomial coefficient, and the weight function is

$$\varrho(x) = C_N^x\, p^x (1-p)^{N-x}. \tag{3.3}$$

Kravchuk functions can be defined [1] as

$$\begin{gathered} \phi_n^{(p)}(x,N) = d_n^{-1}\, k_n^{(p)}(Np+x,N)\, \varrho^{1/2}(x+Np), \\ 0 \le n \le N, \qquad -Np \le x \le (1-p)N. \end{gathered} \tag{3.4}$$

They obey the following difference equation in the variable x:

$$\begin{aligned} &[(1-2p)x - n + 2p(1-p)N]\phi_n^{(p)}(x,N) \\ &\quad = \sqrt{p(1-p)}\left[\alpha(x)\phi_n^{(p)}(x-1,N) + \alpha(x+1)\phi_n^{(p)}(x+1,N)\right], \end{aligned} \tag{3.5}$$

where $\alpha(x) = \{(x+pN)[(1-p)N-x+1]\}^{1/2}$. From (3.2) follows the discrete orthogonality relation

$$\sum_{j=0}^{N} \phi_m^{(p)}(x_j, N)\, \phi_n^{(p)}(x_j, N) = \delta_{m,n} \tag{3.6}$$

for the Kravchuk functions at the points $x_j = (j-pN)$. Notice carefully the domain of the Kravchuk functions in the variable x. Although the sum in (3.6) is only over the $N+1$ discrete points x_j, extending between $-pN$ and $(1-p)N$, $\phi_n^{(p)}(x, N)$ is a well-defined function of x in the slightly larger interval

$$-1-pN \le x \le (1-p)N+1, \tag{3.7}$$

and is *zero* at the endpoints.

The Kravchuk polynomials and their weight function satisfy the following limit relations

$$\lim_{N\to\infty} h^n k_n^{(p)}(pN + h^{-1}\xi, N) = \frac{1}{2^n n!} H_n(\xi), \tag{3.8a}$$

$$\lim_{N\to\infty} h^{-1} \varrho(pN + h^{-1}\xi) = \frac{1}{\sqrt{\pi}}\, e^{-\xi^2}, \qquad h^{-1} = \sqrt{2p(1-p)N}, \tag{3.8b}$$

with the Hermite polynomials $H_n(\xi)$ and their weight function. Therefore, the Kravchuk functions coincide with the harmonic oscillator functions (2.1) in the limit as $N \to \infty$, namely

$$\lim_{N\to\infty} h^{-1/2} \phi_n^{(p)}(h^{-1}\xi, N) = \psi_n(\xi). \tag{3.9}$$

If one writes the Eq. (3.5) in the form

$$\mathbf{H}^{(N)}(x)\, \phi_n^{(p)}(x, N) = \left(n + \tfrac{1}{2}\right)\phi_n^{(p)}(x, N), \tag{3.10}$$

with the difference "Hamiltonian"

$$\mathbf{H}^{(N)}(x) = (1-2p)x + 2p(1-p)N + \tfrac{1}{2} - \sqrt{p(1-p)}\left[\alpha(x)\, e^{-\partial_x} + \alpha(x+1)\, e^{\partial_x}\right], \tag{3.11}$$

then, with the aid of the factorization of $\mathbf{H}^{(N)}(x)$ (for details, see Refs. [1] and [6]) it is not difficult to construct a spectrum-generating algebra for this Hamiltonian, which turns out to be the Lie algebra of the SO(3) group [or its homomorphic group SU(2)].

We recall that the generators of the SO(3) group satisfy the commutation relations $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$ (see, for example Ref. [7]). In the space of an irreducible representation of SO(3), the Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2 = J_+J_- + J_z(J_z - 1)$ is an $\ell(\ell+1)$ multiple of the identity operator. This irreducible space is of dimension $2\ell+1$ and the action of the generators is given by the well-known formulas

$$J_z f_m = m f_m, \qquad J_+ f_m = \alpha^\ell_{m+1} f_{m+1}, \qquad J_- f_m = \alpha^\ell_m f_{m-1}, \tag{3.12a}$$

$$\alpha^\ell_m = \sqrt{(\ell+m)(\ell-m+1)}, \qquad -\ell \le m \le \ell. \tag{3.12b}$$

Thus, the Kravchuk functions $\phi_n^{(p)}(x,N)$, $n=0,1,\dots,N$ form bases for irreducible representations of the group SO(3), corresponding to the eigenvalues $\ell = \frac{1}{2}N$ of the invariant Casimir operator; the eigenvalues of J_z are $m = n - \frac{1}{2}N = n - \ell$. Moreover, the representations corresponding to different values of the parameter p turn out to be unitarily equivalent [1], and therefore it is in fact sufficient to consider a set of functions $\phi_n^{(p)}(x,N)$ with some fixed value of this parameter. It is convenient to choose the value $p=\frac{1}{2}$, since in this case the Kravchuk functions have definite parity with respect to the change of sign of x, *i.e.*,

$$\phi_n^{(1/2)}(-x,N) = (-1)^n \phi_n^{(1/2)}(x,N). \tag{3.13}$$

Also, in the $p=\frac{1}{2}$ case the double commutator of the corresponding Hamiltonian with the variable x is [1]

$$\left[\mathbf{H}^{(N)}(x), \left[\mathbf{H}^{(N)}(x), x\right]\right] = x. \tag{3.14}$$

We recall in this connection that the quantum-mechanical analogue of Newton's equation is $m\dot{v} = -\partial U/\partial x$, where m is the mass, $v = \dot{x} = i\hbar^{-1}[H,x]$ is the velocity operator, and $H(x) = p^2/2m + U(x)$ is the Hamiltonian (see, for example, Ref. [8]). For the harmonic oscillator, *i.e.*, when we have $U(x) = \frac{1}{2}m\omega^2x^2$, this equation takes the form $[H,[H,x]] = (\hbar\omega)^2 x$. Therefore, Eq. (3.14) can be regarded as the difference analogue of the equation of motion of the linear harmonic oscillator in the Schrödinger representation, $H^{(N)}(x)$ being the analogue of $H(x)/\hbar\omega$.

Thus, in the subsequent discussion we shall use the *symmetric Kravchuk functions*

$$\phi_n(x,N) = 2^{n-N/2} k_n(x+\tfrac{1}{2}N, N) \sqrt{\frac{n!\,(N-n)!}{\Gamma\left(\frac{1}{2}N+x+1\right)\Gamma\left(\frac{1}{2}N-x+1\right)}}, \tag{3.15}$$

implying that they are the $p=\frac{1}{2}$ case of the definition (3.4). The symmetric Kravchuk polynomials are given in the Appendix for $n=0,1,\dots,10$. In Figs. 1 we plot the first few Kravchuk functions, comparing them with the corresponding harmonic oscillator functions over the interval (3.7).

Indeed, Kravchuk polynomials have many remarkable mathematical properties: they are related to the Wigner D–functions, the unitary irreducible matrix elements of the SU(2) group. These and other classical orthogonal polynomials of discrete variable have

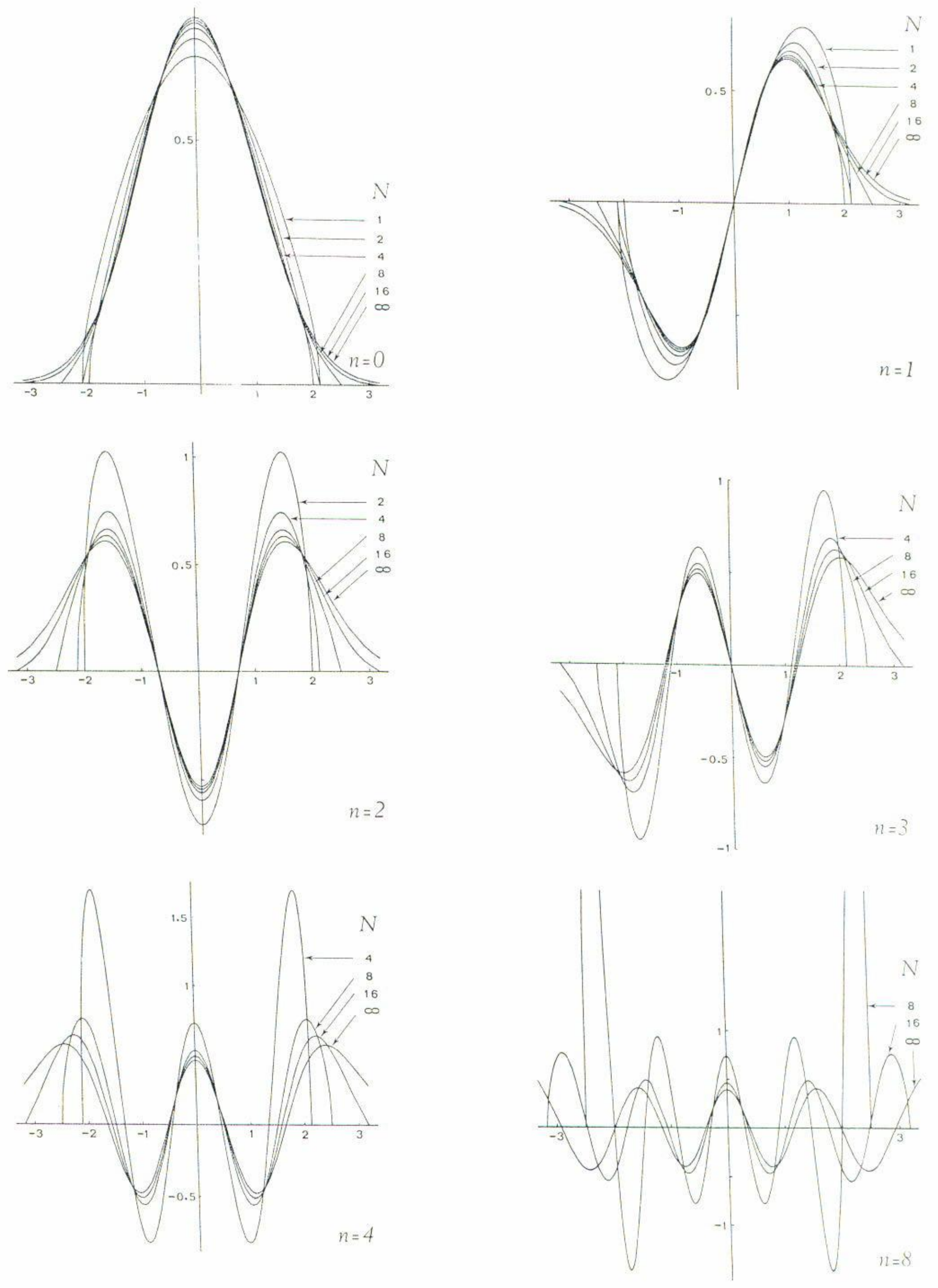

FIGURE 1. Kravchuk functions $h^{-1/2}\phi_n(\xi, N)$ for $n = 0, 1, 2, 3, 4$, and 8, where $h = \sqrt{2/N}$. For each n we show the functions with compatible values of $N \in \{1, 2, 4, 8, 16\}$, and the $N \to \infty$ limit that corresponds to the harmonic oscillator wavefunctions. The Kravchuk functions of parameter N have endpoint zeros at $\xi = \pm(\sqrt{N/2} + \sqrt{2/N})$ which are *beyond* the orthogonality interval $|\xi| \le \sqrt{N/2}$.

been recently of much interest (see Ref. [9]). In this paper we do not need properties beyond those given in this section.

4. FINITE APPROXIMATION THROUGH THE SYMMETRIC KRAVCHUK FUNCTIONS

Let us return finally to the expansion of a function $f(\xi)$ that "lives in a harmonic oscillator environment", *i.e.*, a function whose harmonic oscillator expansion is physically meaningful, but of which we know only the values on a set of $N+1$ equidistant points ξ_j seen above. For such a function one can write the expansion in terms of the symmetric Kravchuk functions (3.15) as

$$f(\xi_j) = \frac{1}{\sqrt{h}} \sum_{n=0}^{N} \kappa_n^{(N)} \, \phi_n(\xi_j/h, N) \qquad (j = 0, 1, \ldots, N). \tag{4.1}$$

The advantage of the Kravchuk basis is the orthonormality property (3.6): when we multiply (4.1) by $\phi_m(\xi_j/h, N)$ and sum over the sample points ξ_j, we obtain

$$\kappa_n^{(N)} = \sqrt{h} \sum_{j=0}^{N} \phi_n(\xi_j/h, N) f(\xi_j). \tag{4.2}$$

Thus, instead of inverting the matrix (2.10) with $h = \sqrt{2/N}$, we have only the task of multiplying, for each $n = 0, 1, \ldots, N$, the sampled values $f(\xi_j)$ by the numerically calculated values of the Kravchuk functions at the points $x_j = \xi_j/h$.

Moreover, having found coefficients $\kappa_n^{(N)}$, it is possible to interpolate the function on discrete points in (4.1) to the whole line segment $[\xi_0, \xi_N]$ by writing

$$f(\xi, N) = \frac{1}{\sqrt{h}} \sum_{n=0}^{N} \kappa_n^{(N)} \, \phi_n(\xi/h, N), \qquad (j = 0, 1, \ldots, N). \tag{4.3}$$

The function $f(\xi, N)$ defined by (4.3) and (4.2) can be called a *finite approximation* to the square-integrable function $f(\xi)$ with given values on a set of $N+1$ equidistant points ξ_j. This approximant $f(\xi, N)$ is *finite* because, as a linear combination of the symmetric Kravchuk functions (3.15), for every finite N it has a finite support $(-h - h^{-1}, h + h^{-1})$, with $h = \sqrt{2/N}$. The function $f(\xi, N)$ is an *approximation* to $f(\xi)$ because its values at the points ξ_j coincide with $f(\xi_j)$ by definition and, therefore,

$$h \sum_{j=0}^{N} f^2(\xi_j, N) = h \sum_{j=0}^{N} f^2(\xi_j) = \sum_{n=0}^{N} \big(\kappa_n^{(N)}\big)^2. \tag{4.4}$$

When N grows, the approximation to $f(\xi)$ becomes better since in the limit $N \to \infty$, the formulas (4.3), (4.2), and (4.4) coincide with (2.4), (2.5), and the Parseval formula

$$\int_{-\infty}^{\infty} d\xi\, f(\xi)^2 = \sum_{n=0}^{\infty} c_n^2, \tag{4.5}$$

respectively, with $c_n = \kappa_n^{(\infty)}$.

To follow the time evolution of the approximating function, one should multiply each summand $\phi_n(\xi_j, N)$ by the usual time dependence $\exp(-iE_n t/\hbar)$, with the energy eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$.

5. Position and momentum functions

The set of values $\{f(\xi_j)\}_{j=0}^{N}$ were placed in a $N+1$ dimensional column vector in the matrix Eqs. (2.8)–(2.10). In this notation, the simplest vector basis are the *position* basis functions

$$\mathbf{\Lambda}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{\Lambda}_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \quad \mathbf{\Lambda}_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{5.1}$$

namely, functions of discrete variable $\xi_j = hx_j$ with the values given by

$$\Lambda_k(\xi_j, N) = \delta_{k,j}. \tag{5.2}$$

It is evident that any other discrete function $f(\xi_j)$ can be expressed through these basis functions, *i.e.*,

$$f(\xi_j) = \sum_{k=0}^{N} f_k^{(N)}\, \Lambda_k(\xi_j, N) = f_j^{(N)}, \qquad (j = 0, 1, \ldots, N). \tag{5.3}$$

Therefore, we can expand the basis functions (5.2) in terms of the Kravchuk functions (3.15) as

$$\Lambda_k(\xi_j, N) = \frac{1}{\sqrt{h}} \sum_{n=0}^{N} \lambda_{k,n}^{(N)}\, \phi_n(\xi_j/h, N), \qquad (j = 0, 1, \ldots, N), \tag{5.4}$$

where the coefficients are

$$\lambda_{k,n}^{(N)} = \sqrt{h}\, \phi_n(\xi_k/h, N). \tag{5.5}$$

From (3.13) and $\xi_{N-j} = -\xi_j$, it follows that $\Lambda_{N-k}(\xi, N) = \Lambda_k(-\xi, N)$. In Figs. 2 we show the interpolation of the values of

$$\Lambda_k(\xi, N) = \sum_{n=0}^{N} \phi_n(\xi_k/h, N)\, \phi_n(\xi/h, N), \tag{5.6}$$

for $k = 0, 1, \ldots, [N/2]$ and continuous ξ. These functions represent the most localized states of the discrete oscillator.

Momentum basis functions are defined exactly in the same way, because Kravchuk functions are self-reproducing under the discrete Fourier transformation [2]. Thus,

$$\tilde{\Lambda}_k(\xi, N) = \sum_{n=0}^{N} i^n\, \phi_n(\xi_k/h, N)\, \phi_n(\xi/h, N). \tag{5.7}$$

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APPENDIX

The symmetric Kravchuk polynomials $\widetilde{k}_n(x, N) = k_n^{(1/2)}(x + \frac{1}{2}N, N)$ satisfy the three-term recurrence relations:

$$(n+1)\widetilde{k}_{n+1}(x, N) = x\widetilde{k}_n(x, N) - \tfrac{1}{4}(N - n + 1)\widetilde{k}_{n-1}(x, N).$$

The first eleven polynomials are:

$$\widetilde{k}_0(x, N) = 1, \qquad \widetilde{k}_1(x, N) = x, \qquad \widetilde{k}_2(x, N) = \tfrac{1}{2}\left[x^2 - \tfrac{1}{4}N\right],$$

$$\widetilde{k}_3(x, N) = \tfrac{1}{3!}\left[x^3 - \tfrac{1}{4}(3N-2)x\right], \quad \widetilde{k}_4(x, N) = \tfrac{1}{4!}\left[x^4 - \tfrac{1}{2}(3N-4)x^2 + \tfrac{3}{16}N(N-2)\right],$$

$$\widetilde{k}_5(x, N) = \tfrac{1}{5!}\left[x^5 - \tfrac{5}{2}(N-2)x^3 + \tfrac{1}{16}(15N^2 - 50N + 24)x\right],$$

$$\widetilde{k}_6(x, N) = \tfrac{1}{6!}\left[x^6 - \tfrac{5}{4}(3N-8)x^4 + \tfrac{1}{16}(45N^2 - 210N + 184)x^2 - \tfrac{15}{64}N(N^2 - 6N + 8)\right],$$

$$\widetilde{k}_7(x, N) = \tfrac{1}{7!}\left[x^7 - \tfrac{7}{4}(3N-10)x^5 + \tfrac{7}{16}(15N^2 - 90N + 112)x^3 \right.$$
$$\left. - \tfrac{3}{64}(35N^3 - 280N^2 + 588N - 240)x\right],$$

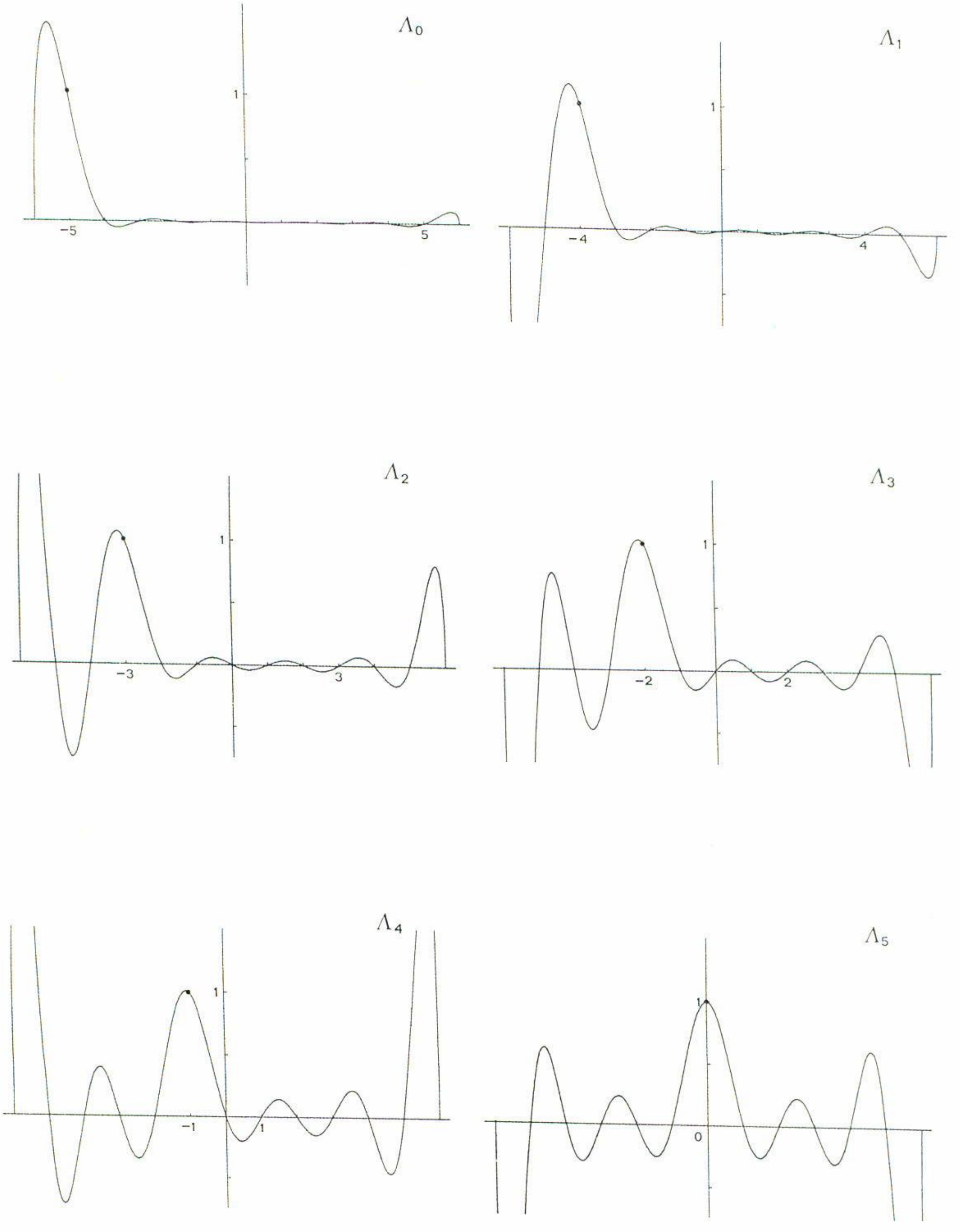

FIGURE 2. The functions $\Lambda_k(\xi, N)$ for $N = 10$, in the interval $|\xi| \leq \sqrt{N/2} + \sqrt{2/N} \approx 2.68\ldots$, for $k = 0, \ldots, [N/2] = 5$. The rest of the functions are obtained from $\Lambda_k(\xi, N) = \Lambda_{N-k}(-\xi)$. Each function $\Lambda_k(\xi, N)$ is equal to 1 (indicated by the bullet) at the point ξ_k and zero at all other points of discrete orthogonality $\xi \in \{0, \pm h, \pm 2h, \pm 3h, \pm 4h, \pm 5h\}$.

$$\widetilde{k}_8(x,N) = \tfrac{1}{8!}\big[x^8 - 7(N-4)x^6 + \tfrac{7}{8}(15N^2 - 110N + 176)x^4$$
$$- \tfrac{1}{16}(105N^3 - 1050N^2 + 2968N - 2112)x^2$$
$$+ \tfrac{105}{256}N(N^3 - 12N^2 + 44N - 48)\big],$$

$$\widetilde{k}_9(x,N) = \tfrac{1}{9!}\big[x^9 - 3(3N-14)x^7 + \tfrac{21}{8}(9N^2 - 78N + 152)x^5$$
$$- \tfrac{1}{16}(315N^3 - 3780N^2 + 13356N - 13088)x^3$$
$$+ \tfrac{9}{256}(105N^4 - 1540N^3 + 7308N^2 - 12176N + 4480)x\big],$$

$$\widetilde{k}_{10}(x,N) = \tfrac{1}{10!}\big[x^{10} - \tfrac{15}{4}(3N-16)x^8 + \tfrac{21}{8}(15N^2 - 150N + 344)x^6$$
$$- \tfrac{5}{32}(315N^3 - 4410N^2 + 18648N - 22976)x^4$$
$$+ \tfrac{9}{256}(525N^4 - 9100N^3 + 52780N^2 - 115600N + 72064)x^2$$
$$- \tfrac{945}{1024}N(N^4 - 20N^3 + 140N^2 - 400N + 384)\big].$$

We note that the generic value for the coefficient of the highest power of x^n is $1/n!$. The coefficient of the next-to-highest power, namely x^{n-2}, is $-[3N-2(n-2)]/24(n-2)!$. Finally, the x-free term of the polynomials of even degree $n=2m$ is $(-1)^m\left(\frac{1}{2}\right)_m\left(\frac{1}{2}N+1-m\right)_m = (-1)^n 2^{-2n}(2n-1)!!\prod_{j=0}^{n-1}(N-2j)$, where $(a)_m = \Gamma(m+a)/\Gamma(a)$ is the Pochhammer symbol and $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$.

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