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Invariants for dissipative systems and Noether's theorem

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ABSTRACT. A recent paper used Noether's theorem to obtain first integrals for dissipative systems. The results were not complete and the physical interpretation of the first integrals was not established. This paper completes the results and provides a simple interpretation of the first integrals.

RESUMEN. Un artículo reciente utilizó el teorema de Noether para obtener primeras integrales de sistemas disipativos. Los resultados estaban incompletos y la interpretación física de las primeras integrales no fue establecida. Este artículo completa los resultados y da una interpretación simple de las integrales primeras. (*Traducción de los editores.*)

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1. INTRODUCTION

In a recent paper [1] Noether's theorem was used to determine first integrals for the physically important systems described by the Lagrangians

$$L = \tfrac{1}{2}F(t)\left[\dot{x}^2 - \omega^2(t)x^2\right] \tag{1}$$

and

$$L = \tfrac{1}{2}F(t)\left[\dot{x}^2 - \omega^2(t)x^2 + 2G(t)V(\beta(t)x)\right], \tag{2}$$

where overdots denote, as usual, differentiation with respect to time, $F(t)$ is an arbitrary real function of time, V is an arbitrary function of its argument and $G(t)$ and $\beta(t)$ are functions of time the form of which has to be determined. González-Acosta and Corona-Galindo [1] found the first integrals

$$I = \tfrac{1}{2}F^2(t)[\sigma\dot{x} - \dot{\sigma}x]^2 + \phi(x/\sigma) \tag{3}$$

and

$$I = \tfrac{1}{2}F^2(t)[\sigma\dot{x} - \dot{\sigma}x]^2 + \phi(x/\sigma) - V(x/\sigma) \tag{4}$$

for Eqs. (1) and (2), respectively, where $\sigma(t)$ is the solution of a nonlinear second order differential equation (Eqs. (21) and (37) in Ref. [1]. We do not quote the equation because it is incorrect) and $\phi(x/\sigma)$ appears to be regarded as a function that can be selected arbitrarily. (See the discussion in Ref. [1] between Eqs. (37) and (39).)

The results quoted in the paragraph above are incomplete. The reason for this is, apparently, due to the method of analysis adopted in [1]. In this note, we supply the correct results using an analysis which we hope will be more transparent to the reader. In addition, we supply an interpretation of the meaning of the first integrals obtained. To avoid unnecessary repetition we unify the two Lagrangians into

$$L = \tfrac{1}{2}F(t)\dot{x}^2 - V(x,t). \tag{5}$$

Following the method of Ref. [1] we use the point symmetry form of Noether's theorem [2]. If the action integral $S = \int L(x,\dot{x},t)\,dt$ is invariant under the infinitesimal transformation generated by

$$G = \xi(x,t)\frac{\partial}{\partial t} + \eta(x,t)\frac{\partial}{\partial x}, \tag{6}$$

then there exists a first integral given by

$$I(x,\dot{x},t) = f(x,t) - \left[L\xi + \frac{\partial L}{\partial \dot{x}}(\eta - \dot{x}\xi)\right], \tag{7}$$

where $f(x,t)$ is a gauge function. The functions ξ, η and f are determined from

$$\dot{f} = \xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial x} + \frac{\partial L}{\partial \dot{x}}(\dot{\eta} - \dot{x}\dot{\xi}) + L\dot{\xi}. \tag{8}$$

We note that the quadratic potential in Eq. (1) has also received attention by Profilo and Soliana [3,4] and Dodonov and Man'ko [5].

2. DETERMINATION OF THE FIRST INTEGRAL

When L in Eq. (5) is substituted into Eq. (8), the velocity dependence is explicit and the coefficients of linearly independent powers of $\dot{x}$ must be separately set equal to zero. This gives the equations

$$\frac{\partial \xi}{\partial x} = 0, \tag{9}$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{2}\frac{\partial \xi}{\partial t} - \frac{1}{2}\frac{\dot{F}}{F}\xi, \tag{10}$$

$$\frac{\partial f}{\partial x} = F\frac{\partial \eta}{\partial t} - V\frac{\partial \xi}{\partial x}, \tag{11}$$

$$\frac{\partial f}{\partial t} = -\xi\frac{\partial V}{\partial t} - \eta\frac{\partial V}{\partial x} - V\frac{\partial \xi}{\partial t}. \tag{12}$$

From Eqs. (9) and (10) it is evident that

$$\xi = a(t), \tag{13}$$

$$\eta = \frac{1}{2}\left(\dot{a} - \frac{\dot{F}}{F}a\right)x + b(t), \tag{14}$$

where $a(t)$ and $b(t)$ are arbitrary functions of time. Since ξ is independent of x, Eq. (11) gives

$$f = \frac{1}{4}F\left(\dot{a} - \frac{\dot{F}}{F}a\right)^{\dot{}} x^2 + F\dot{b}x + c(t), \tag{15}$$

where $c(t)$ is again an arbitrary function of time. Finally Eq. (12) has now the form

$$\frac{1}{4}\left[F\left(\dot{a} - \frac{\dot{F}}{F}a\right)^{\dot{}}\right]^{\dot{}} x^2 + (F\dot{b})^{\dot{}}\dot{x} + \dot{c} = -a\frac{\partial V}{\partial t} - \dot{a}V - \left[\frac{1}{2}\left(\dot{a} - \frac{\dot{F}}{F}a\right)x + b\right]\frac{\partial V}{\partial x}, \tag{16}$$

which, in the absence of an initially assumed structure for V, is a linear partial differential equation for $V(x,t)$.

The solution of Eq. (16) by the method of characteristics requires the solution of the associated Lagrange's system

$$\frac{dt}{1} = \frac{2a\,dx}{F\left(\frac{a}{F}\right)^{\dot{}} x + b} = \frac{d(aV)}{-\frac{1}{4}\left(F\left(F\left(\frac{a}{F}\right)^{\dot{}}\right)^{\dot{}}\right)^{\dot{}} x^2 - \left(F\dot{b}\right)^{\dot{}} x - \dot{c}}. \tag{17}$$

From the first and second terms in Eq. (17) we find that one characteristic is

$$u = x\left(\frac{a}{f}\right)^{-\frac{1}{2}} - \int\left(\frac{a}{F}\right)^{-\frac{1}{2}}\frac{b}{a}\,dt. \tag{18}$$

The other characteristic is found from the first and third terms in Eq. (17) after using Eq. (18) to eliminate x. The integration is straightforward albeit tedious. Equation (18) is then used to eliminate u and this characteristic is

$$\begin{aligned}\nu = aV &+ \frac{1}{8}\left(\frac{a}{F}\right)^{-1}\left[\frac{a^2\dot{F}^2}{F^2} - \dot{a}^2 + 2a\ddot{a} - 2a^2\frac{\ddot{F}}{F}\right]x^2 \\ &+ \left[F\dot{b} - \frac{1}{2}Fb\left(\frac{a}{F}\right)^{-1}\left(\frac{a}{F}\right)^{\dot{}}\right]x - \frac{1}{2}\frac{Fb^2}{a} + c.\end{aligned} \tag{19}$$

From Eqs. (18) and (19) we see that the potential is

$$V = -\frac{1}{8a}\left(\frac{a}{F}\right)^{-1}\left[\frac{a^2\dot{F}^2}{F^2} - \dot{a}^2 + 2a\ddot{a} - \frac{2a^2\ddot{F}}{F}\right]x^2 - \frac{1}{a}\left[F\dot{b} - \frac{1}{2}Fb\left(\frac{a}{F}\right)^{-1}\left(\frac{a}{F}\right)^{\cdot}\right]x$$
$$+\frac{1}{2}\frac{Fb^2}{a^2} - \frac{c}{a} + \frac{1}{a}U\left\{\left(\frac{a}{F}\right)^{-\frac{1}{2}}x - \int\left(\frac{a}{F}\right)^{-\frac{1}{2}}\frac{b}{a}\,dt\right\}, \tag{20}$$

where U is an arbitrary function of its argument, and the corresponding first integral is

$$I = \frac{1}{2}F^2\left[\left(\frac{a}{F}\right)^{\frac{1}{2}}\dot{x} - \left(\left(\frac{a}{F}\right)^{\frac{1}{2}}\right)^{\cdot}x - \frac{b}{a}\left(\frac{a}{F}\right)^{\frac{1}{2}}\right]^2 + U\left\{\left(\frac{a}{F}\right)^{-\frac{1}{2}}x - \int\left(\frac{a}{F}\right)^{-\frac{1}{2}}\frac{b}{a}\,dt\right\}. \tag{21}$$

Note that the arbitrary function $c(t)$ in V does not enter into I. This is not surprising, because an additive function of time in L plays no role in the dynamics. We emphasize that the functions $a(t)$ and $b(t)$ are arbitrary in contrast to the conclusion reached in Ref. [1]. There the authors imposed some structure on the potential V which forced $a(t)$ and $b(t)$ (their $\xi(t)$ and $\gamma(t)$) to be the solutions of second order equations. However, here we have constructed potential and first integral together and the only constraint is one of consistency between the two.

3. THE CASE OF A QUADRATIC POTENTIAL

We notice that Eq. (16) is a linear partial differential equation for V unless its structure is initially specified. To enable a better comparison with the results of Ref. [1] and to rectify an omission in it we consider the special case of a quadratic potential

$$V = F\left[\tfrac{1}{2}\omega^2(t)x^2 + g(t)x\right], \tag{22}$$

where the F is included explicitly as a multiplier. When Eq. (22) is substituted into Eq. (16), the position dependence is explicit and the coefficients of linearly independent powers of x may be separately set equal to zero to give the equations:

$$\frac{1}{4}\left(F\left(F\left(\frac{a}{F}\right)^{\cdot}\right)^{\cdot}\right)^{\cdot} = -a\left(\tfrac{1}{2}F\omega^2\right)^{\cdot} - \dot{a}\left(\tfrac{1}{2}F\omega^2\right) - \tfrac{1}{2}F\omega^2\left(F\left(\frac{a}{F}\right)^{\cdot}\right), \tag{23}$$

$$\left(F\dot{b}\right)^{\cdot} = -a\left(Fg\right)^{\cdot} - \dot{a}(Fg) - F\omega^2 b - \tfrac{1}{2}F\left(\frac{a}{F}\right)^{\cdot}Fg, \tag{24}$$

$$\dot{c} = -bFg, \tag{25}$$

from the coefficients of x^2, x^1 and x^0 respectively. Equation (23) is easily integrated once and on the substitution

$$\frac{a}{F} = \rho^2 \tag{26}$$

reduces to the damped form of the well-known Pinney equation [6],

$$\ddot{\rho} + \frac{\dot{F}}{F}\dot{\rho} + \omega^2\rho = \frac{K}{\rho^3 F^2}, \tag{27}$$

where K is the constant of integration. Equation (24) may be rewritten as

$$\ddot{b} + \frac{\dot{F}}{F}\dot{b} + \omega^2 b = -\frac{1}{\rho F}\left(\rho^3 F^2 g\right) \tag{28}$$

and Eq. (25) defines $c(t)$. The first integral is

$$I = \tfrac{1}{2}F^2(\rho\dot{x} - \dot{\rho}x)^2 + \frac{K}{2}\frac{x^2}{\rho^2} + \rho^2 F^2 g x + c + F\dot{b}x - Fb\dot{x}. \tag{29}$$

The expression in Eq. (29) actually contains three first integrals and not one as implied by Eq. (23) of Ref. [1]. The reason for this is that the solution of Eq. (28) contains two linearly independent solutions of the homogeneous equation

$$\ddot{b} + \frac{\dot{F}}{F}\dot{b} + \omega^2 b = 0, \tag{30}$$

as well as a particular solution of Eq. (28). Specifically, if we write

$$b = Ab_1 + Bb_2 + b_{\mathrm{P}}, \tag{31}$$

the three integrals are

$$I_1 = \tfrac{1}{2}F^2(\rho\dot{x} - \dot{\rho}x)^2 + \frac{K}{2}\frac{x^2}{\rho^2} + \rho^2 F^2 g x - \int F g b_{\mathrm{P}}\, dt + F\left(\dot{b}_{\mathrm{P}}x - b_{\mathrm{P}}\dot{x}\right), \tag{32}$$

which is the generalization of the Lewis-Ermakov invariant [7,8] to the time-dependent damped and forced harmonic oscillator, and

$$I_2 = F\left(\dot{b}_1 x - b_1\dot{x}\right) - \int F g b_1\, dt, \tag{33}$$

$$I_3 = F\left(\dot{b}_2 x - b_2\dot{x}\right) - \int F g b_2\, dt, \tag{34}$$

which are the linear first integrals associated with the initial conditions.

4. Interpretation of the first integral

The Hamiltonian corresponding to the Lagrangian is

$$H = \frac{1}{2}\frac{p^2}{F} + V, \tag{35}$$

where

$$p = F\dot{x}. \tag{36}$$

The expression for the potential in Eq. (20) is considerably simplified if we use Eq. (26) and

$$\int \left(\frac{a}{f}\right)^{-\frac{1}{2}} \frac{b}{a}\, dt = \frac{\alpha}{\rho}. \tag{37}$$

Then

$$V = -\frac{1}{2}\rho(\ddot{\rho}F + \dot{\rho}\dot{F})x^2 - \frac{1}{\rho^2}\left[(\rho^3 F)^{\cdot}\left(\frac{\alpha}{\rho}\right)^{\cdot} + (\rho^3 F)\left(\frac{\alpha}{\rho}\right)^{\cdot\cdot} - \rho^2\dot{\rho}F\left(\frac{\alpha}{\rho}\right)^{\cdot}\right]x$$

$$+\frac{1}{2}\rho^2 F\left(\frac{\alpha}{\rho}\right)^{\cdot 2} + \frac{1}{\rho^2 F}U\left(\frac{x-\alpha}{\rho}\right). \tag{38}$$

The $c(t)$ is omitted since, as we noted above, it plays no role in the dynamics or the structure of the first integral. Evidently a suitable new position variable is the argument of the arbitrary function U, *i.e.*,

$$Q = \frac{x-\alpha}{\rho}. \tag{39}$$

A type II generating function is

$$F_2(x, P, t) = P\left(\frac{x-\alpha}{\rho}\right) + \chi(x,t), \tag{40}$$

which gives

$$p = \frac{P}{\rho} + \frac{\partial\chi}{\partial x}, \tag{41}$$

where the still arbitrary function $\chi(q,t)$ can be chosen at convenience. Under this canonical transformation the Hamiltonian (35) with V as in Eq. (38) is transformed to

$$\overline{H} = \frac{1}{\rho^2 F}\left\{\tfrac{1}{2}P^2 + U(Q)\right\}, \tag{42}$$

provided we put

$$\chi = \frac{1}{2}\frac{F\dot{\rho}}{\rho}x^2 + F\rho\left(\frac{\alpha}{\rho}\right)^{\cdot} x - \int F\rho^2\left(\frac{\alpha}{\rho}\right)^{\cdot 2} dt. \tag{43}$$

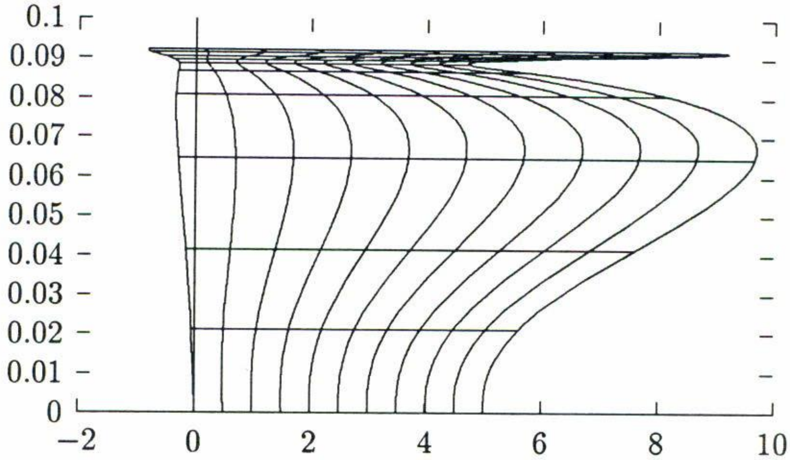

FIGURE. The grid in (Q,T) space of the level lines x = const., t = const. for $x = 1(1)10$ and $t = 0.1(0.1)1.0$ in the case $F(t) = \exp(5t)$, $\alpha(t) = \sin t$ and $\rho(t) = 1 + \cos^2(5t)$.

If we introduce a new time

$$T = \int \left(\rho^2 F\right)^{-1} dt, \tag{44}$$

the Hamiltonian becomes

$$\tilde{H} = \tfrac{1}{2}P^2 + U(Q), \tag{45}$$

which is just the first integral given in Eq. (21).

Thus the first integral is simply the Hamiltonian obtained from the original one under the generalized canonical transformation [9,10]

$$Q = \frac{x-\alpha}{\rho}, \tag{46}$$

$$P = \rho p - F\dot{\rho}x - F\rho^2 \left(\frac{\alpha}{\rho}\right)^{\cdot}, \tag{47}$$

$$T = \int \left(\rho^2 F\right)^{-1} dt. \tag{48}$$

5. CONCLUSION

In this note we have corrected some misconceptions and omissions in Ref. [1]. In particular we have been able to provide a natural interpretation for the first integral. This interpretation is perhaps best viewed as mathematical as, if the variables x and t represent the usual physical displacement and time, it is unlikely that we can assign a physical interpretation to Q and T. The figure illustrates the relationship between the (Q, T) and (x, t) variables for a not untypical choice of the functions $F(t)$, $\alpha(t)$ and $\rho(t)$.

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