A class of conformally flat solutions for null electromagnetic field

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ABSTRACT. For null electromagnetic field, a class of conformally flat solutions to the Einstein-Maxwell equations is obtained. The null field is interpreted as the field due to a plane-wave with propagation vector constituting a geodesic, shear-free, twist-free and hypersurface orthogonal congruences.

RESUMEN. Se obtiene una clase de soluciones conformalmente planas de las ecuaciones de Einstein-Maxwell para campos electromagnéticos nulos. El campo nulo se interpreta como el campo debido a una onda plana cuyo vector de propagación constituye una congruencia geodésica, sin distorsión ni rotación y ortogonal a una hipersuperficie. (*Traducción de la redacción.*)

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1. INTRODUCTION

In their investigations for the electromagnetic fields in general relativity, Debney and Zund [1] stated a theorem on the coupling of the expansion and twist of the principal null congruence of a non-singular (non-null) electromagnetic field. They make no use of the Einstein-Maxwell equations in stating and proving the coupling theorem. Meanwhile, Tariq and Tupper [2] used the Einstein-Maxwell equations to obtain the coupling theorem and found out an example of a space-time satisfying the properties of the coupling theorem (see Ref. [3]).

The expansion θ_ℓ, θ_n and twist ω_ℓ, ω_n of the congruences associated with ℓ^μ, n^μ are given by [4]

$$\theta_\ell = -\tfrac{1}{2}(\rho+\bar{\rho}), \qquad \theta_n = -\tfrac{1}{2}(\mu+\bar{\mu}),$$

$$\omega_\ell^2 = -\tfrac{1}{4}(\rho-\bar{\rho}), \qquad \omega_n^2 = -\tfrac{1}{4}(\mu-\bar{\mu}).$$

In Ref. [1], equations $\rho\bar{\mu} = \bar{\rho}\mu$ and $D\mu+\Delta\rho = 0$ describe the "coupling" of the expansion and twist of the two congruences, and are coupled in the sense $\theta_\ell = 0$ if and only if $\theta_n = 0$ and $\omega_\ell = 0$ if and only if $\omega_n = 0$.

In this paper, we have considered the null electromagnetic fields. It is seen that the coupling of twist and expansion is not possible for this case. A metric describing such situation is obtained which is interpreted as the field due to a plane wave with propagation vector constituting a geodesic, shear-free, twist-free and hypersurface orthogonal congruence. The solution thus presented turns out to be conformally flat.

We shall use the Newman-Penrose formalism [5] for our investigations. Since this formalism is widely known and accepted, it needs not be redefined and the equations quoted directly from it will be prefixed by Newman and Penrose. We first give some basic remarks that are necessary for our investigations. Section 2 contains all the field equations and their simplifications and Sect. 3, the solution of the field equations.

A vacuum metric contains a geodesic ray if and only if there exists a principal null direction of the curvature tensor such that

$$\ell_{[\mu}R_{\alpha]\beta\gamma[\delta}\ell_{\nu]}\ell^{\beta}\ell^{\gamma}=0, \tag{1}$$

i.e., tangent to a congruence of null geodesic:

$$\ell_{\mu;\nu}\ell^{\nu}=0, \qquad \ell^{\mu}\ell_{\mu}=0. \tag{2}$$

Assume that ℓ_μ is a gradient vector,

$$\ell_{\mu}=u_{,\mu}. \tag{3}$$

We choose [5] as coordinates $u=x^1$, $r=x^2$ and x^3, x^4, where r is an affine parameter along the null geodesics and x^3, x^4 label the geodesic on each surface $u=$ constant. The metric has the form

$$g^{\mu\nu}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & g^{24} \\ 0 & g^{23} & & \\ & & g^{ij} & \\ 0 & g^{24} & & \end{bmatrix} \qquad (i,j=3,4). \tag{4}$$

The null vectors ℓ^μ, n^μ and m^μ are

$$\begin{aligned} \ell^{\mu}&=\delta^{\mu}_{2}, \qquad n^{\mu}=\delta^{\mu}_{1}+U\delta^{\mu}_{2}+X^{i}\delta^{\mu}_{i},\\ m^{\mu}&=\omega\delta^{\mu}_{2}+\xi^{i}\delta^{\mu}_{i}. \end{aligned} \tag{5}$$

The metric components are

$$\begin{aligned} g^{22}&=2(U-\omega\bar{\omega}),\\ g^{2i}&=X^{i}-(\xi^{i}\bar{\omega}+\bar{\xi}^{i}\omega),\\ g^{ij}&=-(\xi^{i}\bar{\xi}^{j}+\bar{\xi}^{i}\xi^{j}), \end{aligned} \tag{6}$$

and the operators are

$$\begin{aligned} D&=\frac{\partial}{\partial r}, \qquad \Delta=U\frac{\partial}{\partial r}+\frac{\partial}{\partial u}+X^{i}\frac{\partial}{\partial x^{i}},\\ \delta&=\omega\frac{\partial}{\partial r}+\xi^{i}\frac{\partial}{\partial x^{i}}. \end{aligned} \tag{7}$$

2. The field equations and their simplifications

We recall [6] that a null electromagnetic field is characterized by the following conditions:

$$\phi_0 = \phi_1 = 0 \quad \text{and} \quad \phi_2 = \phi = 0. \tag{8}$$

Under (8), the source-free Maxwell equations (Eq.(A.I) of Ref. [5]) are

$$D\phi = (\rho - 2\epsilon)\phi, \quad \delta\phi = (\tau - 2\beta)\phi, \tag{9a}$$

$$k = \sigma = 0. \tag{9b}$$

The conditions that the principal null congruences are geodesic and the tetrad is parallely propagated along them are

$$\kappa = \epsilon = \pi = \nu = \gamma = \tau = 0. \tag{10}$$

We also assume that the congruence are hypersurface orthogonal and twist-free. This amounts to the following conditions on the spin-coefficients:

$$\rho - \bar{\rho} = \mu - \bar{\mu} = 0. \tag{11}$$

Using Eqs. (9b), (10) and (11), the field equations (Eq. (4.2) of Ref. [5]) have the following form:

$$D\rho = \rho^2, \tag{12a}$$

$$D\alpha = \rho\alpha, \tag{12b}$$

$$D\beta = \rho\beta, \tag{12c}$$

$$D\gamma = \rho\gamma, \tag{12d}$$

$$D\mu = \rho\mu, \tag{12e}$$

$$\Delta\lambda = -2\mu\lambda - \psi_4, \tag{12f}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta), \tag{12g}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta, \tag{12h}$$

$$\delta\gamma - \bar{\delta}\mu = \mu(\alpha + \bar{\beta}) + \bar{\gamma}(\bar{\alpha} - 3\beta), \tag{12i}$$

$$\Delta\mu = -(\mu^2 + \lambda\bar{\lambda}) + \phi\bar{\phi}, \tag{12j}$$

$$\Delta\beta = -\beta\mu - \alpha\bar{\lambda}, \tag{12k}$$

$$\lambda\rho = 0, \tag{12l}$$

$$\Delta\rho = -\rho\mu, \tag{12m}$$

$$\Delta\alpha = \beta\lambda - \mu\alpha, \tag{12n}$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0. \tag{12o}$$

From Eq. (12.l) we have

$$\lambda = 0. \tag{13}$$

Using Eq. (10) in Eq. (9a), we have

$$D\phi = \rho\phi, \qquad \delta\phi = -2\beta\phi. \tag{14}$$

The commutator relations (Eq. (4.4) of Ref. [5]) with Eqs. (9a), (10), (11) and (13), reduce to

$$(\Delta D - D\Delta)\eta = 0, \tag{15a}$$

$$(\delta D - D\delta)\eta = (\bar{\alpha} + \beta)\, D\eta - \rho\, \delta\eta, \tag{15b}$$

$$(\delta\Delta - \Delta\delta)\eta = -(\bar{\alpha} + \beta)\, \Delta\eta + \mu\, \delta\eta, \tag{15c}$$

$$(\delta\bar{\delta} - \bar{\delta}\delta)\eta = -(\bar{\alpha} - \beta)\, \bar{\delta}\eta - (\bar{\beta} - \alpha)\, \delta\eta, \tag{15d}$$

and their complex conjugates. Applying Eq. (14) to Eq. (15), we get

$$\Delta\rho = 0, \tag{16a}$$

$$\delta\rho = (\bar{\alpha} + 4\beta)\rho, \tag{16b}$$

$$\Delta\beta = -\mu\beta, \tag{16c}$$

$$\bar{\delta}\beta = -2(\bar{\beta} - \alpha)\beta, \tag{16d}$$

$$\alpha = -\bar{\beta}, \tag{16e}$$

$$\lambda\beta = 0. \tag{16f}$$

From Eq. (16f), we have

$$\beta = 0, \tag{17a}$$

and therefore, Eq. (16e) gives

$$\alpha = 0. \tag{17b}$$

With Eqs. (13) and (17), the field equations (12a)–(12o) and (14) reduce to

$$D\rho = \rho^2, \tag{18a}$$

$$D\mu = \bar{\delta}\mu = \delta\rho = \Delta\rho = 0, \tag{18b}$$

$$\Delta\mu = -\mu^2 + \phi\bar{\phi}, \tag{18c}$$

$$D\phi = \rho\phi, \tag{18d}$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0. \tag{18e}$$

3. A SOLUTION OF THE EQUATIONS

Using Eq. (7) and replacing η by u, r and x^i in the commutation relations (15) with (17), we get

$$\begin{aligned} DU = DX^2 = 0, \qquad & \delta U - \Delta\omega = \mu\omega, \qquad \delta\bar{\omega} - \bar{\delta}\omega = 0, \\ D\omega = \rho\omega, \qquad & \delta X^i - \Delta\xi^i = \mu\xi^i, \qquad \delta\bar{\xi}^i - \bar{\delta}\xi^i = 0, \\ D\xi^i = \rho\xi^i. & \end{aligned} \tag{19}$$

It can be easily verified that the Eqs. (18) and (19) have their solutions as follows:

$$\begin{aligned} \rho &= -\frac{1}{r}, \\ \mu &= \frac{1}{u} + \phi\bar{\phi}, \\ X^i &= U = \omega = 0, \\ \xi^3 &= A, \qquad \xi^4 = iA, \qquad A = ru. \end{aligned} \tag{20}$$

The tetrad component of the electromagnetic field tensor is

$$\phi = \frac{\phi^0}{r},$$

(ϕ^0 = constant of integration) and the tetrad components of the Weyl tensor are

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0, \tag{21}$$

which shows that the resulting space-time is conformally flat. From Eqs. (6) and (20), the tetrad components of the metric tensor are

$$\begin{aligned} &g^{21} = g^{12} = 1, \\ &g^{22} = g^{23} = g^{24} = 0, \\ &g^{ij} = -2A^2\delta^{ij}, \qquad i,j = 3,4. \end{aligned} \tag{22}$$

Equations (22) represent a class of conformally flat solutions for the null electromagnetic field. The null electromagnetic field is interpreted as the field due to a plane-wave with propagation vector constituting a geodesic, shear-free, twist-free and hypersurface orthogonal congruence. It may be noted that the component of the electromagnetic field is in the radial direction only. Moreover, it is seen that the twist and the expansion of the null congruences can not be coupled together for null electromagnetic field, while this is true for non-null electromagnetic fields [2].

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