

## Cauchy-Kovalevskaya form of the hyperheavenly equations with the cosmological constant

L.E. MORALES\*, J.F. PLEBAŃSKI AND M. PRZANOWSKI\*\*

*Centro de Investigación y de Estudios Avanzados del IPN*

*Apartado postal 14-740, 07000 México, D.F., México*

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ABSTRACT. Legendre transformations leading to the Cauchy-Kovalevskaya form of the hyperheavenly equations with  $\lambda$ , the cosmological constant, are found. The hyperheavenly concept designates a complex space with two Weyl's tensors, one of them is degenerate and the other is arbitrary. The metric so constructed fulfills Einstein's equations in vacuum but with  $\lambda$ . Some questions concerning integration of these equations are considered.

RESUMEN. Se encuentran transformaciones de Legendre que permiten escribir las ecuaciones hipercelestiales con  $\lambda$ , la constante cosmológica, en la forma de Cauchy-Kovalevskaya. El concepto "hipercelestial" denota un espacio complejo con dos tensores de Weyl, donde uno de ellos es degenerado y el otro, arbitrario. La métrica así construida satisface las ecuaciones de Einstein en el vacío, pero con constante cosmológica. Se consideran algunas cuestiones referentes a la integración de estas ecuaciones.

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### 1. INTRODUCTION

Recently a revival of interest in self-dual gravity and heavenly equations is observed. The heavenly equation which arises by reduction of the self-dual complex Einstein equations [1] appears to be a second order nonlinear partial differential equation on one holomorphic function. In Ref. [1] two forms of the heavenly equation are proposed, the *first* or the *second heavenly equation*. Then Grant [2], using the Ashtekar-Jacobson-Smolín equation [3], has found the Cauchy-Kovalevskaya form (and consequently the evolution form) of the heavenly equation. This outstanding result enabled him to find the general solution of the heavenly equation iteratively. It has been quickly observed that the first heavenly equation and Grant's equation are related by a simple Legendre transformation [2,4], and it has been demonstrated that the similar Legendre transformation leads from the second heavenly equation to the one of the Cauchy-Kovalevskaya form [4]. Then it has been proved that this latter equation passes the Painlevé test [5] and it has been shown that this equation as well as Grant's equation and the first or the second heavenly equations can be obtained by some symmetry reduction of the  $sl(\infty; C)$  self-dual Yang-Mills equations [6]. (Compare also with Refs. [7-10].)

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\*Permanent address: Av. Mazatlán 97-A, Col. Condesa, México, D.F., México.

\*\*Permanent address: Institute of Physics, Technical University of Łódź, Wólczańska 219, 93-005 Łódź, Poland.

It is also known [11–13] that assuming the algebraic degeneracy of the antiself-dual (or self-dual) part of the Weyl tensor one can reduce the complex vacuum Einstein equations to one second order nonlinear partial differential equation on one holomorphic function. This equation, called the *hyperheavenly equation*, although much more complicated than the heavenly equation, has many interesting solutions [14,15] and in our opinion the hyperheavenly equation should attract much attention as, in fact, this equation describes all real analytic vacuum algebraically special lorentzian space-times [16]. Then the similar reduction of the complex Einstein equations to one equation called the *hyperheavenly equation with  $\lambda$*  has been also done in the case of nonvanishing cosmological term  $\lambda$  [12,17,18]. The cosmological constant  $\lambda$  is the only arbitrariness the energy tensor has in Einstein's equations. This can be seen immediately since this term has zero divergence automatically.

In the previous paper [19] it has been shown that every hyperheavenly equation ( $\lambda = 0$ ) can be brought to the Cauchy-Kovalevskaya form by some Legendre transformation. Here we generalize the results of Ref. [19] to the case of  $\lambda \neq 0$ .

It is known that the crucial point in reducing complex Einstein equations to the hyperheavenly equation with  $\lambda$  appears to be the existence of an anti-self-dual null string foliation of the complex space-time. This is one of the consequences of the complex Goldberg-Sachs theorem [20–23] that such a foliation does exist for every *hyperheavenly space with  $\lambda$*  (*HH-space with  $\lambda$* ), *i.e.*, for every complex 4-dimensional Einstein space with algebraically special anti-self-dual part of the Weyl tensor. An anti-self-dual null string foliation is determined by the distribution of 2-forms  $\Sigma$  such that

$$*\Sigma = -\Sigma, \quad \Sigma \wedge \Sigma = 0 \quad \text{and} \quad d\Sigma = 0. \tag{1.1}$$

In a local coordinates system  $\{x^i\}$ ,  $i = 1, 2, 3, 4$ , one has

$$\Sigma = \frac{1}{2}\Sigma_{ij} dx^i \wedge dx^j, \quad \Sigma_{ij} = -\Sigma_{ji} \tag{1.2}$$

and

$$\Sigma_{;i}^{ij} \Sigma^{kl} = \Sigma^{ij} \Theta^k, \tag{1.3}$$

where the semi-colon “;” denotes the covariant derivative and  $\Theta^k$  is some vector field. Then one defines the expansion form to be the following 1-form:

$$\Theta := \Theta_i dx^i. \tag{1.4}$$

The null string foliation is said to be *nonexpanding* if  $\Theta = 0$ . If  $\Theta \neq 0$  then it is said to be *expanding*.

Any *HH-space with  $\lambda$*  of the type  $[\text{any}] \otimes [2-1-1]$  or  $[\text{any}] \otimes [\text{III}]$  or  $[\text{any}] \otimes [\text{N}]$  admits one and only one anti-self-dual null string foliation. If  $\lambda \neq 0$ , then for the types  $[\text{any}] \otimes [\text{III}]$  or  $[\text{any}] \otimes [\text{N}]$  this foliation is necessarily expanding but for the type  $[\text{any}] \otimes [2-1-1]$  it can be, *a priori*, expanding or nonexpanding. Any *HH-space with  $\lambda$*  of the type  $[\text{any}] \otimes [\text{D}]$  admits exactly two anti-self-dual null string foliations. When  $\lambda \neq 0$  then, *a priori* each of them can be expanding or nonexpanding. Finally, any *HH-space with  $\lambda$*  of the type  $[\text{any}] \otimes [-]$ ,



*i.e.*, any  $H$ -space with  $\lambda$  admits a variety of anti-self-dual null string foliations such that it constitutes a 3-dimensional complex differentiable manifold called the *projective twistor space*  $PT$  [24,25]. For any  $H$ -space with  $\lambda \neq 0$  every anti-self-dual null string foliation is necessarily expanding. (For details see Ref. [22].)

Our paper is organized as follows. In Sect. 2 we consider the hyperheavenly equations with  $\lambda$  for the case of  $\Theta = 0$ . the Legendre transformation is found which leads to the Cauchy-Kovalevskaya form of this equation. The case  $\Theta \neq 0$  is studied in Sect. 3. In Sect. 4 we give some comments on integration of the equation obtained. Finally, in Sect. 5 some concluding remarks are given.

## 2. HYPERHEAVENLY EQUATION WITH $\lambda$ FOR $\Theta = 0$

We start with the hyperheavenly equation with  $\lambda$  in the spinorial formalism [12,13,18]

$$\frac{1}{2}\theta_{,pApB} \cdot \theta_{,pApB} + \theta_{,pAqA} - \frac{2}{3}f^A p^B \theta_{,pApB} + f^A \theta_{,pA} + \frac{1}{18}(f^A p_A)^2 + \frac{1}{6}p^A p^B f_{A,qB} - n_A p^A + \lambda \cdot \left(-\frac{1}{3}p^A p^B \theta_{,pApB} + p^A \theta_{,pA} - \theta\right) = 0, \quad (2.1)$$

where  $q_A, p^B, A, B = 1, 2$ , are local coordinates,  $\theta = \theta(q_A, p^B)$  is an unknown function,  $f_A = f_A(q_B)$  and  $n_A = n_A(q_B)$  are arbitrary functions of  $q_B$  only;  $\theta_{,pA} \equiv \frac{\partial \theta}{\partial p^A}$ ,  $\theta_{,pAqA} \equiv \frac{\partial^2 \theta}{\partial p^A \partial q^A}$ ,  $f_{A,qB} \equiv \frac{\partial f_A}{\partial q^B}, \dots$ , etc. Spinorial indices  $A, B, \dots$ , etc. are to be manipulated according to the rule

$$\begin{aligned} \Psi_A &= \epsilon_{AB} \Psi^B \cdot \Psi^A = \epsilon^{BA} \Psi_B, \\ (\epsilon_{AB}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon^{AB}). \end{aligned} \quad (2.2)$$

The anti-self-dual part of the Weyl tensor is determined by the quantities [18]

$$\begin{aligned} \bar{C}^{(5)} = 0 = \bar{C}^{(4)}, \quad \bar{C}^{(3)} = -\frac{2}{3}\lambda, \quad \bar{C}^{(2)} = -f^A_{,qA}, \\ \bar{C}^{(1)} = \left(f^A - \frac{\partial}{\partial q_A}\right) (2n_A + f^B_{,qB} \cdot p_A) + 2\lambda n_A p^A. \end{aligned} \quad (2.3)$$

We take two constant spinors

$$L_A := (-1, 0) \quad \text{and} \quad M_A := (0, 1) \Rightarrow M^A L_A = 1. \quad (2.4)$$

Define the local coordinates  $p, q, x, y$ , and the function  $F = F(p, q, x, y)$  by

$$p := M_A q^A, \quad q := L_A q^A, \quad x := L_A p^A, \quad y := M^A p_A,$$

*i.e.*,

$$p^A = \begin{pmatrix} -x \\ -y \end{pmatrix}, \quad q_A = \begin{pmatrix} p \\ q \end{pmatrix}, \quad p_A = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad q^A = \begin{pmatrix} -q \\ p \end{pmatrix}; \quad (2.5)$$

$$\begin{aligned} F &:= \theta - \frac{2}{3}L_A L_B M_C M_D p^A p^C f^{(B} p^{D)} - \frac{\lambda}{6}L_A L_B M_C M_D p^A p^B p^C p^D \\ &= \theta + \frac{1}{3} \cdot (f \cdot y + g \cdot x)xy - \frac{1}{6}\lambda x^2 y^2; \quad (f, g) := f^A \end{aligned} \quad (2.6)$$

Then by straightforward but long calculations one gets the hyperheavenly equation with  $\lambda$  (2.1) in the form

$$\begin{aligned} &F_{,xx}F_{,yy} - F_{,xy}^2 + F_{,px} + F_{,qy} + 2 \cdot (fy + gx)F_{,xy} - fF_{,x} - gF_{,y} - \\ &\frac{1}{2}(fy + gx)^2 - \frac{1}{2} \cdot [f_{,p}y^2 + g_{,q}x^2 + (f_{,q} + g_{,p})xy] + n_1x + n_2y + \\ &\lambda \cdot [-2xyF_{,xy} + xF_{,x} + yF_{,y} - F - \frac{1}{2}\lambda x^2 y^2 + (fy + gx)xy] = 0. \end{aligned} \quad (2.7)$$

To simplify Eq. (2.7) we consider two cases: (i)  $\overline{C}^{(2)} = 0$  and (ii)  $\overline{C}^{(2)} \neq 0$ . First we deal with the case (i):

$$\overline{C}^{(2)} = 0. \quad (2.8)$$

Here the gauge exists such that [13]

$$f = 0 = g, \quad n_A = nq_A = (n \cdot p, n \cdot q), n = n(x, y). \quad (2.9)$$

Therefore Eq. (2.7) reads

$$\begin{aligned} &F_{,xx}F_{,yy} - F_{,xy}^2 + F_{,px} + F_{,qy} + n \cdot (px + qy) + \\ &\lambda \cdot (-2xyF_{,xy} + xF_{,x} + yF_{,y} - F - \frac{1}{2}\lambda x^2 y^2) = 0. \end{aligned} \quad (2.7i)$$

Now consider the case (ii):

$$\overline{C}^{(2)} \neq 0. \quad (2.10)$$

Then one can choose the gauge such that

$$f^A = \rho_0 q^A, \quad \rho_0 = \text{const.} \neq 0 \Rightarrow f = -\rho_0 q, \quad g = \rho_0 p, \quad \text{and} \quad n_A = 0. \quad (2.11)$$

Substituting (2.11) into (2.7) we get

$$\begin{aligned} &F_{,xx}F_{,yy} - F_{,xy}^2 + F_{,px} + F_{,qy} + \rho_0 \cdot [2 \cdot (px - qy)F_{,xy} + qF_{,x} - pF_{,y} - \frac{1}{2}\rho_0 \cdot (px - qy)^2] + \\ &\lambda \cdot [-2xyF_{,xy} + xF_{,x} + yF_{,y} - F - \frac{1}{2}\lambda x^2 y^2 + \rho_0 \cdot (px - qy)xy] = 0. \end{aligned} \quad (2.7ii)$$

In order to find appropriate Legendre transformations leading to the hyperheavenly equations with  $\lambda$  of the Cauchy-Kovalevskaya form we write down Eqs. (2.7i) and (2.7ii) in the differential form language [4,19]. Thus for Eqs. (2.7i) and (2.7ii) we get

$$dF_{,x} \wedge F_{,y} + dF_{,p} \wedge dy - dF_{,q} \wedge dx + n \cdot (px + qy) dx \wedge dy + \lambda \cdot (-2xy dF_{,y} \wedge dy + x dF \wedge dy - y dF \wedge dx - F dx \wedge dy - \frac{1}{2} \lambda x^2 y^2 dx \wedge dy) \} \wedge dp \wedge dq = 0 \quad (2.12i)$$

and

$$\{dF_{,x} \wedge dF_{,y} + dF_{,p} \wedge dy - dF_{,q} \wedge dx + \rho_0 \cdot [2 \cdot (px - qy) dF_{,y} \wedge dy + q dF \wedge dy + p dF \wedge dx - \frac{1}{2} \rho_0 \cdot (px - qy)^2 dx \wedge dy] + \lambda \cdot [-2xy dF_{,y} \wedge dy + x dF \wedge dy - y dF \wedge dx - F dx \wedge dy - \frac{1}{2} \lambda x^2 y^2 dx \wedge dy + \rho_0 \cdot (px - qy)xy dx \wedge dy] \} \wedge dp \wedge dq = 0, \quad (2.12ii)$$

respectively.

Perform the following Legendre transformation (compare with Refs. [4,19]; it is assumed that  $F_{,xx} \neq 0$ ):

$$z = -F_{,x} \Rightarrow x = x(p, q, z, t), \tag{2.13}$$

$$H = H(p, q, z, t) = F(p, q, x(p, q, z, t), t) + z \cdot x(p, q, z, t),$$

where we put  $t \equiv y$ . Then one finds the relations

$$F_{,p} = H_{,p}, \quad F_{,q} = H_{,q}, \quad x = H_{,z}, \quad F_{,t} = H_{,t}. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12i) and (2.12ii) we obtain

$$\{-dz \wedge dH_{,t} + dH_{,p} \wedge dt - dH_{,q} \wedge dH_{,z} + n \cdot (pH_{,z} + qt) dH_{,z} \wedge dt + \lambda \cdot [-2tH_{,z} dH_{,t} \wedge dt - tH_{,t} dt \wedge dH_{,z} - H dH_{,z} \wedge dt - \frac{1}{2} \lambda t^2 H_{,z}^2 dH_{,z} \wedge dt] \} \wedge dp \wedge dq = 0, \quad H_{,zz} \neq 0, \tag{2.15i}$$

and

$$\{-dz \wedge dH_{,t} + dH_{,p} \wedge dt - dH_{,q} \wedge dH_{,z} + \rho_0 \cdot [2 \cdot (pH_{,z} - qt) dH_{,t} \wedge dt - (qz + pH_{,t}) dH_{,z} \wedge dt - \frac{1}{2} \rho_0 \cdot (pH_{,z} - qt)^2 dH_{,z} \wedge dt] + \lambda \cdot [-2tH_{,z} dH_{,t} \wedge dt - tH_{,t} dt \wedge dH_{,z} - H dH_{,z} \wedge dt - \frac{1}{2} \lambda t^2 H_{,z}^2 dH_{,z} \wedge dt + \rho_0 \cdot (pH_{,z} - qt)tH_{,z} dH_{,z} \wedge dt] \} \wedge dp \wedge dq = 0; \quad H_{,zz} \neq 0, \tag{2.15ii}$$

respectively.



Finally, Eqs. (2.15i) and (2.15ii) can be written down in the following Cauchy-Kovalevskaya form:

$$\begin{aligned}
 H_{,tt} - H_{,zp} + H_{,zq}H_{,zt} - H_{,zz}H_{,tq} - n \cdot (pH_{,z} + qt)H_{,zz} + \\
 \lambda \cdot [2tH_{,z}H_{,zt} + (-tH_{,t} + \frac{1}{2}\lambda t^2 H_{,z}^2 + H)H_{,zz}] = 0, \quad H_{,zz} \neq 0 \quad (2.16i)
 \end{aligned}$$

and

$$\begin{aligned}
 H_{,tt} - H_{,zp} + H_{,zq}H_{,zt} - H_{,zz}H_{,tq} + \rho_0 \cdot \{2 \cdot (qt - pH_{,z})H_{,zt} + \\
 [qz + pH_{,t} + \frac{1}{2}\rho_0 \cdot (pH_{,z} - qt)^2]H_{,zz}\} + \lambda \cdot \{2tH_{,z}H_{,zt} + \\
 [-tH_{,t} + \frac{1}{2}\lambda t^2 H_{,z}^2 - \rho_0 \cdot (pH_{,z} - qt)tH_{,z} + H]H_{,zz}\} = 0, \quad H_{,zz} \neq 0, \quad (2.16ii)
 \end{aligned}$$

respectively.

[Note that if  $F_{,xx} = 0$  and one is not able to perform the Legendre transformation (2.13) then the coordinate transformation

$$t = x + p, \quad z = x - p, \quad q \rightarrow q, \quad y \rightarrow y \quad (2.17)$$

leads to the Cauchy-Kovalevskaya form of Eqs. (2.7i) and (2.7ii).]

It is obvious that defining

$$A := H_{,t} \quad (2.18)$$

one gets for (2.16i) and (2.16ii) the evolution equations on two functions  $H$  and  $A$ . From (2.3) it follows that if  $\lambda \neq 0$  then  $C^{(3)} = -\frac{2}{3}\lambda \neq 0$  and, consequently, the anti-self-dual part of the Weyl tensor is of the type D iff

$$2 [C^{(2)}]^2 - 3C^{(3)}C^{(1)} = 0 \quad (2.19)$$

or of the type [2-1-1] iff

$$2 [C^{(2)}]^2 - 3C^{(3)}C^{(1)} \neq 0. \quad (2.20)$$

First consider the case (i). Here we have [see Eqs. (2.3), (2.8) and (2.9)]

$$C^{(2)} = 0, \quad C^{(1)} = -2 \cdot (pn_{,p} + qn_{,q} + 2n) - 2\lambda n \cdot (px + qy). \quad (2.21)$$

Then the condition (2.19) is satisfied on an open set iff

$$n = 0. \quad (2.21)$$

Now for the case (ii) one gets [see Eqs. (2.3), (2.10) and (2.11)]

$$\bar{C}^{(2)} = -2\rho_0 \neq 0, \quad \bar{C}^{(1)} = 2\rho_0^2 \cdot (px + qy), \tag{2.22}$$

and it is evident that the condition (2.19) cannot be satisfied on any open set.

Gathering all that one arrives at the conclusion: *If  $\lambda \neq 0$  then Eq. (2.16i) with  $n = 0$  corresponds to the type [any]  $\otimes$  [D] and Eq. (2.16i) with  $n \neq 0$  or Eq. (2.16ii) correspond to the type [any]  $\otimes$  [2-1-1].* (The case  $\lambda = 0$  has been done in [19].)

This closes the analysis of the nonexpanding heavenly equation with  $\lambda$ .

### 3. HYPERHEAVENLY EQUATIONS WITH $\lambda$ FOR $\Theta \neq 0$

Here the original equation in the spinorial formalism reads [12,18]

$$\frac{1}{4}\varphi^4(\varphi^{-2}\theta_{,pB})_{,pA} \cdot (\varphi^{-2}\theta_{,pB})_{,pA} + \varphi^{-1}\theta_{,pAq^A} - \mu_0\varphi^4\partial_\varphi[\varphi^{-1}\partial_\varphi(\varphi^{-1}\theta)] + \nu\eta - \gamma - \frac{\lambda}{6}\varphi^{-1}\theta_{,\varphi\varphi} = 0, \tag{3.1}$$

where  $\theta = \theta(q_A, p^B)$  is an unknown function,  $\mu_0$  is a constant,  $\nu$  and  $\gamma$  are some functions of  $q_A$  only and

$$\varphi := \mathcal{J}_A n^A, \quad \eta := K^A p_A, \tag{3.2}$$

where  $\mathcal{J}_A$  and  $K_A$  are the constant spinors such that

$$K_A \mathcal{J}_B - K_B \mathcal{J}_A = \epsilon_{AB} \Leftrightarrow K^A \mathcal{J}_A = 1. \tag{3.3}$$

Similarly as in the previous papers [4,19] we define new coordinates  $\{p, q, \eta, \varphi\}$ :

$$p := \mathcal{J}_A q^A, \quad q := K_A q^A; \tag{3.4}$$

and  $\eta$  and  $\varphi$  are defined by (3.2). Then, in terms of the differential forms Eq. (3.1) can be written as follows:

$$\begin{aligned} & \{ \varphi^4 d(\varphi^{-2}\theta_{,\varphi}) \wedge d[\varphi^{-2} \cdot (\theta_{,\eta} - \mu_0\eta\varphi^2 - \frac{\lambda}{6}\eta\varphi^{-1})] - \varphi^{-1} d\theta_{,p} \wedge d\varphi + \\ & \varphi^{-1} d\theta_{,q} \wedge d\eta + \mu_0\varphi^4 d(\varphi^{-3}\theta) \wedge d\eta + (\nu\eta - \gamma) d\varphi \wedge d\eta + \frac{\lambda}{2}\varphi^{-2}\eta d\theta_{,\eta} \wedge d\eta - \\ & \frac{\lambda}{3}\varphi^{-2} d\theta \wedge d\eta \} \wedge dp \wedge dq = 0. \end{aligned} \tag{3.5}$$

Define a new function  $F = F(p, q, \eta, \varphi)$  by

$$F = \theta - \frac{1}{2} \cdot (\mu_0\varphi^2 + \frac{\lambda}{6}\varphi^{-1})\eta^2. \tag{3.6}$$

Substituting (3.6) into (3.5) we get

$$\begin{aligned} & \{ \varphi^4 d[\varphi^{-2}(F_{,\varphi} + \mu_0 \varphi \eta^2 - \frac{\lambda}{12} \varphi^{-2} \eta^2)] \wedge d(\varphi^{-2} F_{,\eta}) - \varphi^{-1} dF_{,p} \wedge d\varphi + \\ & \varphi^{-1} dF_{,q} \wedge d\eta + \mu_0 \varphi^4 d[\varphi^{-3} \cdot (F + \frac{1}{2} \mu_0 \eta^2 \varphi^2 + \frac{\lambda}{12} \eta^2 \varphi^{-1})] \wedge d\eta + \\ & (\nu \eta - \gamma) d\varphi \wedge d\eta + \frac{\lambda}{2} \varphi^{-2} \eta d[F_{,\eta} + (\mu_0 \varphi^2 + \frac{\lambda}{6} \varphi^{-1}) \eta] \wedge d\eta - \\ & \frac{\lambda}{3} \varphi^{-2} d[F + \frac{1}{2} \cdot (\mu_0 \varphi^2 + \frac{\lambda}{6} \varphi^{-1}) \eta^2] \wedge d\eta \} \wedge dp \wedge dq = 0. \end{aligned} \tag{3.7}$$

Perform the following Legendre transformation:

$$\begin{aligned} z &= -F_{,\eta} \Rightarrow \eta = \eta(p, q, z, \varphi) \\ \tilde{H} &= \tilde{H}(p, q, z, \varphi) = F(p, q, \eta(p, q, z, \varphi), \varphi) + z \cdot \eta(p, q, z, \varphi). \end{aligned} \tag{3.8}$$

(It is assumed that  $F_{,\eta\eta} \neq 0$ .) Then (3.7) and (3.8) yield

$$\begin{aligned} & \{ \varphi^4 d(\varphi^{-2} \tilde{H}_{,\varphi}) \wedge d(-\varphi^{-2} z) + \varphi^4 d[(\mu_0 \varphi^{-1} - \frac{\lambda}{12} \varphi^{-4}) \tilde{H}_{,z}^2] \wedge d(-\varphi^{-2} z) - \\ & \varphi^{-1} d\tilde{H}_{,p} \wedge d\varphi + \varphi^{-1} d\tilde{H}_{,q} \wedge \tilde{H}_{,z} + \\ & \mu_0 \varphi^4 d[\varphi^{-3} \cdot (\tilde{H} - z \tilde{H}_{,z}) + \frac{1}{2} \cdot (\mu_0 \varphi^{-1} + \frac{\lambda}{6} \varphi^{-4}) \tilde{H}_{,z}^2] \wedge d\tilde{H}_{,z} + \\ & (\nu \tilde{H}_{,z} - \gamma) d\varphi \wedge d\tilde{H}_{,z} + \frac{\lambda}{2} \varphi^{-2} \tilde{H}_{,z} d[-z + (\mu_0 \varphi^2 + \frac{\lambda}{6} \varphi^{-1}) \tilde{H}_{,z}] \wedge d\tilde{H}_{,z} - \\ & \frac{\lambda}{3} \varphi^{-2} d[\tilde{H} - z \tilde{H}_{,z} + \frac{1}{2} \cdot (\mu_0 \varphi^2 + \frac{\lambda}{6} \varphi^{-1}) \cdot \tilde{H}_{,z}^2] \wedge d\tilde{H}_{,z} \} \wedge dp \wedge dq = 0, \quad \tilde{H}_{,zz} \neq 0. \end{aligned} \tag{3.9}$$

Finally, substituting  $t := \ln \varphi$  and  $H = H(p, q, z, t) := \tilde{H}(p, q, z, e^t)$  one finds that Eq. (3.9) leads to the following equation of the Cauchy-Kovalevskaya form:

$$\begin{aligned} & H_{,tt} - e^t H_{,zp} + H_{,zq} H_{,zt} - H_{,zz} H_{,tq} + 2z H_{,zt} - 3H_{,t} + \\ & e^{2t} \cdot \{ (\gamma - \nu H_{,z}) \cdot H_{,zz} + \mu_0 \cdot [2H_{,z} H_{,zt} + (\frac{1}{2} \mu_0 H_{,z}^2 + z H_{,z} - H_{,t} + 3H) H_{,zz}] \} + \\ & \frac{\lambda}{3} e^{-t} \cdot \{ -2H_{,z} H_{,zt} + H_{,z}^2 + [H_{,t} - z H_{,z} + (\frac{\lambda}{6} e^{-t} - \mu_0 e^{2t}) H_{,z}^2] H_{,zz} = 0, \quad H_{,zz} \neq 0. \end{aligned} \tag{3.10}$$

As before (see Sect. 2) defining  $A := H_{,t}$  one brings Eq. (3.10) to the system of two evolution equations on two functions,  $H$  and  $A$ .

(Note that if  $F_{,\eta\eta} = 0$  then the simple coordinate transformation

$$t = x + p, \quad z = x - p, \quad q \mapsto q, \quad y \mapsto y \tag{3.11}$$

leads from Eq. (3.7) to the equation of the Cauchy-Kovalevskaya form.) Equation (3.10) is the generalization of Eqs. (3.13) and (3.29) of the previous work [19] for the case of  $HH$  spaces with  $\lambda$ .



Now we study our Eq. (3.10) from the point of view of the Petrov-Penrose classification. The anti-self-dual part of the Weyl tensor is determined by the quantities [18]

$$\begin{aligned} \bar{C}^{(5)} = 0 = \bar{C}^{(4)}, \quad \bar{C}^{(3)} = -2\mu_0\varphi^3, \quad \bar{C}^{(2)} = -2\nu\varphi^5, \\ \bar{C}^{(1)} = -2\varphi^7 \cdot \left[ \varphi \cdot \nu_{,p} + \frac{\partial}{\partial q}(\gamma - \nu\eta + 3\mu_0\theta) + 2\nu\theta_{,q} \right]. \end{aligned} \tag{3.12}$$

First we consider the type [any]  $\otimes$  [2-1-1].

a) [any]  $\otimes$  [2-1-1]

Here one can choose the gauge such that [13]

$$\mu_0 \neq 0, \quad \nu = 0 = \gamma. \tag{3.13}$$

Then from (3.6), (3.8), (3.12) and (3.13) we get

$$\begin{aligned} \bar{C}^{(3)} = -2\mu_0\varphi^3 \neq 0, \quad \bar{C}^{(2)} = 0, \\ \bar{C}^{(1)} = -6\mu_0\varphi^7\theta_{,q} = -6\mu_0\varphi^7H_{,q}. \end{aligned} \tag{3.14}$$

Consequently, in the present case  $\bar{C}^{(1)} \neq 0$  and it is equivalent to the condition

$$H_{,q} \neq 0. \tag{3.15}$$

b) [any]  $\otimes$  [D]

Now the gauge exists such that (3.13) holds but, as for the type [any]  $\otimes$  [D] if  $\bar{C}^{(2)} = 0$  then  $\bar{C}^{(1)} = 0$ , the formulas (3.14) yield

$$H_{,q} = 0, \quad \text{i.e.,} \quad H = H(p, z, t). \tag{3.16}$$

One observes that with (3.16) assumed the hyperheavenly equation with  $\lambda$  (3.10) appears to be linear with respect to the second derivatives of  $H$ .

c) [any]  $\otimes$  [III]

In this case we choose the gauge such that

$$\mu_0 = 0, \quad \nu = \nu_0 = \text{const.} \neq 0, \quad \gamma_{,q} = 0. \tag{3.17}$$

Now one can bring Eq. (3.10) to a system of evolution equations by the following substitution (compare with Refs. [2,4,19]):

$$A := H_{,t}, \quad B := H_{,z}. \tag{3.18}$$

Thus we get

$$\begin{aligned}
 A_{,t} &= \left[ 3 + B_{,z} \cdot \left( -\frac{\lambda}{3}e^{-t} + \partial_q \right) - \left( 2z - \frac{2\lambda}{3}e^{-t}B + B_{,q} \right) \partial_z \right] A \\
 &\quad + e^t B_{,p} + \left[ e^{2t} \cdot (\nu_0 B - \gamma) + \frac{\lambda}{3}e^{-t}B \cdot \left( z - \frac{\lambda}{6}e^{-t}B \right) \right] B_{,z} \\
 &\quad - \frac{\lambda}{3}e^{-t}B^2, \\
 B_{,t} &= A_{,z}, \quad B_{,z} \neq 0.
 \end{aligned}
 \tag{3.19}$$

d)  $[any] \otimes [N]$

Now one has the gauge

$$\mu_0 = 0, \quad \nu = 0.
 \tag{3.20}$$

Then (3.12) with (3.20) yield

$$\bar{C}^{(3)} = 0 = \bar{C}^{(2)}, \quad \bar{C}^{(1)} = -2\varphi^7 \gamma_{,q}.
 \tag{3.21}$$

As  $\bar{C}^{(1)} \neq 0$  one finds from (3.21) the following restriction on  $\gamma$ :

$$\gamma_{,q} \neq 0.
 \tag{3.22}$$

In the present case the substitution (3.18) leads to the system of evolution equations (3.19) with  $\nu_0 = 0$  and  $\gamma_{,q} \neq 0$ .

e)  $[any] \otimes [-]$  i.e. the  $H$ -space with  $\lambda$

Here we have

$$\mu_0 = 0, \quad \nu = 0, \quad \gamma = 0.
 \tag{3.23}$$

The system of evolution equations for the present case is given by (3.19) with  $\nu_0 = 0$  and  $\gamma = 0$ . Thus one gets

$$\begin{aligned}
 A_{,t} &= \left[ 3 + B_{,z} \cdot \left( -\frac{\lambda}{3}e^{-t} + \partial_q \right) - \left( 2z - \frac{2\lambda}{3}e^{-t}B + B_{,q} \right) \partial_z \right] A \\
 &\quad + e^t B_{,p} + \frac{\lambda}{3}e^{-t}B \cdot \left[ -B + \left( z - \frac{\lambda}{6}e^{-t}B \right) B_{,z} \right], \\
 B_{,t} &= A_{,z}, \quad B_{,z} \neq 0.
 \end{aligned}
 \tag{3.24}$$

4. COMMENTS ABOUT INTEGRATION

The general solution of the hyperheavenly equations with  $\lambda$  (2.16i,ii) or (3.10) can be given by the power series [2,4,19]

$$H = \sum_{j=0}^{\infty} \frac{1}{j!} a_j(p, q, z) \cdot t^j \tag{4.1}$$

where  $a_0 = a_0(p, q, z) = H(p, q, z, 0)$ ,  $a_1 = a_1(p, q, z) = H_{,t}(p, q, z, 0)$  are Cauchy data on the hypersurface  $t = 0$  and all remaining coefficients  $a_j, j \geq 2$ , of (4.1) are determined by  $a_0$  and  $a_1$ .

Now, using the method proposed by Grant [2] (see also [4,19]), one can also find the formal solutions of our equations. We show how it can be done, for example, in the case of the  $H$ -space with  $\lambda$ . This case has been studied in the point (e) of Sect. 3.

Define

$$\begin{aligned} \hat{T}_\lambda(t) &:= 3 + B_{,z}(p, q, z, t) \cdot \left(-\frac{\lambda}{3}e^{-t} + \partial_q\right) - \left(2z - \frac{2\lambda}{3}e^{-t} \cdot B(p, q, z, t)\right. \\ &\quad \left.+ B_{,q}(p, q, z, t)\right)\partial_z \\ g_\lambda(t) &:= e^t B_{,p}(p, q, z, t) + \frac{\lambda}{3}e^{-t} B(p, q, z, t) \cdot \left[-B(p, q, z, t)\right. \\ &\quad \left.+ \left(z - \frac{\lambda}{6}e^{-t} B(p, q, z, t)\right) \cdot B_{,z}(p, q, z, t)\right] \end{aligned} \tag{4.2}$$

Then the system of evolution equations (3.24) takes the form

$$\begin{aligned} A_{,t} &= \tilde{T}_\lambda(t) \cdot A + g_\lambda(t), \\ B_{,t} &= A_{,z}, \quad B_z \neq 0 \end{aligned} \tag{4.3}$$

The formal solution of (4.3) reads [2,4,19]

$$\begin{aligned} A &= \exp \left\{ \int_0^t dt_1 \hat{T}_\lambda(t_1) \right\} a_1 + \int_0^t dt_1 g_\lambda(t_1) \\ &\quad + \sum_{n=2}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \hat{T}_\lambda(t_1) \cdots \hat{T}_\lambda(t_{n-1}) g_\lambda(t_n) \\ B &= \int_0^t dt_1 A_{,z}(t_1), \end{aligned} \tag{4.4}$$

where

$$\exp \left\{ \int_0^t dt_1 \hat{T}_\lambda(t_1) \right\} := 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \hat{T}_\lambda(t_1) \cdots \hat{T}_\lambda(t_n). \tag{4.5}$$



(*Remark:* To avoid misunderstanding we use in (4.4) a slightly different form of the series involving  $g_\lambda(t)$  than it has been done in the previous papers [4,19].)

As

$$\begin{aligned} B_{,p} &= \int_0^t dt_1 A_{,zp}(t_1), & B_{,q} &= \int_0^t dt_1 A_{,zq}(t_1), \\ B_{,z} &= \int_0^t dt_1 A_{,zz}(t_1), \end{aligned} \tag{4.6}$$

one can understand the formal solution (4.4) in the sense of the successive approximation procedure.

Of course we should look for another, more effective methods of integration of our hyperheavenly equations with  $\lambda$ . This question will be considered in a separate paper.

## 5. CONCLUSIONS

In this paper we have found the Cauchy-Kovalevskaya form of all hyperheavenly equations with  $\lambda$ . The results obtained are the natural generalizations of the results contained in the previous works [4,19]. Moreover, our considerations concerning the  $H$ -spaces with  $\lambda$  (see the point (e) of Sect. 3 and also Sect. 4), give some solution to the problem stated by Grant [2]. At the end of his paper he writes: "It would be interesting to see if a similar transformation can be used to turn the problem of conformally self-dual metrics with nonzero cosmological constant into an initial value problem".

Of course the open question is if starting with another form of the heavenly equation with  $\lambda$  [26] one can find a more convenient Cauchy-Kovalevskaya form of this equation. The same question concerns also the hyperheavenly equations with  $\lambda$  describing the type [any]  $\otimes$  [D] [13,27,28].

As the main problems which are to be considered in the next future we mention the following ones: *i*) searching for new solutions of the equations obtained, *ii*) looking for the hamiltonian formulation of the evolution hyperheavenly equations with  $\lambda$ , *iii*) studying the integrability of our equations. We intend to consider these problems in separate papers.

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