Hamiltonian structures for massless free fields*

G.F. Torres del Castillo
Departamento de Física Matemática, Instituto de Ciencias
Universidad Autónoma de Puebla, 72000 Puebla, Pue., México

AND

D. Acosta Avalos
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1152, Puebla, Pue., México
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Abstract. By using the fact that a massless free field can be treated as a collection of independent harmonic oscillators, it is shown that there exists an infinite number of Hamiltonian structures and of Hamiltonian functionals for the massless free field equations. The case of the electromagnetic field and of the Weyl neutrino field are treated explicitly. It is also shown that an n–dimensional isotropic harmonic oscillator admits an infinite number of Hamiltonian (symplectic) structures for $n > 1$.

Resumen. Usando el hecho de que un campo libre sin masa puede tratarse como una colección de osciladores armónicos independientes, se muestra que existe una infinidad de estructuras y funcionales hamiltonianas para las ecuaciones de un campo libre sin masa. Los casos del campo electromagnético y del campo de neutrinos de Weyl se tratan explícitamente. Se muestra también que un oscilador armónico isótropo n–dimensional admite una infinidad de estructuras hamiltonianas (simplécticas) para $n > 1$.

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1. Introduction

It is well-known that the source-free electromagnetic field can be expanded in plane waves and that, by virtue of the Maxwell equations, the coefficients of this expansion vary harmonically in time, which allows one to relate the evolution of the electromagnetic field with that of a set of independent harmonic oscillators in such a way that the sum of the energies of these oscillators amounts to the usual expression for the energy of the electromagnetic field (see, *e.g.*, Refs. [1–3]).

In recent years, it has been found that various (linear and nonlinear) evolution equations can be written in a form analogous to that of Hamilton's equations of classical mechanics for systems with a finite number of degrees of freedom. Specifically, it has been shown that several evolution equations can be expressed as

$$\dot{\phi}_\alpha = D_{\alpha\beta}\frac{\delta H}{\delta\phi_\beta}, \tag{1}$$

*Dedicated to Professor Jerzy Plebański on the ocassion of his 65th birthday.

where the ϕ_α are variables that represent the state of the system, $H[\phi_\alpha]$ is a suitable functional of the field quantities ϕ_α, called Hamiltonian functional, $\delta H/\delta\phi_\beta$ denotes the functional derivative, and the $D_{\alpha\beta}$ are differential operators. Here and in the forthcoming a dot denotes partial differentiation with respect to the time and there is summation over repeated indices. The operators $D_{\alpha\beta}$ define a Hamiltonian structure if the Poisson bracket defined by

$$\{F,G\} \equiv \int \frac{\delta F}{\delta\phi_\alpha} D_{\alpha\beta} \frac{\delta G}{\delta\phi_\beta}\, dv, \tag{2}$$

where F and G are functionals of the ϕ_α satisfying suitable boundary conditions, is skew-symmetric and satisfies the Jacobi identity. In fact, there exist evolution equations that can be written in the form (1) in two or more distinct ways (see, *e.g.*, Ref. [4] and the references cited therein).

In this paper we show that the coefficients of the expansion of a massless free field in plane waves can be related to a set of independent harmonic oscillators in an infinite number of different ways, which leads to an infinite number of different Hamiltonian structures. We also show that, in a similar manner, an n–dimensional isotropic harmonic oscillator can be related to a set of n one-dimensional harmonic oscillators in an infinite number of different ways. In Sect. 2 we obtain Hamiltonian structures for the source-free electromagnetic field for which the corresponding Hamiltonian functional can be positive definite or indefinite. Section 3 contains a similar analysis for the Weyl neutrino equation. In Sect. 4 we obtain a class of symplectic structures and of Hamiltonians for an n–dimensional isotropic harmonic oscillator parameterized by $\mathrm{GL}(n,C)$ modulo the unitary (or a pseudo-unitary) group in n dimensions.

2. SOURCE-FREE ELECTROMAGNETIC FIELD

The source-free Maxwell equations in empty space can be written as

$$\nabla\cdot\mathbf{F} = 0, \qquad \nabla\times\mathbf{F} = \frac{i}{c}\frac{\partial\mathbf{F}}{\partial t}, \tag{3}$$

with the complex vector field $\mathbf{F}$ defined by

$$\mathbf{F} \equiv \mathbf{E} + i\mathbf{B}. \tag{4}$$

Assuming that $\mathbf{F}$ satisfies periodic boundary conditions at the walls of a rectangular box of volume Ω, we expand $\mathbf{F}$ in a three-dimensional Fourier series

$$\mathbf{F}(\mathbf{x}) = \Omega^{-1/2} \sum_{\mathbf{k}} \big(c_{\mathbf{k}}\mathbf{e}_{\mathbf{k}}^{(1)} + d_{\mathbf{k}}\mathbf{e}_{\mathbf{k}}^{(2)}\big) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5}$$

where the (real) unit vectors $\mathbf{e}_{\mathbf{k}}^{(1)}$ and $\mathbf{e}_{\mathbf{k}}^{(2)}$ are orthogonal to $\mathbf{k}$ and orthogonal to each other, so that $\mathbf{e}_{\mathbf{k}}^{(1)}$, $\mathbf{e}_{\mathbf{k}}^{(2)}$, and $\mathbf{k}$ form a right-handed set. Making use of the complex combinations

$$\boldsymbol{\epsilon}_{\mathbf{k}}^{(1)} \equiv -\frac{1}{\sqrt{2}}\big(\mathbf{e}_{\mathbf{k}}^{(1)} + i\mathbf{e}_{\mathbf{k}}^{(2)}\big), \qquad \boldsymbol{\epsilon}_{\mathbf{k}}^{(2)} \equiv \frac{1}{\sqrt{2}}\big(\mathbf{e}_{\mathbf{k}}^{(1)} - i\mathbf{e}_{\mathbf{k}}^{(2)}\big), \tag{6}$$

Eq. (5) can also be written as

$$\mathbf{F}(\mathbf{x}) = \Omega^{-1/2}\sum_{\mathbf{k}}\big(a_{\mathbf{k}}\boldsymbol{\epsilon}_{\mathbf{k}}^{(1)} + b_{\mathbf{k}}\boldsymbol{\epsilon}_{\mathbf{k}}^{(2)}\big)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{7}$$

From Eqs. (6–7) it follows that

$$a_{\mathbf{k}} = \Omega^{-1/2}\int \boldsymbol{\epsilon}_{\mathbf{k}}^{(1)*}\cdot\mathbf{F}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,d^3x, \qquad b_{\mathbf{k}} = \Omega^{-1/2}\int \boldsymbol{\epsilon}_{\mathbf{k}}^{(2)*}\cdot\mathbf{F}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,d^3x. \tag{8}$$

By substituting Eq. (7) into the second of Eqs. (3) one finds that the time dependence of the (complex) expansion coefficients $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ is given by

$$\dot{a}_{\mathbf{k}} = -i\omega a_{\mathbf{k}}, \qquad \dot{b}_{\mathbf{k}} = i\omega b_{\mathbf{k}}, \tag{9}$$

where $\omega \equiv |\mathbf{k}|\,c$.

Equations (9) show that the real and imaginary parts of $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ can be related with the coordinates and momenta of two independent harmonic oscillators of frequency ω, which allows us to find a Hamiltonian that reproduces Eqs. (9) and, therefore, the evolution equations contained in Eqs. (3).

2.1. Positive definite Hamiltonians

For each allowed vector $\mathbf{k}$ we introduce four real variables $q_{\mathbf{k}}$, $p_{\mathbf{k}}$, $\overline{q}_{\mathbf{k}}$, and $\overline{p}_{\mathbf{k}}$ satisfying the equations of motion

$$\dot{q}_{\mathbf{k}} = p_{\mathbf{k}}, \quad \dot{p}_{\mathbf{k}} = -\omega^2 q_{\mathbf{k}}, \quad \dot{\overline{q}}_{\mathbf{k}} = \overline{p}_{\mathbf{k}}, \quad \dot{\overline{p}}_{\mathbf{k}} = -\omega^2\overline{q}_{\mathbf{k}}, \tag{10}$$

which follow from Hamilton's equations with a Hamiltonian

$$H = \tfrac{1}{2}\sum_{\mathbf{k}}\big[p_{\mathbf{k}}^2 + \omega^2 q_{\mathbf{k}}^2 + \overline{p}_{\mathbf{k}}^2 + \omega^2\,\overline{q}_{\mathbf{k}}^2\big], \tag{11}$$

assuming that $q_{\mathbf{k}}$, $p_{\mathbf{k}}$ and $\overline{q}_{\mathbf{k}}$, $\overline{p}_{\mathbf{k}}$ are canonically conjugate variables. Equations (10) are equivalent to

$$\dot{p}_{\mathbf{k}} + i\omega\dot{q}_{\mathbf{k}} = i\omega(p_{\mathbf{k}} + i\omega q_{\mathbf{k}}), \qquad \dot{\overline{p}}_{\mathbf{k}} + i\omega\dot{\overline{q}}_{\mathbf{k}} = i\omega(\overline{p}_{\mathbf{k}} + i\omega\overline{q}_{\mathbf{k}}), \tag{12}$$

(*cf.* Eqs. (9)); therefore we can write

$$\begin{aligned} a^*_{\mathbf{k}} &= g_{11}(\mathbf{k})(p_{\mathbf{k}} + i\omega q_{\mathbf{k}}) + g_{12}(\mathbf{k})(\overline{p}_{\mathbf{k}} + i\omega\overline{q}_{\mathbf{k}}), \\ b_{\mathbf{k}} &= g_{21}(\mathbf{k})(p_{\mathbf{k}} + i\omega q_{\mathbf{k}}) + g_{22}(\mathbf{k})(\overline{p}_{\mathbf{k}} + i\omega\overline{q}_{\mathbf{k}}), \end{aligned} \tag{13}$$

where the $g_{ij}(\mathbf{k})$ are complex-valued functions of $\mathbf{k}$ such that $\det(g_{ij}) \neq 0$ and the * denotes complex conjugation. From Eqs. (13) we now obtain

$$\begin{aligned} p_{\mathbf{k}} + i\omega q_{\mathbf{k}} &= f_{11}(\mathbf{k})a^*_{\mathbf{k}} + f_{12}(\mathbf{k})b_{\mathbf{k}}, \\ \overline{p}_{\mathbf{k}} + i\omega\overline{q}_{\mathbf{k}} &= f_{21}(\mathbf{k})a^*_{\mathbf{k}} + f_{22}(\mathbf{k})b_{\mathbf{k}}, \end{aligned} \tag{14}$$

where the matrix $f \equiv (f_{ij})$ is the inverse of $g \equiv (g_{ij})$.

Since $q_{\mathbf{k}}$, $p_{\mathbf{k}}$ and $\overline{q}_{\mathbf{k}}$, $\overline{p}_{\mathbf{k}}$ are assumed to be canonically conjugate variables, Eqs. (13) yield

$$\begin{aligned} \{a_{\mathbf{k}}, a_{\mathbf{k}'}\} &= \{a_{\mathbf{k}}, b^*_{\mathbf{k}'}\} = 0, \\ \{b^*_{\mathbf{k}}, b^*_{\mathbf{k}'}\} &= \{a^*_{\mathbf{k}}, a^*_{\mathbf{k}'}\} = 0, \\ \{a^*_{\mathbf{k}}, b_{\mathbf{k}'}\} &= \{b_{\mathbf{k}}, b_{\mathbf{k}'}\} = 0, \\ \{a^*_{\mathbf{k}}, a_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(gg^\dagger)_{11}, \\ \{a^*_{\mathbf{k}}, b^*_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(gg^\dagger)_{12}, \\ \{b_{\mathbf{k}}, a_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(gg^\dagger)_{21}, \\ \{b_{\mathbf{k}}, b^*_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(gg^\dagger)_{22}, \end{aligned} \tag{15}$$

where $g^\dagger$ is the adjoint of g; therefore, using Eq. (7), one obtains the following Poisson brackets among the cartesian components of $\mathbf{F}$ and $\mathbf{F}^*$ (at equal times):

$$\{F_i(\mathbf{x}), F_j(\mathbf{x}')\} = 2c\,\Omega^{-1}\sum_{\mathbf{k}}(gg^\dagger)_{21}(\mathbf{k})\epsilon_{ijm}k_m e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{x}')}, \tag{16}$$

where we have used the relation

$$\epsilon^{(1)}_{\mathbf{k},i}\epsilon^{(2)}_{\mathbf{k},j} - \epsilon^{(1)}_{\mathbf{k},j}\epsilon^{(2)}_{\mathbf{k},i} = i\epsilon_{ijm}\frac{k_m}{|\mathbf{k}|}, \tag{17}$$

and the subscripts i, j denote the cartesian components of the vector on which they appear; similarly,

$$\{F_i(\mathbf{x}), F^*_j(\mathbf{x}')\} = 2i\,\Omega^{-1}\sum_{\mathbf{k}}\omega\left[(gg^\dagger)_{11}(\mathbf{k})\epsilon^{(1)}_{\mathbf{k},i}\epsilon^{(2)}_{\mathbf{k},j} - (gg^\dagger)_{22}(\mathbf{k})\epsilon^{(1)}_{\mathbf{k},j}\epsilon^{(2)}_{\mathbf{k},i}\right]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \tag{18}$$

which can be reduced by means of Eq. (17) and

$$\epsilon^{(1)}_{\mathbf{k},i}\epsilon^{(2)}_{\mathbf{k},j} + \epsilon^{(1)}_{\mathbf{k},j}\epsilon^{(2)}_{\mathbf{k},i} = -\left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right). \tag{19}$$

Using Eqs. (11) and (14), one finds that the Hamiltonian is

$$\begin{aligned} H = \tfrac{1}{2}\sum_{\mathbf{k}} \big[& (f^\dagger f)_{11}(\mathbf{k}) a_{\mathbf{k}} a^*_{\mathbf{k}} + (f^\dagger f)_{12}(\mathbf{k}) a_{\mathbf{k}} b_{\mathbf{k}} + (f^\dagger f)_{21}(\mathbf{k}) a^*_{\mathbf{k}} b^*_{\mathbf{k}} \\ & + (f^\dagger f)_{22}(\mathbf{k}) b_{\mathbf{k}} b^*_{\mathbf{k}} \big], \end{aligned} \tag{20}$$

which can be expressed in terms of $\mathbf{F}$ and $\mathbf{F}^*$ making use of Eqs. (8). Equations (16), (18), and (20) show that the Hamiltonian structure induced by the relations (13) depends on $gg^\dagger$ (note that, since $f = g^{-1}$, it follows that $f^\dagger f = (gg^\dagger)^{-1}$); therefore, two matrix functions $g(\mathbf{k})$ and $\tilde{g}(\mathbf{k})$ lead to the same Hamiltonian structure, and to the same Hamiltonian, if and only if

$$\tilde{g}(\mathbf{k}) = g(\mathbf{k}) U(\mathbf{k}), \tag{21}$$

where $U(\mathbf{k})$ is a unitary matrix that may depend on $\mathbf{k}$.

A simple example of the present case, where the Hamiltonian coincides with the energy of the electromagnetic field, is obtained with the choice $gg^\dagger = 4\pi I$, where I denotes the unit matrix, which corresponds to

$$g(\mathbf{k}) = \sqrt{4\pi}\, I, \tag{22}$$

modulo the transformations (21). Then, Eqs. (16) and (18) give

$$\begin{aligned} \{F_i(\mathbf{x}), F_j(\mathbf{x}')\} &= 0, \\ \{F_i(\mathbf{x}), F^*_j(\mathbf{x}')\} &= 8\pi c\, i\epsilon_{ijm} \frac{\partial}{\partial x_m}\delta(\mathbf{x} - \mathbf{x}'), \end{aligned} \tag{23}$$

which amount to [*cf.* Eq. (4)]

$$\begin{aligned} \{E_i(\mathbf{x}), E_j(\mathbf{x}')\} &= \{B_i(\mathbf{x}), B_j(\mathbf{x}')\} = 0, \\ \{E_i(\mathbf{x}), B_j(\mathbf{x}')\} &= -4\pi c\, \epsilon_{ijm} \frac{\partial}{\partial x_m}\delta(\mathbf{x} - \mathbf{x}'), \end{aligned}$$

and from Eqs. (8) and (19–20) one finds that

$$H = \frac{1}{8\pi}\int \mathbf{F}^* \cdot \mathbf{F}\, d^3x = \frac{1}{8\pi}\int (E^2 + B^2)\, d^3x. \tag{24}$$

In the same way as the Hamiltonian, which is the generator of displacements in the time, has an infinite number of distinct expressions, depending on the Hamiltonian structure chosen, the generator of any canonical transformation will have a different expression for each Hamiltonian structure. It is known that a differentiable function $G(q_i, p_i)$ of the canonical variables q_i, p_i, is the generator of a one-parameter group of canonical transformations parameterized by a variable s, defined by

$$\frac{dq_i}{ds} = \frac{\partial G}{\partial p_i}, \qquad \frac{dp_i}{ds} = -\frac{\partial G}{\partial q_i}. \tag{25}$$

Conversely, the generator of a given one-parameter group of canonical transformations can be obtained from Eqs. (25).

For instance, the Maxwell equations (3) are invariant under the duality rotations, which are given by $\mathbf{F} \mapsto \mathbf{e}^{is}\mathbf{F}$. The Poisson brackets (23) and the Hamiltonian (24) are also invariant under these transformations; this means that the duality rotations are canonical transformations and that their generator is a constant of motion. From Eqs. (8) we see that $\mathbf{F} \mapsto \mathbf{e}^{is}\mathbf{F}$ corresponds to $a_\mathbf{k} \mapsto e^{is}a_\mathbf{k}$ and $b_\mathbf{k} \mapsto e^{is}b_\mathbf{k}$ or, equivalently, by virtue of Eqs. (14) with $f(\mathbf{k}) = \frac{1}{\sqrt{4\pi}} I$,

$$p_\mathbf{k} + i\omega q_\mathbf{k} \mapsto e^{-is}(p_\mathbf{k} + i\omega q_\mathbf{k}), \qquad \overline{p}_\mathbf{k} + i\omega \overline{q}_\mathbf{k} \mapsto e^{is}(\overline{p}_\mathbf{k} + i\omega \overline{q}_\mathbf{k}). \tag{26}$$

From Eqs. (25–26) it is readily seen that the generator of the duality rotations (with respect to the Hamiltonian structure defined by Eqs. (23)) is

$$\begin{aligned} G &= \tfrac{1}{2}\sum_\mathbf{k} \tfrac{1}{\omega}\left[\overline{p}_\mathbf{k}^2 + \omega^2 \overline{q}_\mathbf{k}^2 - p_\mathbf{k}^2 - \omega^2 q_\mathbf{k}^2\right] \\ &= \tfrac{1}{8\pi}\sum_\mathbf{k} \tfrac{1}{\omega}\left[b_\mathbf{k} b_\mathbf{k}^* - a_\mathbf{k} a_\mathbf{k}^*\right] = -\frac{1}{8\pi c}\int \mathbf{A}\cdot\mathbf{F}^*\, d^3x, \end{aligned} \tag{27}$$

where we have used Eqs. (8) and (17), and where $\mathbf{A}$ is a (complex) vector potential for $\mathbf{F}$, $\mathbf{F} = \nabla \times \mathbf{A}$.

It may be noticed that if $(gg^\dagger)_{21}(\mathbf{k})$ is different from zero, then Eq. (16) shows that the translations are not canonical transformations and, therefore, a generator of these transformations, with respect to such a Hamiltonian structure, cannot exist.

2.2. Indefinite Hamiltonians

Instead of Eqs. (10), we can assume that the canonical coordinates $q_\mathbf{k}$, $p_\mathbf{k}$, $\overline{q}_\mathbf{k}$, and $\overline{p}_\mathbf{k}$ satisfy the equations of motion

$$\dot{q}_\mathbf{k} = p_\mathbf{k}, \quad \dot{p}_\mathbf{k} = -\omega^2 q_\mathbf{k}, \quad \dot{\overline{q}}_\mathbf{k} = -\overline{p}_\mathbf{k}, \quad \dot{\overline{p}}_\mathbf{k} = \omega^2 \overline{q}_\mathbf{k}, \tag{28}$$

corresponding to the Hamiltonian

$$H = \tfrac{1}{2}\sum_\mathbf{k}\left[p_\mathbf{k}^2 + \omega^2 q_\mathbf{k}^2 - \overline{p}_\mathbf{k}^2 - \omega^2 \overline{q}_\mathbf{k}^2\right]. \tag{29}$$

From Eqs. (28) now we have

$$\dot{p}_{\mathbf{k}} + i\omega\dot{q}_{\mathbf{k}} = i\omega(p_{\mathbf{k}} + i\omega q_{\mathbf{k}}), \quad \dot{\overline{p}}_{\mathbf{k}} - i\omega\dot{\overline{q}}_{\mathbf{k}} = i\omega(\overline{p}_{\mathbf{k}} - i\omega\overline{q}_{\mathbf{k}}), \tag{30}$$

and comparison with Eqs. (9) suggests the relations

$$\begin{aligned} a^*_{\mathbf{k}} &= g_{11}(\mathbf{k})(p_{\mathbf{k}} + i\omega q_{\mathbf{k}}) + g_{12}(\mathbf{k})(\overline{p}_{\mathbf{k}} - i\omega\overline{q}_{\mathbf{k}}), \\ b_{\mathbf{k}} &= g_{21}(\mathbf{k})(p_{\mathbf{k}} + i\omega q_{\mathbf{k}}) + g_{22}(\mathbf{k})(\overline{p}_{\mathbf{k}} - i\omega\overline{q}_{\mathbf{k}}), \end{aligned} \tag{31}$$

where $g(\mathbf{k}) = (g_{ij}(\mathbf{k}))$ is a complex nonsingular matrix; hence,

$$\begin{aligned} p_{\mathbf{k}} + i\omega q_{\mathbf{k}} &= f_{11}(\mathbf{k})a^*_{\mathbf{k}} + f_{12}(\mathbf{k})b_{\mathbf{k}}, \\ \overline{p}_{\mathbf{k}} - i\omega\overline{q}_{\mathbf{k}} &= f_{21}(\mathbf{k})a^*_{\mathbf{k}} + f_{22}(\mathbf{k})b_{\mathbf{k}}, \end{aligned} \tag{32}$$

where $f(\mathbf{k}) \equiv (g(\mathbf{k}))^{-1}$.

Making use of the matrix

$$\eta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{33}$$

one finds that the Poisson brackets among the $a_{\mathbf{k}}, b_{\mathbf{k}}$ and their complex conjugates can be written in a form analogous to that of Eqs. (15):

$$\begin{aligned} \{a_{\mathbf{k}}, a_{\mathbf{k}'}\} &= \{a_{\mathbf{k}}, b^*_{\mathbf{k}'}\} = 0, \\ \{b^*_{\mathbf{k}}, b^*_{\mathbf{k}'}\} &= \{a^*_{\mathbf{k}}, a^*_{\mathbf{k}'}\} = 0, \\ \{a^*_{\mathbf{k}}, b_{\mathbf{k}'}\} &= \{b_{\mathbf{k}}, b_{\mathbf{k}'}\} = 0, \\ \{a^*_{\mathbf{k}}, a_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(g\eta g^\dagger)_{11}, \\ \{a^*_{\mathbf{k}}, b^*_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(g\eta g^\dagger)_{12}, \\ \{b_{\mathbf{k}}, a_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(g\eta g^\dagger)_{21}, \\ \{b_{\mathbf{k}}, b^*_{\mathbf{k}'}\} &= 2i\omega\delta_{\mathbf{k}\mathbf{k}'}(g\eta g^\dagger)_{22}. \end{aligned} \tag{34}$$

Hence, the Poisson brackets among the components of $\mathbf{F}$ and $\mathbf{F}^*$ can be obtained from Eqs. (16) and (18) by replacing $gg^\dagger$ by $g\eta g^\dagger$:

$$\begin{aligned} \{F_i(\mathbf{x}), F_j(\mathbf{x}')\} &= 2c\,\Omega^{-1}\sum_{\mathbf{k}}(g\eta g^\dagger)_{21}(\mathbf{k})\epsilon_{ijm}k_m e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{x}')}, \\ \{F_i(\mathbf{x}), F^*_j(\mathbf{x}')\} &= 2i\,\Omega^{-1}\sum_{\mathbf{k}}\omega\Big[(g\eta g^\dagger)_{11}(\mathbf{k})\epsilon^{(1)}_{\mathbf{k},i}\epsilon^{(2)}_{\mathbf{k},j} - (g\eta g^\dagger)_{22}(\mathbf{k})\epsilon^{(1)}_{\mathbf{k},j}\epsilon^{(2)}_{\mathbf{k},i}\Big]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \end{aligned} \tag{35}$$

On the other hand, substituting Eqs. (32) into Eq. (29) one obtains the following expression for the Hamiltonian:

$$H = \tfrac{1}{2}\sum_{\mathbf{k}}\big[(f^{\dagger}\eta f)_{11}(\mathbf{k})a_{\mathbf{k}}a_{\mathbf{k}}^{*} + (f^{\dagger}\eta f)_{12}(\mathbf{k})a_{\mathbf{k}}b_{\mathbf{k}} + (f^{\dagger}\eta f)_{21}(\mathbf{k})a_{\mathbf{k}}^{*}b_{\mathbf{k}}^{*} + (f^{\dagger}\eta f)_{22}(\mathbf{k})b_{\mathbf{k}}b_{\mathbf{k}}^{*}\big] \tag{36}$$

[*cf.* Eq. (20)]. It may be noticed that $f^{\dagger}\eta f = (g\eta g^{\dagger})^{-1}$, and from Eqs. (35) and (36) we conclude that two matrix-valued functions $g(\mathbf{k})$ and $\tilde{g}(\mathbf{k})$ yield the same Hamiltonian structure and the same Hamiltonian if and only if there exists a U(1, 1)–valued function of $\mathbf{k}$, $U(\mathbf{k})$, such that

$$\tilde{g}(\mathbf{k}) = g(\mathbf{k})U(\mathbf{k}) \tag{37}$$

(*i.e.*, $U(\mathbf{k})$ satisfies, $U(\mathbf{k})\,\eta\,(U(\mathbf{k}))^{\dagger} = \eta$).

Taking, for example, $(g\eta g^{\dagger})(\mathbf{k}) = (2|\mathbf{k}|)^{-1}\eta$, which corresponds to $g(\mathbf{k}) = (2|\mathbf{k}|)^{-1/2}I$, up to the transformations (37), from Eqs. (17), (19), (35), and (36) we get

$$\begin{aligned}&\{F_i(\mathbf{x}), F_j(\mathbf{x}')\} = 0,\\ &\{F_i(\mathbf{x}), F_j^{*}(\mathbf{x}')\} = -ic\,\Omega^{-1}\sum_{\mathbf{k}}\left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\end{aligned} \tag{38}$$

and

$$H = \int \mathbf{F}^{*}\cdot\nabla\times\mathbf{F}\,d^3x \tag{39}$$

(*cf.* Ref. [5]). In the present case, one finds that the generator of the duality rotations is given by

$$G = -\frac{1}{c}\int \mathbf{F}^{*}\cdot\mathbf{F}\,d^3x, \tag{40}$$

which, except for a constant factor, coincides with the Hamiltonian (24) (*cf.* also Ref. [5]).

3. THE WEYL NEUTRINO EQUATION

The Weyl neutrino equation for the two-component neutrino field, ψ, is given by

$$\dot{\psi} = c\boldsymbol{\sigma}\cdot\nabla\psi, \tag{41}$$

where the σ_i are the Pauli matrices (see, *e.g.*, Refs. [6,7]). The spinor field ψ can be expanded in a form analogous to Eq. (7),

$$\psi(\mathbf{x}) = \Omega^{-1/2} \sum_{\mathbf{k}} \left(b_{\mathbf{k}} \epsilon_{\mathbf{k}}^{(1)} + a_{\mathbf{k}} \epsilon_{\mathbf{k}}^{(2)} \right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{42}$$

where, in the present case, $\epsilon_{\mathbf{k}}^{(1)}$ and $\epsilon_{\mathbf{k}}^{(2)}$ denote the two-component spinors [6]

$$\epsilon_{\mathbf{k}}^{(1)} = \begin{pmatrix} e^{-i\varphi/2} \cos \frac{\theta}{2} \\ e^{i\varphi/2} \sin \frac{\theta}{2} \end{pmatrix}, \qquad \epsilon_{\mathbf{k}}^{(2)} = \begin{pmatrix} -e^{-i\varphi/2} \sin \frac{\theta}{2} \\ e^{i\varphi/2} \cos \frac{\theta}{2} \end{pmatrix}, \tag{43}$$

and θ, φ are the polar and azimuth angles of $\mathbf{k}$. It is easy to see that

$$(\mathbf{k} \cdot \boldsymbol{\sigma}) \epsilon_{\mathbf{k}}^{(1)} = |\mathbf{k}| \, \epsilon_{\mathbf{k}}^{(1)}, \qquad (\mathbf{k} \cdot \boldsymbol{\sigma}) \epsilon_{\mathbf{k}}^{(2)} = -|\mathbf{k}| \, \epsilon_{\mathbf{k}}^{(2)}, \tag{44}$$

therefore, substituting Eq. (42) into Eq. (41) one finds that

$$\dot{a}_{\mathbf{k}} = -i\omega a_{\mathbf{k}}, \qquad \dot{b}_{\mathbf{k}} = i\omega b_{\mathbf{k}}, \tag{45}$$

[*cf.* Eqs. (9)]. The two-component spinors (43) satisfy the orthonormality condition

$$\epsilon_{\mathbf{k},\alpha}^{(i)*} \epsilon_{\mathbf{k},\alpha}^{(j)} = \delta_{ij}, \tag{46}$$

where the subscripts α, β, which range and sum over $1, 2$ denote the components of the spinor on which they appear. Hence, from Eq. (42) it follows that

$$a_{\mathbf{k}} = \Omega^{-1/2} \int \epsilon_{\mathbf{k},\alpha}^{(2)*} \psi_\alpha(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \, d^3x, \qquad b_{\mathbf{k}} = \Omega^{-1/2} \int \epsilon_{\mathbf{k},\alpha}^{(1)*} \psi_\alpha(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \, d^3x. \tag{47}$$

Following the same steps as in Sect. 2, we can obtain an infinite number of different Hamiltonian structures for the evolution equations (41) with a positive (or negative) definite Hamiltonian or with an indefinite Hamiltonian, by relating the amplitudes $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ with the coordinates and momenta of two independent harmonic oscillators as in Eqs. (13) and (31). The differences with the preceding case come from the algebraic relations satisfied by the spinors (43), replacing Eqs. (17) and (19), namely

$$\begin{aligned}
\epsilon_{\mathbf{k},\alpha}^{(1)} \epsilon_{\mathbf{k},\beta}^{(1)*} - \epsilon_{\mathbf{k},\alpha}^{(2)} \epsilon_{\mathbf{k},\beta}^{(2)*} &= \frac{\mathbf{k}}{|\mathbf{k}|} \cdot \boldsymbol{\sigma}_{\alpha\beta}, \\
\epsilon_{\mathbf{k},\alpha}^{(1)} \epsilon_{\mathbf{k},\beta}^{(1)*} + \epsilon_{\mathbf{k},\alpha}^{(2)} \epsilon_{\mathbf{k},\beta}^{(2)*} &= \delta_{\alpha\beta}, \\
\epsilon_{\mathbf{k},\alpha}^{(1)} \epsilon_{\mathbf{k},\beta}^{(2)} - \epsilon_{\mathbf{k},\alpha}^{(2)} \epsilon_{\mathbf{k},\beta}^{(1)} &= \epsilon_{\alpha\beta},
\end{aligned} \tag{48}$$

where $\epsilon_{\alpha\beta}$ is the Levi-Civita symbol. Thus, we get [*cf.* Eq. (16)]

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = 2i\epsilon_{\alpha\beta}\,\Omega^{-1}\sum_{\mathbf{k}}\omega(gSg^\dagger)_{21}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{x}')}, \tag{49}$$

where $S = I$, η, or $-I$, according to whether the Hamiltonian is positive definite, indefinite, or negative definite, respectively. Similarly,

$$\{\psi_\alpha(\mathbf{x}), \psi^*_\beta(\mathbf{x}')\} = 2i\Omega^{-1}\sum_{\mathbf{k}}\omega\big[(gSg^\dagger)_{22}(\mathbf{k})e^{(1)}_{\mathbf{k},\alpha}\epsilon^{(1)*}_{\mathbf{k},\beta} - (gSg^\dagger)_{11}(\mathbf{k})\epsilon^{(2)}_{\mathbf{k},\alpha}\epsilon^{(2)*}_{\mathbf{k}\beta}\big] \times e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{50}$$

[*cf.* Eq. (18)] and the Hamiltonian is given by

$$H = \tfrac{1}{2}\sum_{\mathbf{k}}\big[(f^\dagger Sf)_{11}(\mathbf{k})a_{\mathbf{k}}a^*_{\mathbf{k}} + (f^\dagger Sf)_{12}(\mathbf{k})a_{\mathbf{k}}b_{\mathbf{k}} + (f^\dagger Sf)_{21}(\mathbf{k})a^*_{\mathbf{k}}b^*_{\mathbf{k}} + (f^\dagger Sf)_{22}(\mathbf{k})b_{\mathbf{k}}b^*_{\mathbf{k}}\big], \tag{51}$$

with $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ given by Eqs. (47).

3.1. Example of a positive definite Hamiltonian

Taking $S = I$ in order to get a positive definite Hamiltonian, we choose $gg^\dagger = I$ (which corresponds to $g = I$ modulo the transformations (21)), from Eqs. (47–51) we obtain

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = 0,$$
$$\{\psi_\alpha(\mathbf{x}), \psi^*_\beta(\mathbf{x}')\} = 2c\boldsymbol{\sigma}_{\alpha\beta}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$H = \tfrac{1}{2}\sum_{\mathbf{k}}(a_{\mathbf{k}}a^*_{\mathbf{k}} + b_{\mathbf{k}}b^*_{\mathbf{k}}) = \tfrac{1}{2}\int\psi^*_\alpha\psi_\alpha\,d^3x.$$

3.2. Example of an indefinite Hamiltonian

Taking now $S = \eta$ and $g\eta g^\dagger = (2\hbar\omega)^{-1}\eta$, which is obtained with $g = (2\hbar\omega)^{-1/2}I$ up to transformations of the form (37), Eqs. (47–51) give

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = 0,$$

$$\{\psi_\alpha(\mathbf{x}), \psi^*_\beta(\mathbf{x}')\} = \frac{1}{i\hbar}\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$H = \sum_{\mathbf{k}} \hbar\omega(a_{\mathbf{k}}a_{\mathbf{k}}^* - b_{\mathbf{k}}b_{\mathbf{k}}^*) = \int \psi_\alpha^* i\hbar c \boldsymbol{\sigma}_{\alpha\beta} \cdot \nabla\psi_\beta \, d^3x.$$

4. Hamiltonian structures for the isotropic harmonic oscillator

Since an n–dimensional isotropic harmonic oscillator can be considered as a set of n independent one-dimensional harmonic oscillators, we can generalize Eqs. (13) and (31) relating these n one-dimensional harmonic oscillators with another set of n (auxiliary) one-dimensional harmonic oscillators. Let q_i, p_i be the coordinates and momenta of the n harmonic oscillators corresponding to an n–dimensional isotropic harmonic oscillator, which satisfy the equations of motion

$$\dot{p}_i + i\omega\dot{q}_i = i\omega(p_i + i\omega q_i), \quad (i = 1, 2, \ldots, n). \tag{52}$$

We now introduce $2n$ coordinates $\overline{q}_j$, $\overline{p}_j$, $(j = 1, 2, \ldots, n)$ such that

$$\dot{\overline{p}}_j + \zeta_j i\omega\dot{\overline{q}}_j = i\omega(\overline{p}_j + \zeta_j i\omega\overline{q}_j), \quad (j \text{ not summed}), \tag{53}$$

[*cf.* Eq. (30)], where each ζ_j is $+1$ or -1. Assuming that the coordinates $\overline{q}_j$, $\overline{p}_j$ are canonically conjugate, *i.e.*,

$$\{\overline{q}_j, \overline{p}_m\} = \delta_{jm}, \tag{54}$$

Eqs. (53) can be obtained from the Hamiltonian

$$H = \tfrac{1}{2}\sum_{j=1}^{n} \zeta_j \left[\overline{p}_j^2 + \omega^2\overline{q}_j^2\right] \tag{55}$$

[*cf.* Eq.(29)].

Equations (52) are reproduced if the variables q_i, p_i are related with $\overline{q}_j$, $\overline{p}_j$ through the linear transformation given by

$$p_j + i\omega q_j = \sum_{m=1}^{n} g_{jm}(\overline{p}_m + \zeta_m i\omega\overline{q}_m) \tag{56}$$

or, equivalently,

$$\overline{p}_j + \zeta_j i\omega\overline{q}_j = \sum_{m=1}^{n} f_{jm}(p_m + i\omega q_m), \tag{57}$$

where $g = (g_{ij})$ in an arbitrary nonsingular complex $n \times n$ matrix and $f = (f_{ij}) = g^{-1}$.

Substituting Eqs. (57) into Eq. (55) we find that, in terms of the original coordinates, the Hamiltonian (55) is given by

$$H = \tfrac{1}{2} \sum_{k,\, m=1}^{n} \left(\sum_{j=1}^{n} (f^\dagger)_{mj}\, \zeta_j\, f_{jk} \right) (p_m - i\omega q_m)(p_k + i\omega q_k). \tag{58}$$

Introducing the diagonal matrix

$$\zeta \equiv \mathrm{diag}(\zeta_1, \zeta_2, \ldots, \zeta_n), \tag{59}$$

Eq. (58) can be rewritten as

$$\begin{aligned} H &= \tfrac{1}{2} \sum_{k,\, m=1}^{n} (f^\dagger \zeta f)_{mk} (p_m - i\omega q_m)(p_k + i\omega q_k) \\ &= \tfrac{1}{2} \sum_{k,\, m=1}^{n} \big[(\mathrm{Re}(f^\dagger \zeta f)_{mk})\, (p_m q_k + \omega^2 q_m q_k) \\ &\qquad + (\mathrm{Im}(f^\dagger \zeta f)_{mk})\, \omega (q_m p_k - q_k p_m) \big], \end{aligned} \tag{60}$$

where we have used the fact that $(f^\dagger \zeta f)$ is hermitian. Thus, we see that the Hamiltonian, which is a constant of motion, is a linear combination of the constants of motion $p_m p_k + \omega^2 q_m q_k$ and $q_m p_k - q_k p_m$. From Eqs. (54) and (56) one finds that the Poisson brackets for the original variables q_i, p_i are given by

$$\begin{aligned} \{q_j, p_k\} &= \mathrm{Re}(g \zeta g^\dagger)_{jk}, \\ \{q_j, q_k\} &= -\frac{1}{\omega}\, \mathrm{Im}(g \zeta g^\dagger)_{jk}, \\ \{p_j, p_k\} &= -\omega\, \mathrm{Im}(g \zeta g^\dagger)_{jk}. \end{aligned} \tag{61}$$

Equations (60) and (61) show that two complex nonsingular matrices g and $\tilde{g}$ give rise to the same Hamiltonian (or symplectic) structure and to the same Hamiltonian if and only if they are related through

$$\tilde{g} = gU, \tag{62}$$

where U is an arbitrary complex matrix such that

$$U \zeta U^\dagger = \zeta, \tag{63}$$

which means that U is a unitary or pseudo-unitary $n \times n$ matrix. In particular, $\zeta = I$, $g = I$, lead to the usual expression for the Hamiltonian of an n–dimensional isotropic harmonic oscillator and the matrices U, appearing in Eq. (62), that leave invariant this Hamiltonian and the corresponding symplectic structure, according to Eq. (63) are the $n \times n$ unitary matrices.

5. CONCLUDING REMARKS

A massless field of an arbitrary spin greater than zero can be expanded in terms of circularly polarized plane waves as in Eqs. (7) and (42), and from the massless free field equations one obtains equations analogous to (9) and (45); therefore, as in the cases treated in Sects. 2 and 3 one can obtain an infinite number of Hamiltonian structures for the corresponding evolution equations. For instance, in the case of the Einstein vacuum field equations linearized about the Minkowski metric, by choosing the matrix $g(\mathbf{k})$ as in Secs. 2.1 and 3.1, one gets the Hamiltonian structure given in Ref. [8].

Since each Hamiltonian constructed by the procedure presented here is a constant of motion, one obtains in this manner an infinite number of constants of motion (which may not be in involution with respect to a given Hamiltonian structure of the class considered above). It should be remarked that, in general, these Hamiltonians will not correspond to the energy of the field [see, *e.g.*, Eq. (39)]. The operators $D_{\alpha\beta}$ appearing in Eqs. (1–2) can be read off from the Poisson brackets among the field components [such as Eqs. (23)], since Eq. (2) implies that $\{\phi_\alpha(\mathbf{x}), \phi_\beta(\mathbf{x}')\} = D_{\alpha\beta}(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')$ (*cf.* also Ref. [8]).

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