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Galileo, Bernoulli, Leibniz and Newton around the brachistochrone problem

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ABSTRACT. The brachistochrone problem, having challenged the talents of Newton, Leibniz and many others, plays a central role in the history of physics. Their solutions not only give implicit information as to their mathematical skills and cleverness, but also are worthwhile because of their heuristic content. We emphasize several physical and mathematical details around this problem, reviewing for this the geometrical and mechanical methods of Huygens applied to the cycloid. The solutions of Leibniz and Bernoulli are presented, followed by Newton's Theorem on cycloids and his solution sent to Charles Montague. A geometrical approach to this problem, as counterexample against the contention of Leibniz that it may only be solved through the mastering of 'his' calculus, is given. Under the light of such solutions and of the historical frame, we discuss how Galileo was involved, with this problem, into the priority dispute between Newton and Leibniz.

RESUMEN. El problema de la braquistócrona, habiendo retado los ingenios de Newton, Leibniz y muchos otros, juega un papel central en la historia de la física. Las soluciones con que aquellos respondieron al reto no sólo dan información implícita respecto de sus talentos matemáticos y astucia, sino que también resultan de gran valor por su contenido heurístico. Resaltamos algunos detalles físico-matemáticos alrededor de este problema, repasando los métodos geométrico-mecánicos de Huygens, relativos a la cicloide. Se presentan las soluciones de Leibniz y Bernoulli, seguidas de un teorema de Newton relativo a la cicloide, y de la solución que envió a Charles Montague. Se brinda una solución geométrica de este problema, en calidad de ejemplo, contra la aseveración de Leibniz de que este problema sólo puede ser resuelto mediante el dominio de su cálculo. A la luz de tales soluciones y del marco histórico, analizamos como fue involucrado Galileo, mediante este problema, en la controversia sobre la paternidad del cálculo entre Newton y Leibniz.

PACS: 01.65.+G; 02.40.-k; 03.20.+i

1. INTRODUCTION

The main object of this work is to analyze the brachistochrone problem in its own historical frame, which, as known, was proposed by John Bernoulli in 1696 as a challenge to the best mathematicians. The details are reviewed in Sect. 2. We present in Sect. 3 the cycloid's geometrical properties, while the mechanical ones in Sect. 4, to appreciate the solutions published by Leibniz and Newton.

The solution of Leibniz is presented next in Sect. 5. It may be remarked that its main assertion, non-justified and obtained from "calculus", is used to establish, by means of geometrical reasoning, that the curve must be an arc of cycloid.

The method discovered by John Bernoulli, presented in Sect. 6, is entirely based in the differential and integral calculus and in Fermat's least time principle.

We reproduce in Sect. 7.1 a theorem by Newton, related to the motion of heavy bodies along cycloids. This is all we can do toward the understanding of Newton's solution, since there is no record of the method he followed to face Bernoulli's challenge. It should be remarked that he already calls "shortest", in this theorem, the time to fall along an arc of cycloid. Newton's solution, as published, is presented in Sect. 7.2.

We present a geometrical approach to this problem, *i.e.*, not based in a differential equation, in section Sect. 8. It is adapted from that of Bernoulli and depends thus on the mentioned work of Fermat, which was established on geometrical grounds.

The brachistocrone problem is, however, connected with the controversy on the priority dispute between Newton and Leibniz, thus involving even Galileo, as will be shown in Sect. 9. The presented solutions give implicit information not only as to their mathematical skills and cleverness, but also on historical facts.

2. BERNOULLI CHALLENGES THE MATHEMATICIANS

In June 1696, Bernoulli presented a challenge to the mathematicians by publishing a new problem: *Problema novum ad cujus solutionem Mathematici invitantur* in the famous journal *Acta Eruditorum Lipsiæ* [1], stated in the following words: "Given two points A and B in a vertical plane, assign a path AMB to the moving body M, along which the body will arrive to point B, falling by its own gravity and beginning from A, in the least time". Next, Bernoulli adds that the path, although known to the geometricians, is not a straight line, and that he will indicate that path, if nobody would do so that year. Those six months elapsed however, without receiving a satisfactory solution. Westfall claims that this challenge was meant to be directed to Newton: "Recall that earlier in 1696 Bernoulli had expressed the opinion that Newton had filched the method that he first published in Wallis' Opera from Leibniz papers. Manifestly, both Bernoulli and Leibniz interpreted the silence from June to December as a demonstration that the problem had baffled Newton. They intended now to demonstrate their superiority publicly" [2].

In a letter dated Jan. 30 $169\frac{6}{7}$, Newton wrote to Charles Montague, then president of the Royal Society [3], that he had received from Groningen, the previous day, two problems proposed by a great mathematician, and transcribed the whole letter, where Bernoulli reported that his last June challenge to the mathematicians had received no solution and that Leibniz had written him, not only asserting that he had solved the problem, but also requesting the deadline to be extended to Easter and the problem to be republished between the French and the Italian. Bernoulli adds that he had accepted and decided to make public this extension. In this letter, however, Bernoulli restates the problem "Find the path connecting two fixed points, chosen at different heights, not in the same vertical, along which a moving body, falling by its own gravity and starting from the higher point, will descend most quickly to the lower one" and adds a second purely mathematical problem, which says according to Newton's interpretation: "Given a fixed point P, a curve is sought, such that for each straight $\overline{PKL}$ cutting it in two points K and L, the sum of the distances $\overline{PK}$ and $\overline{PL}$, risen to a given power n, be a constant". Newton adds then, in this same letter to Montague, the solution of both problems. This letter, included in his "Collected Papers" [3], seems anonymous, since no signature is apparent. Four months later, in May, an excerpt from a paper originally published in England, in the January issue of the *Philosophical Transactions*, is published in the *Acta Eruditorum Lipsiæ*, [4] with the title: "Epistola Missa ad Prænobilem Virum D. Carolum Montague Armigerum, Scaccarii Regii apud Anglos Cancellarium, et Societatis Regiæ Præsidem: in qua solvuntur duo problemata Mathematica a Johanne Bernoullio Mathematico Celeberrimo proposita". The solutions presented in this excerpt, also anonymous, are those of Newton. This fact seems to show that the letter from Newton to Montague is, really, anonymous. This observation helps to understand several assertions, to be made below. Westfall [2] adds: "In addition to Leibniz's solution, Bernoulli received two others, one from the Marquis de l'Hospital in France and an anonymous one from England. Disabused on Newton's skill in mathematics, Bernoulli recognized the author through the authority the paper displayed —'as the lion is recognized from his print'—in his classic phrase, in Latin of course: 'tanquam ex ungue leonem' ". It should be remarked that the same May Issue of the *Acta Eruditorum* also published the paper "Solutio Problematum Fraternorum" by Jakob Bernoulli, senior brother of John Bernoulli [9].

Leibniz presented all the received solutions [5] in a paper bearing the title: "G.G.L. Communicatio suæ pariter, duarumque alienarum ad edendum sibi primum a Dn. Jo. Bernoullio, deinde a Dn. Marchione Hospitalio communicatarum solutionum problematis curvæ celerrimi descensus a Dn. Jo. Bernoullio Geometris publice propositi, una cum solutione sua problematis alterius ab eodem postea propositi" in the mentioned May issue of the *Acta Eruditorum Lipsiæ*, as promised by Bernoulli [3], and claimed [5] that "Newton could solve this problem if he only undertook the task".

3. CYCLOIDS' GEOMETRY ACCORDING TO HUYGENS

The state of the art on cycloids in 1696–7, on either geometrical or mechanical properties, is exposed in the book *Concerning the Motion of Oscillating Clocks or of Pendula Adapted to Clocks* published in 1673 by Huygens [6]. As known, a cycloid is the path described by a chosen point E from the circumference of a circle —the generating circle— as it rolls without slipping along a straight line $\overline{E_0E_3}$, called the basis (Fig. 1). To distinguish between the cases when the circle stays above, as is the case in Fig. 1, or below the straight line, the curve's concavity being directed downwards or upwards, we shall speak, following Huygens, of a downwards-facing or of an upwards-facing cycloid respectively. Figure 1 shows the generating circle at an arbitrary position K, the chosen point being at E_1; and at the center D, the chosen point falling on the same circle's diameter $\overline{DE_2}$, called the axis of the cycloid. The arc $[E_0E_1E_2E_3]$ is the downwards-facing cycloid. That the length of the arc $[E_1K]$ equals that of segment $\overline{E_0K}$, follows from the non-slipping condition. The two main properties of the cycloid may be stated with the help of Fig. 1. Let $\overline{E_1NGF}$ be a parallel line to the basis $\overline{E_0E_3}$, meeting the generating circle DGE_2 at G. Draw the segment $\overline{GE_2}$.

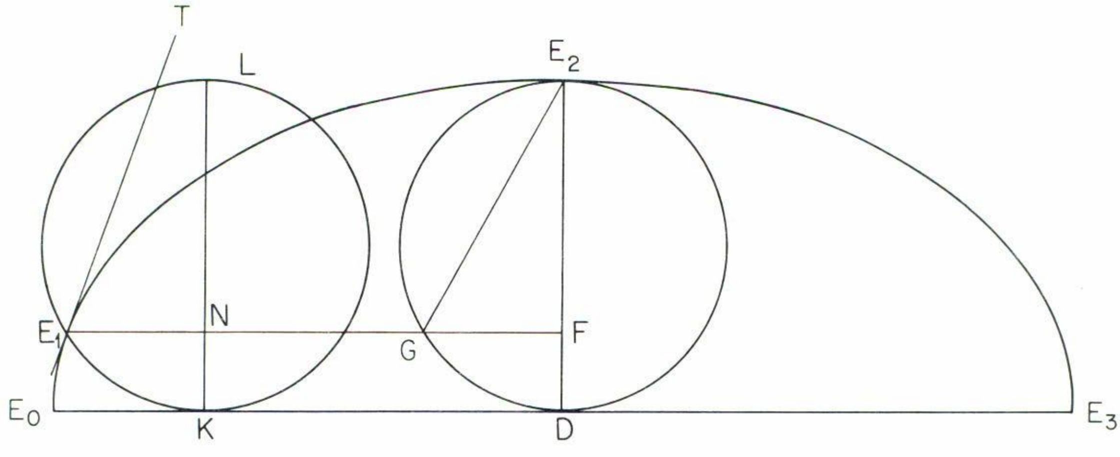

FIGURE 1. The two main properties of the cycloid: i) The length of arc $[GE_2]$ equals that of segment $\overline{E_1G}$. ii) A tangent $\overline{E_1T}$ at any point E_1 of the cycloid is parallel to $\overline{GE_2}$.

1. The length of arc $[GE_2]$ equals that of segment $\overline{E_1G}$. This fact may be shown [6] from the non slipping condition.
2. To draw a tangent at any point E_1 of the cycloid, it is enough to draw $\overline{E_1T}$ parallel to $\overline{GE_2}$.

These two properties were established by Huygens, reasoning along the classical grounds of geometry [6–7]. However, he also points a "dynamic" argument, which we recall here because of its brevity: when the generating circle is at K, its instantaneous center of rotation is precisely point K. This means that $\overline{E_1K}$ (this segment and segment $\overline{GD}$, to be mentioned lated, are not drawn in the figure, since it is a reproduction of the original one) is the instantaneous radius, *i.e.*, a normal to the cycloid, itself parallel to the segment $\overline{GD}$. Since $\overline{GE_2}$ is perpendicular to $\overline{GD}$, and the tangent $\overline{E_1T}$ is perpendicular to its normal, *i.e.*, to $\overline{E_1K}$, it follows that $\overline{E_1T}$ is parallel to $\overline{GE_2}$.

4. CYCLOID'S MECHANICS ACCORDING TO HUYGENS

4.1 Galileo's Hypothesis

Huygens develops all the mechanical properties of the cycloid, by adopting three laws of mechanics, establishing first the laws of accelerated motion along an incline. In this work we shall not repeat the reasonings which lead him to the establishment of such laws, but shall restrict ourselves to the discussions directly linked with the cycloid. We shall, together with Bernoulli [3], "... to avoid ambiguities..." adopt Galileo's hypothesis: *The speed of a falling body varies as the square root of the height.*

4.2 Huygens' Proposition XXIII

The main mechanical proposition by Huygens, bearing number XXIII in his treatise, is related to the falling of a body along an arc of an upwards-facing cycloid. Huygens gives

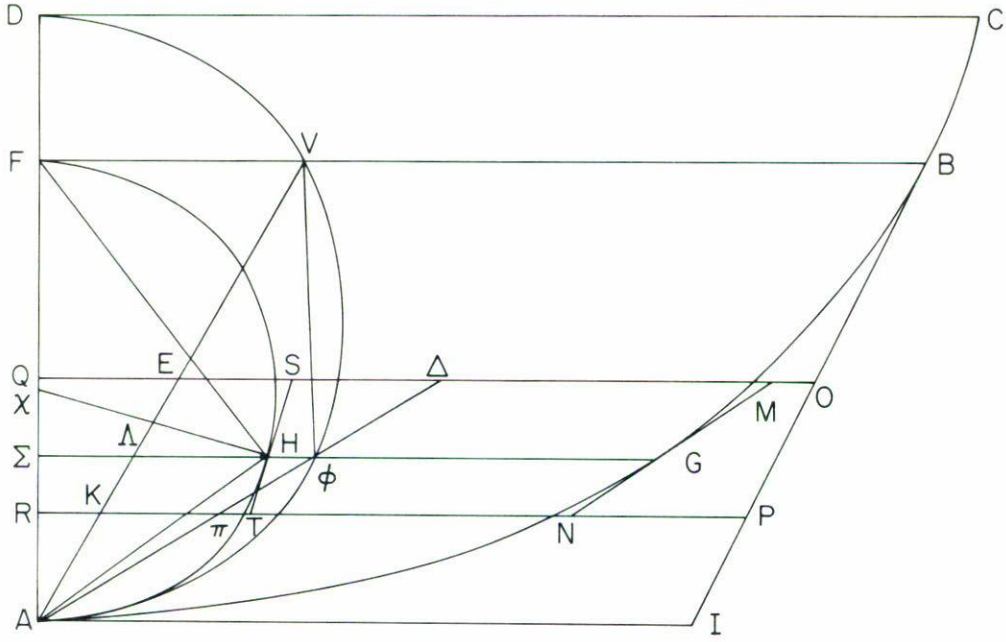

FIGURE 2. Huygen's main (local) mechanical proposition. A body, placed at point B of the cycloid, is to be left under the joint action of gravity and of the cycloid. As the body passes through point G, it has already attained a certain velocity. Huygens compares the time to go through the segment $\overline{MN}$, tangent to the cycloid at G, moving with the speed attained by falling from B to G, against the time it would take to go through the segment $\overline{OP}$, from the tangent to the cycloid at B, moving with the speed attained by falling from B to I.

a sharp geometrical description of the experiment to be performed (Fig. 2): A body, placed at point B of the cycloid, is to be left under the joint action of gravity and of the cycloid. As the body passes through point G, it has already attained a certain velocity. Given a pair of horizontal lines $\overline{QO}$ and $\overline{RP}$, Huygens compares the time to go through the segment $\overline{MN}$, tangent to the cycloid at G, moving with the speed attained by falling from B to G, against the time it would take to go through the segment $\overline{OP}$, from the tangent to the cycloid at B, moving with the speed attained by falling from B to I. Remark that the time to go through segment $\overline{OP}$ plays a rôle similar to a time-unit.

Let ABC be an upwards-facing cycloid, Fig. 2, whose axis $\overline{AD}$ is vertical. From any point B in that curve draw the tangent $\overline{BI}$, cutting the horizontal $\overline{AI}$ in I and $\overline{BF}$, perpendicular to the axis $\overline{AD}$, cutting the circumference $D\phi A$ in V. Let X be the middle point of $\overline{FA}$ and FHA be a half-circle. From any point G of the curve BA draw $\overline{G\Sigma}$, parallel to $\overline{BF}$, cutting the circumference FHA in H, the circumference $D\phi A$ in ϕ, the segment $\overline{AV}$ in Λ and the axis $\overline{AD}$ in Σ. Through the points G and H draw tangents to the corresponding curves, whose segments contained between the horizontal lines $\overline{MS}$ and $\overline{NT}$ be $\overline{MN}$ and $\overline{ST}$. Let $\overline{OP}$ and $\overline{QR}$ be the segments cut by $\overline{MS}$ and $\overline{NT}$ from the tangent $\overline{BI}$ and from the axis $\overline{DA}$.

In these conditions, the time $t_{BG}(\overline{MN})$ to go through the segment $\overline{MN}$, with the speed it acquires by falling along the arc $[BG]$, is to the time $t_{\overline{BI}}(\overline{OP})$ to go through the segment

$\overline{OP}$, with the speed acquired by falling the whole tangent $\overline{BI}$, as twice the ratio of $\overline{ST}$ to $\overline{QR}$:

$$\frac{t_{BG}(\overline{MN})}{t_{\overline{BI}}(\overline{OP})} = 2\frac{\overline{ST}}{\overline{QR}}. \tag{1}$$

Demonstration: Since the time to go through a given segment is proportional to its length and inversely proportional to its speed we have:

$$\frac{t_{BG}(\overline{MN})}{t_{\overline{BI}}(\overline{OP})} = \frac{MN}{OP}\frac{\mathcal{V}_{FA}}{\mathcal{V}_{F\Sigma}}, \tag{2}$$

where $\mathcal{V}_{FA}$ ($\mathcal{V}_{F\Sigma}$) represents the speed reached by falling $\overline{FA}$ ($\overline{F\Sigma}$). Since the attained speeds vary as the square root of the fallen heights (Galileo's hypothesis, Sect. 4.1), we may write the speeds ratio as

$$\frac{\mathcal{V}_{FA}}{\mathcal{V}_{F\Sigma}} = \sqrt{\frac{\overline{FA}}{\overline{F\Sigma}}} = \frac{\overline{FA}}{\overline{FH}} = 2\frac{\overline{FX}}{\overline{FH}}. \tag{2'}$$

Concerning the segments' ratio $\overline{MN}/\overline{OP}$:

$$\frac{\overline{MN}}{\overline{OP}} = \frac{\overline{\Pi\Delta}}{\overline{EK}} = \frac{\overline{A\phi}}{\overline{A\Lambda}},$$

since $\overline{A\phi} \parallel \overline{NM}$ y $\overline{AV} \parallel \overline{OP}$. The triangle $A\phi\Lambda$ being similar to triangle $A\phi V$, we can write

$$\frac{\overline{A\phi}}{\overline{A\Lambda}} = \frac{\overline{AV}}{\overline{A\phi}} = \sqrt{\frac{\overline{AF}}{\overline{A\Sigma}}} = \frac{\overline{AF}}{\overline{AH}} = \frac{\overline{FH}}{\overline{H\Sigma}}. \tag{2''}$$

Substitution of Eqs. (2′–2″) into Eq. (2) gives:

$$\frac{t_{BG}(\overline{MN})}{t_{\overline{BI}}(\overline{OP})} = \frac{\overline{FH}}{\overline{H\Sigma}} \cdot 2 \cdot \frac{\overline{FX}}{\overline{FH}} = 2\frac{\overline{XH}}{\overline{H\Sigma}} = 2\frac{\overline{ST}}{\overline{QR}}.$$

Q.E.D.

4.3 Huygens' Propositions XXIV to XXVI

The preceding result is local, being relative to point G (Fig. 2). Huygens works, however, in proposition XXIV with a finite arc $[BE]$, as can be seen in Fig. 3, copy of the original one published in his treatise, except for the lettering, changed in order to present the main ideas of Huygens in a simpler way. In this proposition, the time $t_B(\text{arc}\,[BE])$ to go through an arc $[BE]$, mapped by means of horizontal lines to the arc $[FH]$ of the

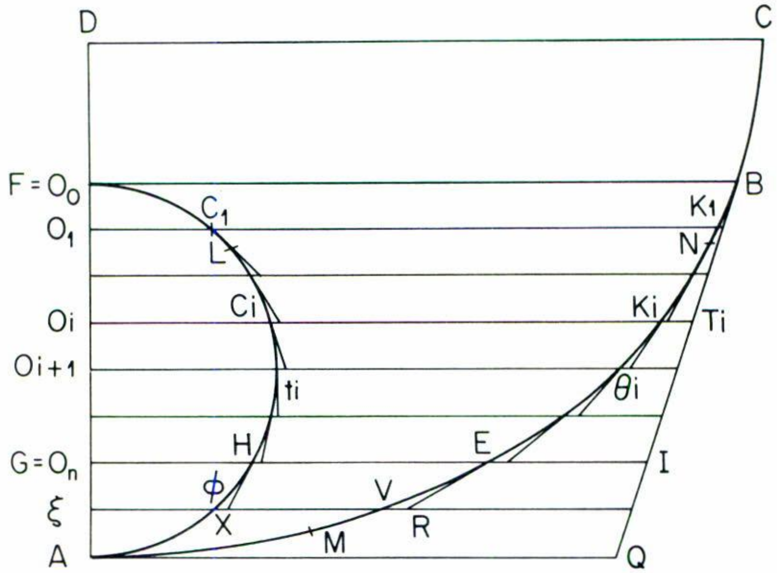

FIGURE 3. Huygen's main mechanical propositions: i) The time $t_B(\text{arc}\,[BE])$ to go through an arc $[BE]$, mapped by means of horizontal lines to the arc $[FH]$ of the semicircle FHA, is shown to be directly proportional to the length of arc $[FH]$. ii) The time to go through arc $[BA]$ is independent of point B (isochronous property). iii) The ratio of the time $t_B(\text{arc}\,[BE])$ to go through arc $[BE]$ to the time $t_B(\text{arc}\,[EA])$ to go through arc $[EA]$, having gone through arc $[BE]$, equals the ration of the lengths of arc $[FH]$ that of arc $[HA]$.

previously drawn semicircle FHA, is shown to be directly proportional to the length of arc $[FH]$.

Let, as in Proposition XXIII, ABC be an upwards facing cycloid (Fig. 3), with vertical axis $\overline{AD}$. Having chosen any point B from the curve, draw the horizontal $\overline{BF}$, the semicircle FHA and the tangent $\overline{BQ}$, cutting the horizontal $\overline{AQ}$ in Q. Let $\overline{GE}$ be any horizontal line below $\overline{BF}$, cutting the cycloid in E, the tangent line $\overline{BQ}$ in I, the semicircle FHA in H and the axis $\overline{AD}$ in G.

In these conditions the time $t_B(\text{arc}\,[BE])$ to go through arc $[BE]$, falling from B, is to the time $t_{BQ}(\overline{BI})$ to go through the tangent $\overline{BI}$ with the speed it reaches by falling from B to Q, as twice the ratio of arc $[FH]$ to the segment $\overline{FG}$, that is,

$$\frac{t_B(\text{arc}\,[BE])}{t_{\overline{BQ}}(\overline{BI})} = 2\frac{\text{arc}\,[FH]}{\overline{FG}}. \tag{3}$$

Huygens' demonstration of this Proposition is very long, although interesting from the historical point of view, since it prefigures the techniques of integration, to appear later in the works of Newton and Leibniz, except that, instead of taking a limit, as in calculus, the reductio ad absurdum is used. Any way, the influence is clear, since Leibniz chose Huygens as professor to learn mathematics.

The main ideas of Huygens, however, may be simply sketched by employing indexed notation.

Let $\{O_i,\, i = 0, 1, \ldots, n\}$ be a partition of segment $\overline{FG}$, whose elements are equally spaced, with $O_0{=}F$ and $O_n{=}G$. Draw a set of horizontal lines $\{\mathcal{L}_i\}$ through O_i, $i = 0, \ldots, n$, such that $\mathcal{L}_i$ cuts the semicircle FHA in C_i, the cycloid in K_i and the tangent $\overline{BQ}$ in T_i. from C_i and K_i draw downwards the tangents $\overline{C_it_i}$, $\overline{K_i\theta_i}$, to the corresponding curves.

If the number n of intervals is large enough, we can safely identify the time $t_B(K_iK_{i+1})$ to go through arc $[K_iK_{i+1}]$, starting from point B, with the time $t_{BK_i}(\overline{K_i\theta_i})$ to go through segment $\overline{K_i\theta_i}$ with the speed reached by falling from B to K_i. According to Eq. (1), we have

$$\frac{t_{BK_i}\left(\overline{K_i\theta_i}\right)}{t_{\overline{BQ}}\left(\overline{T_iT_{i+1}}\right)} = 2\frac{\overline{C_it_i}}{\overline{O_iO_{i+1}}},$$

which may be rewritten as

$$t_{BK_i}(\overline{K_i\theta_i}) = 2\overline{C_it_i}\cdot\frac{t_{\overline{BQ}}(\overline{T_iT_{i+1}})}{\overline{O_iO_{i+1}}} = 2\overline{C_it_i}\cdot\frac{t_{\overline{BQ}}(\overline{BI})}{\overline{FG}}. \tag{4}$$

The last equality being due to the equal speed with which both segments $\overline{T_iT_{i+1}}$ and $\overline{BI}$ are gone through. By summing Eq. (4) from $i = 0$ to $i = n - 1$ we get Huygens' result, since the sum of segments C_it_i approaches the length of arc $[FH]$.

Huygens' Proposition XXV states the cycloid's isochronous property: *The time to go through arc* $[BA]$ (Fig. 3), *is independent of point* B. This result follows both from Eq. (3), which gives $t_B(\text{arc}\,[BA])/t_{\overline{BQ}}(\overline{BQ}) = 2\cdot\pi$, and from the fact that the time $t_{\overline{BQ}}(\overline{BQ})$ equals the time $t_{\overline{DA}}(\overline{DA})$. This last assertion may be established using Galileo's hypothesis, noting that $\overline{BQ}$ is a cycloid's tangent.

Huygens' Proposition XXVI, which is the basis of Newton's theorem on cycloids, states that *the ratio of the time* $t_B(\text{arc}\,[BE])$ *to go through arc* $[BE]$ *to the time* $t_B(\text{arc}\,[EA])$ *to go through arc* $[EA]$, *having gone through arc* $[BE]$, *equals the ratio of the lengths of arc* $[FH]$ *to that of arc* $[HA]$:

$$\frac{t_B(\text{arc}\,[BE])}{t_B(\text{arc}\,[EA])} = \frac{\text{arc}\,[FH]}{\text{arc}\,[HA]}. \tag{5}$$

This Proposition may be established algebraically from proposition XXIV, noting that $t_B(\text{arc}\,[BA]) = t_B(\text{arc}\,[BE]) + t_B(\text{arc}\,[EA])$.

We may now follow, having thus established this geometrical properties of the cycloid, the works of Leibniz, Bernoulli and Newton.

5. The solution of Gottfried Wilhelm Leibniz

Leibniz, who signs *GGL* because of his Latinized name, presents a solution [5], whose main assertion is not justified. He even explains that Bernoulli wished to publish his own

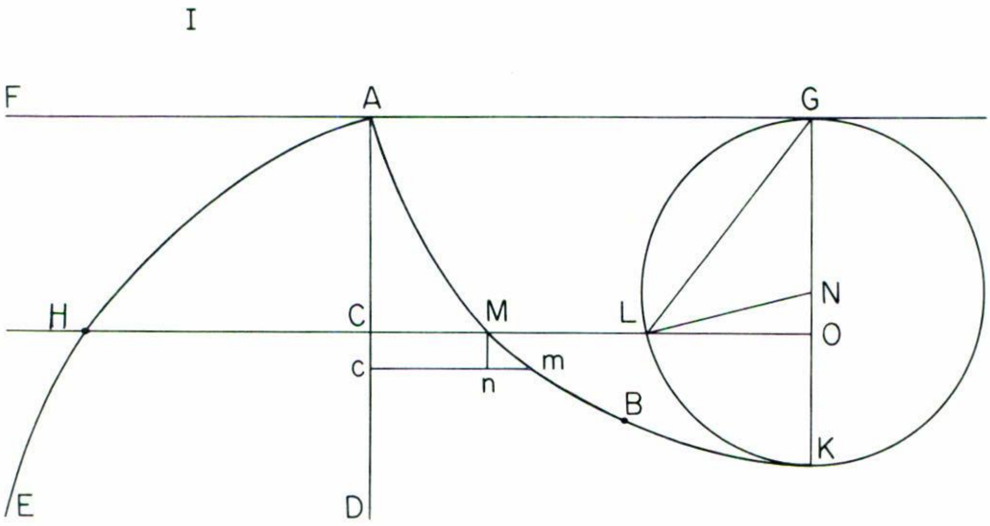

FIGURE 4. Johann Bernoulli's and Leibniz' figure to establish that a cycloid is a brachistochrone.

method, thus indicating, by the way of contrast, that he did not. This seems to mean it optional to explain the mathematical details, a fact which may be confirmed by reading Newton's solution, as shown later in this paper, which also lacks information on the leading mathematical criterions. Remark that Leibniz explains carefully the steps following his main assertion:

> ... calculus has given me the sought curve to be the figure representing the circular segments (lineam quæsitam esse figuram segmentorum circularium repræsentatricem)...

This is his main assertion, about whose origin he indicates only to be the calculus. He continues:

> ... Certainly, if the curve ABK, [Fig. 4], is of such nature that, having drawn the circle, it cuts its lower point K, that it 'touches' at G the horizontal straight line through A and, having drawn normally the intersections with the vertical axis $\overline{AC}$ in C, with the curve at M, with the circle at L, and with its vertical diameter $\overline{GK}$ at O, that its ordinates $\overline{CM}$ are proportional to the circular segments and that the rectangle formed by the circle's semiradius and the ordinate $\overline{CM}$ equals the segment enclosed between arc $[GL]$ and the cord $\overline{GL}$, then AB, the arc intercepted between the two given points, is the curve along which a body, under its gravity, may come fastest from A to B.
>
> ... That this curve is a cycloid may be easily shown: Since the segment $\overline{OC}$ equals the semicircle GLK, and the segment $\overline{LM}$ equals the arc $[LK]$, the sum of segments $\overline{OL} + \overline{CM}$ will be equal to the arc $[GL]$. From the circle's center N draw $\overline{NL}$. It is obvious that the rectangle under the semiradius and $\overline{OL} + \overline{CM}$ equals the circular sector $GNLG$, and that the rectangle under the semiradius and under $\overline{OL}$ equals the triangle GNL. Then the rectangle under the semiradius and under $\overline{CM}$ equals the segment GLG, that is, to the difference from the circular sector and the triangle GNL.

From this text, we may remark that Leibniz shows a cycloid to fulfill his main assertion but that he does not indicate how to choose the right cycloid which solves the problem.

6. THE METHOD OF JOHANN BERNOULLI

Between the solutions of the brachistochrone problem published by Leibniz that year, only one, the longest, written by Bernoulli, is provided with explanations as to the mathematical principles followed. In the paper [8] "Curvatura radii in diaphnis non uniformibus, Solutioque Problematis a se in Actis 1696, p. 269, propositi, de invenienda Linea Brachystochrona, id est, in qua grave a dato puncto ad datum punctum brevissimo tempore decurrit, et de curva Synchrona seu radiorum unda construenda", Bernoulli makes use of Fermat's principle, adapted to the motion of bodies, and the calculus of Newton and Leibniz. Two points should be remarked from the following excerpt: first, the asymmetrical statement of Fermat's principle, and second, the assertion of Bernoulli about the geometric nature of Fermat's demonstration:

> ...Fermat established, in a letter addressed to De La Chambre, that a light ray, going from a rarer to a denser medium, must refract towards the normal, so that the ray, supposed to advance from the illuminating to the illuminated point, (qui a puncto luminante ad punctum illuminatum successive procedere supponitur) follows the path of shortest time. From this principle, Fermat established that the sine of the angle of incidence is to the sine of the angle of refraction as the direct ratio of the media's rarities, or as the inverse ratio of the media's densities, that is, in the direct ratio of the speeds with which the ray penetrates the media (id est, in ipsa ratione velocitatum, quibus radius media penetrat). This principle was concisely shown later to hold by the keen Leibniz in *Acta Eruditorum Lipsiæ*, 1682, p. 185. and, more recently, by the celebrated Huygens in his treatise *De lumine, i.e., Concerning light*, p. 40. They established, in a sounder basis, the physical principle, or rather metaphysical of Fermat, who was satisfied with his geometric demonstration....

Bernoulli adapts then Fermat's principle to the mechanical motion of bodies, mapping their changing velocity with a stratified optical medium:

> ...If now we do not consider the medium of uniform density, but composed of a large number of horizontal layers, whose interstices are filled with a transparent material, of increasing or decreasing rarity according to a certain law, it is evident that a light ray, which we consider as a particle, will not go along a straight line, but along a certain curve (a fact already pointed out by Huygens in the same treatise, de Lumine, but without determining the curve). This curve is of such nature that the particle goes through the arc between any two of its points in the shortest time, while its speed continually increases or decreases according to the rarity changes...
>
> ...It is also clear, since the sine of the refraction angle varies in each point as the medium's rarity, or as the particle's speed, that the curve is such that *The sines of the angles of inclination measured from the vertical vary everywhere in the same ratio as the speeds.* Having established these facts, no difficulty remains: The brachistochrone curve is the path followed by a light ray going through a medium, whose rarities are in the same ratio as the speeds of a vertically falling body...
>
> ...In this general manner may be solved our problem, for any acceleration law that we might establish. It has been reduced to determine the curvature of a light ray going through a medium whose rarity varies arbitrarily. Let FGD, [Fig. 4], be the medium, limited by the horizontal $\overline{FG}$; A, the radiating point; $\overline{AD}$, the vertical; AHE, the axis of the given curve, so that $\overline{HC}$ stands for the medium's rarity at depth $\overline{AC}$, or the speed of the light ray or of the particle at M, and AMB, the sought path. Let x stand for the distance $\overline{AC}$; t, for $\overline{HC}$;

y, for $\overline{CM}$; dx, for the differential $\overline{Cc}$; dy, for the differential $\overline{nm}$; dz, for the differential Mm; and a, for an arbitrary constant. The arc $[Mm]$ will be taken to represent the curved light ray; $\overline{mn}$, the sine of the refraction angle, that is, of the curve's inclination measured from the vertical. For the previously indicated reasons, $\overline{mn}$ is directly proportional to $\overline{HC}$, that is $dy : t = dz : a$, implying this equation: $a\,dy = t\,dz$ or $aa\,dy^2 = tt\,dz^2 = tt\,dx^2 + tt\,dy^2$, which reduced gives the general differential equation $dy = t\,dx/\sqrt{aa - tt}$ for the sought curve...

Bernoulli applies next such results to the falling bodies, under Galileo's hypothesis:

...The given curve AHE is a parabola, that is, $tt = ax$ & $t = \sqrt{ax}$, which after substitution into the general equation gives $dy = dx\sqrt{x/(a-x)}$, from which I conclude that the sought curve, the brachistochrone, is the cycloid. Let the circle GLK of diameter a be rolled along $\overline{AG}$, beginning at point A. The circle's point K will describe a cycloid, which is found to have the same differential equation $dy = dx\sqrt{x/(a-x)}$, after substituting x for $\overline{AC}$ and y for $\overline{CM}$. This may be discovered a priori and analytically in the following manner:

$$dx\sqrt{\frac{x}{(a-x)}} = \frac{x\,dx}{\sqrt{ax-xx}} = \frac{-a\,dx + 2x\,dx}{2\sqrt{ax-xx}} + \frac{a\,dx}{2\sqrt{ax-xx}},$$

however, $a\,dx - 2x\,dx\big/(2\sqrt{ax-xx})$ is a differential quantity whose sum is given by $\sqrt{ax-xx}$, that is, $\overline{LO}$, and $2\,dx\big/(2\sqrt{ax-xx})$ is the differential of the arc $[GL]$ itself, so that the summed equation from $dy = dx\sqrt{x/(a-x)}$ will have y or $\overline{CM} = \text{arc}\,[GL] - \overline{LO}$, that is, $\overline{MO} = \overline{CO} - \text{arc}\,[GL] + \overline{LO}$. But $\overline{CO} - \text{arc}\,[GL] = \text{arc}\,[LK]$, since $\overline{CO}$ equals the semicircle GLK, so that we have $\overline{MO} = \text{arc}\,[LK] + \overline{LO}$, and having subtracted the common $\overline{LO}$ we obtain $\overline{ML} = \text{arc}\,[LK]$, showing the curve KMA to be a cycloid...

...We still have to show, so that the problem is completely satisfied, how to describe from the chosen point, the brachistochrone or cycloid, passing through the other given point. This is solved as easy as this: Draw [Fig. 5] through the given points A and B a straight line $\overline{AB}$, and, along the horizontal $\overline{AL}$, any cycloid beginning at A, intersecting the straight line $\overline{AB}$ in R. In the same ratio as $\overline{AR}$ is to $\overline{AB}$, take the diameter of the generating circle of cycloid ARS to a fourth, which will be the diameter of the generating circle of the sought cycloid ABL, which must pass through B...

This last detail, published in May 1697, should be compared with Newton's solution, published the previous January.

7. Sir Isaac Newton and the Brachistochrone Problem

There is no record of the method followed by Newton to face Bernoulli's challenge and solve the problem, although we believe that he reasoned along geometrical, non-analytical techniques, since not only he sent his solution, established in such terms, but also wrote a theorem [3], closely connected with our problem, recorded in a especial chapter from his *Collected Papers* [3] entirely devoted to the Bernoulli problems "De problematibus Bernoullianis". In this theorem he already gives the name of shortest to the time to fall along an arc of cycloid. Another reason favoring our hypothesis is the natural reaction against the contention of Leibniz, to be presented later, that the solution may only be

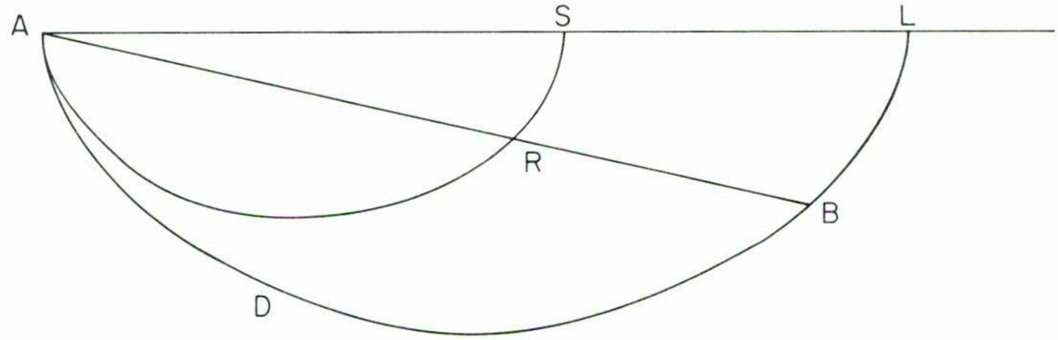

FIGURE 5. Bernoulli's figure to choose the cycloid going through the given points A and B.

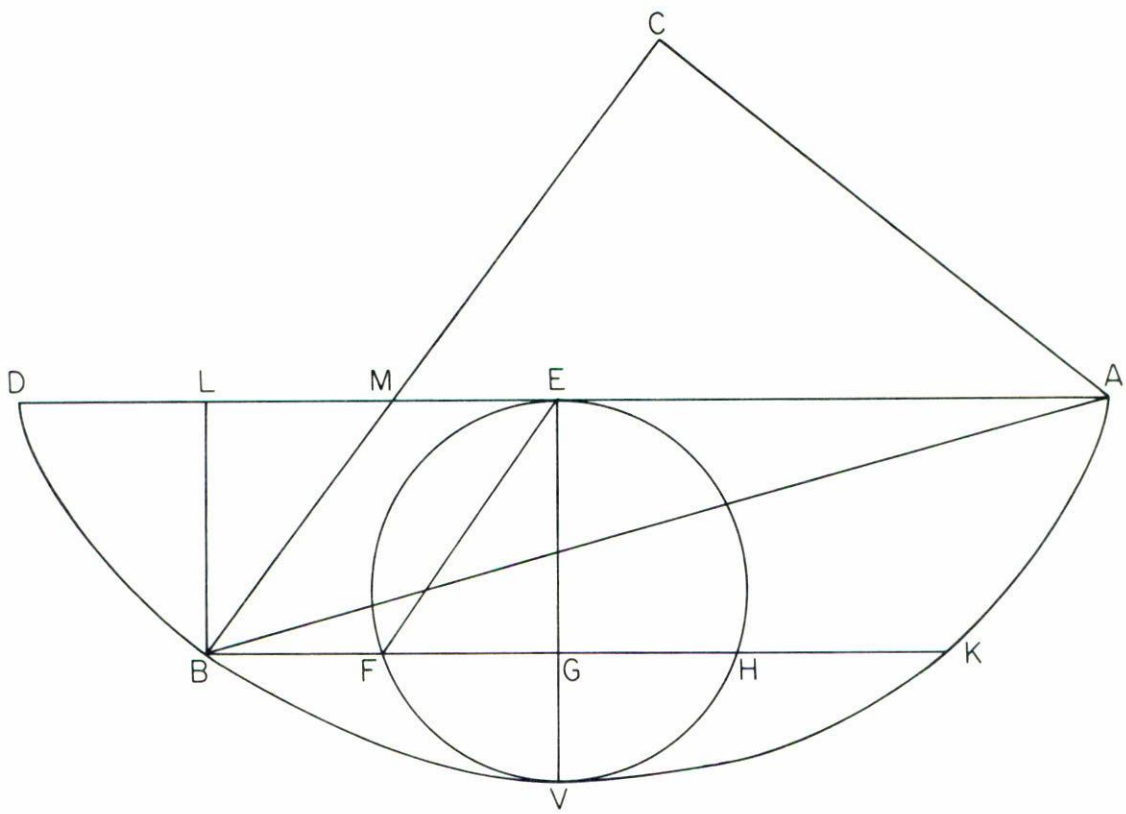

FIGURE 6. Newton's Theorem on Cycloids: About the ratio of time to slide along a straight line, drawn through the given points A and B, to the shortest, of sliding by the force of gravity, from one of these points to the other along an arc of cycloid.

obtained through the calculus, which he, *GGL*, had discovered. In the following two paragraphs, the previously mentioned theorem, related to Fig. 6, and his solution to the brachistochrone problem, related to Fig. 7 are presented.

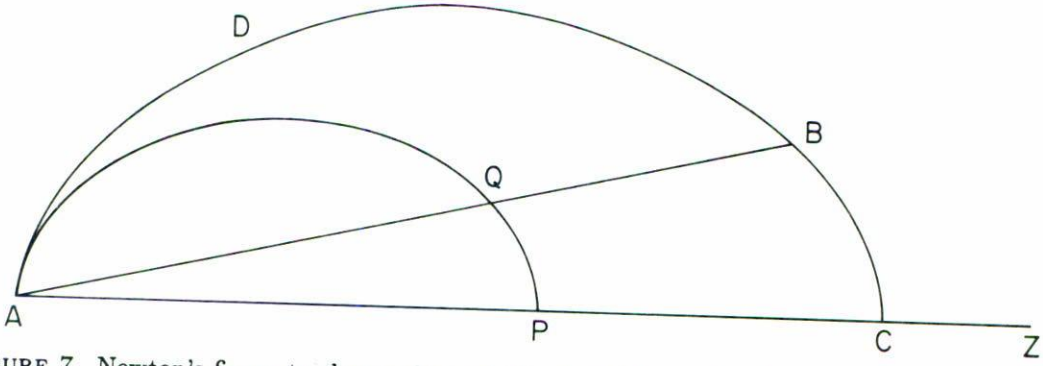

FIGURE 7. Newton's figure to choose the cycloid going through the given points A and B.

7.1 Newton's theorem on cycloids

The next excerpt has been directly translated from Newton's work [3]. The results on geometrical and mechanical properties of the cycloid, previously presented in Sects. 3 and 4, allow us to follow his reasoning, without difficulty:

About the ratio of the time to slide along a straight line, drawn through the given points, to the shortest, of sliding by the force of gravity, from one of these points to the other along an arc of cycloid.

Theorem

...If in a cycloid AVD, whose basis $\overline{AD}$ is parallel to the horizon, the vertex V being directed downwards, any straight line $\overline{AB}$ is drawn from A, intersecting the cycloid at B, and from B is drawn the straight line $\overline{BC}$, normal in B to the cycloid, and the perpendicular $\overline{AC}$ to $\overline{BC}$ is drawn from A, I assert that the time for a heavy body to go through straight line $\overline{AB}$, starting from rest, is to the time to go through arc $[AVB]$ as the straight line $\overline{AB}$ is to the straight line $\overline{AC}$.

...Draw $\overline{BL}$ through B, parallel to the cycloid's axis $\overline{VE}$, and $\overline{BK}$ parallel to the basis $\overline{AD}$, intersecting the axis at G, the generating circle at points F and H, and finally the cycloid at K. Draw the straight line $\overline{EF}$, which is, according to cycloid's nature, parallel to line $\overline{BC}$. It follows that $\overline{BM}$ equals $\overline{EF}$ and $\overline{EM}$ equals $\overline{BF}$; which, because of the cycloid, equals the arc $[VF]$; and hence, $\overline{AM}$ equals the arc $[EHVF]$.

...According to Proposition XXV, from the second part of the book *Horologium Oscillatorium* [6] by Huygens, the time for a body to go through AV, starting at rest, is to the free fall time along $\overline{EV}$ as the semicircle is to the diameter, and according to the last Proposition of the indicated part, the time to go through VB, after going through AV, (which certainly equals the time to go through KV after going through AK) is to the time to go through AV as the arc $[VF]$ is to the semicircle; and for this reason, it is to the free fall time along $\overline{EV}$, as FV is to the diameter, so that the time to go through arc AVB is to the free-fall time along $\overline{EV}$ as the arc $[EHVF]$ is to the diameter $\overline{EV}$. But the free-fall time along $\overline{EV}$ is to the free-fall time along $\overline{LB}$ (or $\overline{EG}$) as $\overline{EV}$ is to $\overline{EF}$. Then, from the equality, the time to go through AVB is to the free-fall time along $\overline{LB}$ as the arc $[EHVF]$ is to the cord $\overline{EF}$, that is, as the straight line $\overline{AM}$ is to the straight line $\overline{MB}$. However, the free fall time along $\overline{LB}$

is to the time to go through the straight line $\overline{AB}$, as $\overline{LB}$ is to $\overline{AB}$. Then, the ratio of the time to go through AVB to the time to go through $\overline{AB}$ is composed from the ratios $\overline{AM}$ to $\overline{MB}$ and $\overline{LB}$ to $\overline{BA}$, and for this reason equals the ratio of $\overline{AM} \times \overline{LB}$ to $\overline{MB} \times \overline{BA}$. But $\overline{AM} \times \overline{LB}$ equals $\overline{MB} \times \overline{AC}$, since both are equal to twice the area of the triangle ABM. And then, the time to go through arc $[AVB]$, starting from rest, is to the time to go through $\overline{AB}$, as $\overline{MB} \times \overline{AC}$ to $\overline{MB} \times \overline{BA}$, that is, as $\overline{AC}$ is to $\overline{AB}$. Q.E.D.

...Similarly proceeds the demonstration if the point B lies between A and V.

7.2 The solution of Isaac Newton

The anonymous solution which Newton sent to Charles Montague was published in England in the January Issue of $169\frac{6}{7}$, in the *Philosophical Transactions.* It was also published later, according to Bernoulli's promise, in the May issue of 1697, in the *Acta Eruditorum Lipsiæ* [4]. There is still another copy in Newton's *Collected Papers* [3]. The solution runs in the following terms (Fig. 7):

> From the given point A draw the infinite straight line $\overline{APCZ}$, parallel to the horizon, and above this line describe any cycloid AQP, intersecting the straight line $\overline{AB}$, produced if necessary, at point Q, as well as the cycloid ABC, whose basis and height are to the basis and height of the first cycloid as $\overline{AB}$ is to $\overline{AQ}$. An the last cycloid will pass through point B and be that curved line along which a heavy body will reach most quickly the point B by the force of its gravity. Q.E.I.

Remark that Q.E.I. means what was to be found.

8. GEOMETRICAL SOLUTION TO THE BRACHISTOCHRONE PROBLEM

A geometrical solution of this problem is given, as a natural reaction against the contention of Leibniz [5], that the solution may only be obtained through the calculus which he had discovered. It is adapted from that of Bernoulli and depends thus on the previously mentioned least-time principle of Fermat. It does not depend, then, on calculus, since Fermat died in 1665 and Leibniz began taking lessons of mathematics in 1672. Furthermore, recall that Bernoulli calls that work *geometric* [8], underestimating Fermat's demonstration. Our starting point is Bernoulli's assertion: *The sines of the angles of inclination, measured from the vertical, vary everywhere in the same ratio as the speeds.* This may be written

$$\sin\theta = kC, \tag{6}$$

where θ is the inclination angle, measured from the vertical, k, a constant and C, the speed of the falling body, which varies as the square root of the height, according to Galileo's hypothesis. First, we must write the square root of the height using geometrical means. Let us represent the fallen heights $\overline{GO}$ along the diameter $\overline{GK}$ (Fig. 8). Draw the horizontal $\overline{OL}$ through O and the cords $\overline{LG}$ and $\overline{LK}$. Since the triangles LGK and LGO are similar, we can write $\overline{LG}/\overline{GK} = \overline{GO}/\overline{LG}$ It follows that

$$\overline{LG}^2 = \overline{GK}\,\overline{GO}. \tag{7}$$

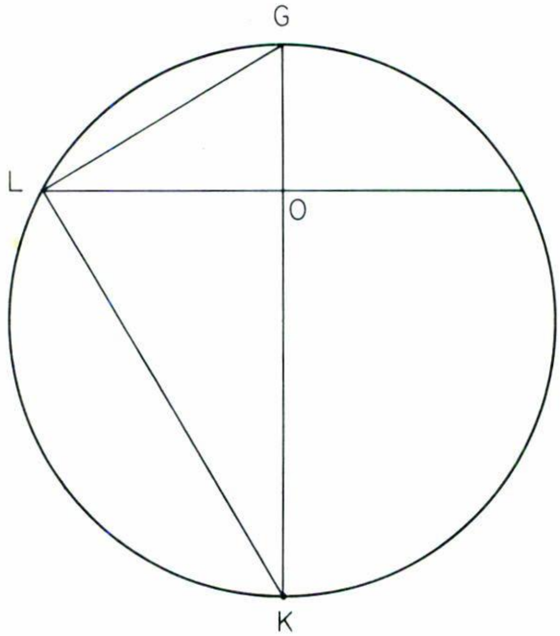

FIGURE 8. Geometrical representation of the speed of a falling body. Segment $\overline{GO}$ stands for the fallen height while its speed is proportional to segment $\overline{GL}$ (Galileo's hypothesis).

Equation (7) establishes that the segment $\overline{LG}$ varies as the square root of segment $\overline{GO}$. We identify thus the body speed C with segment $\overline{LG}$. We must now impose that $\sin\theta$ is directly proportional to the segment $\overline{LG}$. This means according to Fig. 8, that we should identify the angle θ with the angle LKG. The physical interpretation of Fig. 8 is very easy: When the body has fallen the distance $\overline{GO}$ the inclination angle should be LKG, while the straight line $\overline{LK}$ should be parallel to its instantaneous tangent. According to the second property, concerning the drawing of tangents to a cycloid, mentioned in the III paragraph of this paper, we may conclude the figure to be a cycloid. This fact does not solve completely the problem. A criterion must be given, to choose the right cycloid. This, however, is given by Newton's solution.

9. FINAL COMMENTS

Although Bernoulli calls this problem new (novum) [1], his friend Leibniz writes that it was originally stated by Galileo [5] and explains that he, GGL, had discovered the calculus, that Galileo, in spite of being a very clever man (Fuit sane Galilæ us Vir ingenii judiciique maximi) had not been able to solve it, for the art of analysis was not developed in his time, and even less its superior discipline, the calculus (quod ipsius tempore Ars Analytica nondum satis promota esset, pars autem ejus superior seu infinitesimalis adhuc in tenebris jaceret, solutiones hujusmodi sperare non debuit) but that Bernoulli had done it, (Sed Dn. Jo. Bernoullius, meo calculo profundius inspecto, ejus ope optatam solutionem obtinuit).

It is not difficult to identify the work of Galileo alluded by Leibniz. It is the scholium of Theorem XXII, Proposition XXXVI, of his book *Discorsi e Dimostrazione matematiche intorno a due nuove scienze*, which states "...from the facts already shown, it seems possible to infer that the fastest motion from one end to the other does not take place along the shortest line, but along a circular arc...". This scholium, according to C. Solís and J. Sádaba [10] has been interpreted as statement of the Brachistochrone Problem. Galileo, however, states a different fact. To see it, let us recall his Theorem XXII, Proposition XXXVI [10] "If a cord is drawn from the lowest point of a vertical circle, so as to embrace an arc not larger than a quadrant, and if from the ends of such a cord two additional ones are drawn to any point of the corresponding arc, the descent along these two cords takes less time than along the first cord only". After the theorem's demonstration, in the mentioned scholium, Galileo continues his reasoning. He starts with an inscribed polygonal-line in the circle's arc and continues the process started in the theorem, *i.e.*, substituting any fixed segment (cord) of the polygonal line, by a pair of cords, drawn from its ends to any common point of the arc, building thus a new polygonal-line, which not only is closer to the arc, but also, whose descent takes less time. It is from this result that Galileo, comparing the time along a polygonal-line against that along the circle's arc (*not along any arc!*), —obtained by continuing this process indefinitely— establishes the mentioned assertion. We see that Galileo stated a very different result

In the above mentioned text [5], Leibniz repeats five times that Galileo had identified incorrectly the 'catenary', only once that he made the mistake of taking the brachistochrone to be an arc of circle, only once that he had mistaken both problems. On the other side, he prizes three times the infinitesimal calculus and its possibilities, not without recalling that he, *GGL*, had discovered it. These observations are particularly interesting, especially after reading the beginning of his text: "...The proposal of problems to the geometricians, nowadays customary, is certainly useful, provided that it is not made for jactation, (cum non sit animo suos profectus jactandi) but to stimulate others, so that, as each one applies his methods, the art of discovering grows...".

The key to understand the assertions of Leibniz is his enormous need to declare, by all means, that the calculus had been discovered by him, and that it is the fundamental tool to solve the problem, so as to be favored by the public opinion. This need is enough to understand his assertions concerning Galileo, and the very especial situation, which not only prevented him to identify Newton's solution, but also allowed for the implicit assertion that Newton had not solved the problems.

There is still another fact which should be remarked: In the *Acta Eruditorum Lipsiæ*, the Journal which published the different solutions of the brachistochrone problem, there is a page out of text, where all the geometrical figures are collected. Both Leibniz and Bernoulli present their reasonings using the same figure, which we have reproduced in our Fig. 4. However, Newton not only asserts the solution to be a cycloid, but indicates also how to choose the right cycloid, using the figure which we have reproduced in our Fig. 7, first published in the January issue of the *Philosophical Transactions* Journal. We think this to be the reason for Bernoulli to add, not only a second figure, which we have reproduced in our Fig. 5, identical to Newton's, but also, to include the same method to choose the right cycloid.

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