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Fractal solid mechanics

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ABSTRACT. The purpose of present article is to review a new general approach to solid mechanics, named, fractal solid mechanics. The attention is focused on systematic account of the proposed basic concepts, as well as on the most important result of fractal solid mechanics. Special attention is paid to the thermodynamic theory of elasticity of multifractals which is effective for modeling various types of behavior patterns of deformed materials with multifractal microstructure. The fractal concepts in fracture mechanics are considered. It is shown, that the nature of fractal geometry of fracture of a solid is associated with fundamental phenomenon of transverse strains of solid (Poisson's effect). This is manifested by the self-similarity of self-affinity of heterogeneous stresses in irreversibly deformed solids. Some of the most useful analytical and computer models are discussed. The result of theoretical predictions are compared with experimental data. It is shown that the proposed approach is very effective for adequate description of various behavior patterns and some other phenomena in deformed solids.

RESUMEN. El presente trabajo se fundamenta en el nuevo método para la física del estado sólido denominada mecánica fractal del sólido. Se acentúa la atención de un análisis sistemático de las concepciones y desarrollo de los principios fundamentales planteados por la teoría general de fractales y una serie de modelos no lineales aplicables a materiales de elevada elasticidad (elastómeros), plasticidad y fractura de materiales con estructura multifractal. Se demuestra que la naturaleza de la geometría fractal se basa en el fenómeno de deformación transversal del sólido (efecto de Poisson). Esto puede ser explicado por la afinidad de los campos de tensión-deformación, cuando el sólido presenta deformación irreversible. Se analizan una serie de importantes modelos analíticos y computacionales. Los resultados de los planteamientos teóricos se comparan con datos experimentales. Se demuestra que el método propuesto es efectivo para describir y explicar una serie de comportamientos y algunos fenómenos en el sólido deformado, y el disímil comportamiento de cuerpos deformables.

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1. INTRODUCTION

Now it is obvious that the classical approximation of homogeneous continuum is not adequate for real heterogeneous solids and, therefore, is not usable as a general model. In the most general case, natural or artificial heterogeneous materials consist of domains of different materials (phases) or of the same material in different states. Furthermore, crystals and polycrystals, polymers and composites, rocks and porous media, grids and multibar systems can be considered as media with microstructure [1–4]. And finally, recently, it has been determined experimentally that after irreversible deformation even initially homogeneous continuous solid exhibits hierarchical block structure [4–6].

A characteristic feature of all materials with microstructure is the existence of scale parameters L_i which are related to microgeometry or long-range interacting forces. The nature of hierarchical block structure of deformed solid is determined by its fundamental property, namely, its resistance to shear, that causes the difference between the characteristic spatial scales of the regions of localization and dissipation of energy pumped into a deformed body by an external action [7,8]. That is why in recent years, new physical and mathematical models of material media, which can be considered as far-reaching generalizations of classical theories of elasticity, viscoelasticity and plasticity have been intensely developed. The Cosserat continuum [9] was historically one of the first models of elastic media which could not be described within the scope of classical elasticity. However, the works of E. and F. Cosserat (1909) remained practically unknown for a long time, and only around 1960 did the generalized models of the Cosserat continuum began to be developed. Now, they are known as oriented media, micromorphic, multipolar, asymmetric, couple-stress, etc. theories [1]. Explicit or implicit nonlocality is the characteristic feature of all such theories. The latter, in its turn, displays itself in that the theories contain parameters which have the dimension of length. These scale parameters can have different nature such as distance between particles in discrete structures, the dimension of a grain or a cell, characteristic radius of correlation, or acting at certain distance forces, etc. [1–4]. A. Griffith pioneered the use of characteristic scale parameters in the theories of plasticity and fracture of solids. Now, various characteristic scale parameters of length's unit are used in the most of modern theories of irreversible deformations and fracture of solids [2–4,10]. However, the foundation of these theories, as well as of couple-stress theories of elasticity, is the euclidean geometry.

At the same time, in the last ten years it has become clear, due to efforts of many scientists (see, for example, Refs. [4,11–15]), that a large amount of objects and processes in nature most be characterized by non-euclidean geometry and have scaling behavior. The concept of fractals has recently become popular in natural sciences. This concept is introduced on the basis of Hausdorff-Besicovitch dimension which may exceed topological dimension of the object and may assume fractional values. The concept of dimensionality, investigated actively by mathematicians from the beginning of the present century, has been brought to the attention of physicists in the monograph of B. Mandelbrot [11], which appeared in 1975 in French, and then in 1977 in English. This book can be regarded as an excellent example of scientific advertising of popularization, in this case advertising of new concepts and models. This is, in particular, reflected in identification of special class of objects, which are called fractals, whose metric (Hausdorff-Besicovitch) dimension is different from its topological dimension. Another fundamental property of fractals, distinguishing them from homogeneous euclidean objects, is their scaling invariance (self-similarity) [11–15]. Fractals found in nature, such as colloidal aggregates, aerogel, polymers, dendritic particles, porous media, surfaces and spatial distributions of cracks in solids, fracture surfaces, etc., differ from regular fractals (Cantor set, Peano and Koch curves, Sierpinski gasket and carpet, etc.) because they exhibit only statistical self-similarity in a limited range of spatial scaling lengths $L_0 < L < L_M$ (see, for example, Refs. [11–15]). The majority of them can be regarded as an ensemble of fractals of different dimensions characterized by different weights. Such objects are called multifractals and have a spectrum of Rényi dimensions D_q with $-\infty < q < \infty$ [15]. The self-similarity

property of multifractals is more complicated than that of homogeneous fractals and is described in terms of multiscaling [16]. A recent thermodynamical formulation of multifractals, that of Feigenbaum *et al.* [17], maps the multifractal measure into some popular physical models, such as Ising model, etc. [18].

Topology and geometry of natural objects, ranging in size from atomic scale to the size of universe is central to the models we make in order to "understand" and "describe" nature. The last decade has seen recognition of the important, and sometimes decisive, influence of the fractal topology on the elastic and fracture properties of solids [4,19–30]. The unity of laws governing dynamics of nonlinear systems, as well as relationships between the dimensions of phase paths and Lyapounov exponents are among the obvious reasons for the growing interest in the dynamics of fractals and in the fractal dynamics of deformed solids. Among these one should mention particularly the topics of elasticity of self-similar (statistically self-similar, in general case) structures, closely related to the problems of rubber elasticity of polymers and nonlinear elasticity of composites, and problems of fractal geometry of fracture of solids.

Two fundamental laws of reversible deformation of statistically self-similar multifractal structures were proposed in our works [19,20]. A general thermodynamical theory of elasticity of fractals and multifractals was developed in our works [20,21] on the basis of these laws. It was shown that the transverse deformation exponent of an elastically isotropic multifractal is defined uniquely by its metric dimension. Theoretical predictions agree well with experimental data, and with the results of computer simulations. The model of irreversible deformations of solids with multifractal microstructure was proposed in our works [22,23]. Later on we have used these concepts for solving some problems of fracture mechanics [21,24,25]. It was shown that the appearance of fractal topology during fracture of solids is the consequence of Poisson's effect. Some conclusions concerning interpretations of solutions to the problems of linear theory elasticity and crack problems were obtained on the basis of fractal concept. Various models were proposed for determination of fractal geometry, dynamics, and fractal kinetics of the destruction of solids in Refs. [31–40]. General results of these works provide the basis on which the fractal solid mechanics could be developed.

This review is focused on systematic account of the basic concepts, fundamental principles and the most important results of fractal solid mechanics. Also some of the most useful analytical and computer models of fractal solid mechanics are discussed.

2. THE CONCEPT OF FRACTALS AND THE BASIS OF MULTIFRACTAL ANALYSIS OF DEFORMED SOLIDS

After the introduction of fractal geometry by B. Mandelbrot the key problem was to understand why nature gives rise to fractal structures. This implies the formulation of models of fractal growth based on physical phenomena and subsequent understanding of their mathematical structure in the same sense as the renormalization group has allowed to understand Ising type models. The models of the limited diffusion aggregation and the more general dielectric breakdown model, based on iterative processes governed by a Laplace equation and stochastic field, have clear physical meaning and they spontaneously

evolve into random fractal structures of great complexity. However, from theoretical point of view, it is not possible to describe them within the framework of usual concepts. Fractal geometry and multifractal analysis are most widely used in statistical topography, physical kinetics, statistical physics, fluid mechanics, geology, geophysics, and astrophysics, polymer physics, and become increasingly more important in material science and solid mechanics, including fracture mechanics. These models are also used in quantum mechanics, relativistic quantum theory, electrodynamics, relativity theory, etc. Methods of fractal geometry find increasing use in the description of microstructure of structural materials, especially of composite materials. The most adequate description of fracture surfaces can be given in terms of fractals. For the study of physicochemical, electrophysical, and mechanical properties of disordered, disperse, and porous materials different fractal and multifractal models are also widely used.

In recent years, it has been established that several physical quantities describing random systems do not obey conventional scaling laws. Prominent examples are the probability density of random walks on random fractals, such as percolation clusters, voltage drops in random resistor networks, growth probabilities of limited diffusion aggregation, growth dynamics of viscous fingering at high capillary numbers, distribution of impurities on the surfaces and in the bulk, invariant measure of strange attractors in chaotic dynamical systems, and some phenomena in deformed media, specifically the energy dissipation in turbulence, clusters of microscopic defects in deformed solids, cascades of debris generated by explosions or shocks, etc. (see Ref. [4]). All these quantities have very broad distribution, and their moments cannot be described by a single exponent but an infinite hierarchy of exponents is needed to characterize them. The point is that this description holds whatever the linear scale is —be it global, on the scale of meters, or on the microscopic scale. This phenomenon is called multifractality and was first found in the context of turbulence. Multifractals are fractal sets which are not self-similar. Multifractal measure are related to the study of distribution of physical or other quantities on a geometric support. The support may be an ordinary plane, the surface of a sphere or of a volume, or it could itself be a fractal. Concepts underlying recent development of what is now called multifractals were originally introduced by Mandelbrot in the discussion of turbulence and extended by Mandelbrot, Grassberger, Aharony, Meaking, Anisimov, Feder, Balankin, Hentschel and Procaccia, Grassberger and Procaccia, Badii and Politi, Arcangelis *et al.*, Rammal *et al.*, Halsey *et al.*, Feigenbaum *et al.*, and other authors to many other areas (see Refs. [4,14,15]). Analysis of experimental data and introduction of the dimension function $f(\alpha)$ by Frish and Parisi and Jensen *et al.* gave the most remarkable agreement between observation and simple theoretical model of multifractals. They demonstrated the usefulness of multifractals in describing experimental observations. Related works are described by Glazier *et al.*, Bensimon *et al.*, Halsey *et al.*, Meakin *et al.*, and Nittmann *et al.* The connection between multifractality and multiscaling was investigated by Coniglio and Zannetti, and Vavriv and Ryabov [15,16]. It has been proposed that multifractals can be characterized by an infinite spectrum of generalized dimensions D_q, where $-\infty < q < -\infty$, also called the Rényi dimensions [12–15].

In his attempt to generalize the concept of entropy of a probability distribution, the Hungarian mathematician A. Rényi introduced the following expression based on the

moments of order q of the probabilities P_i:

$$I_q = (q-1)^{-1} \ln \sum_{i=1}^{N} P_i^q, \tag{1}$$

where q is not necessarily an integer. In the limit $q \to 1$ this definition becomes the well-known information entropy

$$I_1 = -\sum_{i=1}^{N} P_i \ln P_i, \tag{2}$$

of a discrete probability distribution, also called Shannon entropy. The definition in Eq. (1) can therefore be considered, as was Rényi's intent, as generalized entropy. For definition of generalized dimensions D_q, one must choose the portion of the multifractal with boxes of size r, and define $P_i(r)$ as the probability of finding the point of structure in the i–th box. It should be noted that

$$\sum_{i=1}^{N(r)} P_i(r) = 1. \tag{3}$$

The set $\mu = \{P_i\}$ is called multifractal probability measure or mass density, of the subset $F_i(r) \subset F$ contained inside the i–th covering box with the edge r. Consequently, the Rényi dimension of order q is given by the expression

$$D_q = \lim_{r \to 0} \left\{ \frac{I_q(r)}{\ln r} \right\} = -(q-1)^{-1} \lim_{r \to 0} \left\{ \sum_{i=1}^{N(r)} \frac{P_i^q(r)}{\ln r} \right\}, \tag{4}$$

where the parameter q ranges from $-\infty$ to ∞.

One can readily see that in general case, D_0 is equal to metric dimension evaluated by means of box-counting algorithm, and called fractal dimension

$$D_{\mathrm{F}} \equiv D_0 = D_{\mathrm{B}}.$$

Generalized dimension of order $q = 1$ is equal to information dimension D_{I}; and dimension D_2 is equal to the correlation integral exponent D_{C}, also called correlation dimension. Note that, generalized Rényi dimensions satisfy the relation

$$D_{q'} \leq D_q, \quad \text{for} \quad q' > q;$$

the equality being obtained in the case of uniform sets, *i.e.*, such that the probability measure is constant,

$$P_i \equiv \frac{1}{N(r)},$$

TABLE I. Rényi dimension D_q, special values and limits of sequences of mass exponents $\tau(q)$ and $f(\alpha)$ curve for a multifractal measure $M = \{P_i\}$, supported by a set with fractal dimension $D_F = \dim_H M$ ($\dim_H M$ = Hausdorff-Besicovitch dimension), for different moments of order q of M (see Refs. [12–14]).

q	Dimension D_q	$\tau(q)$	$\alpha = -\frac{d\tau}{dq}$	$f(\alpha) = q \cdot \alpha + \tau(q)$
0	Fractal dimension $D_F = D_0 = \dim_H M$	$\dim_H M$	α_0	$f_{max} = \dim_H M$
1	Information dimension $D_I = D_1$	0	$\alpha_1 = -\frac{I(r)}{\ln r}$	$f_I = \alpha_1 = I$ *
2	Correlation dimension $D_C = D_2$	$-D_C$	$-D_C - \left(\frac{dD_q}{dq}\right)_{q=2}$	$2\alpha_2 - D_C$
$+\infty$	Uper limit D_∞	$\sim -q \cdot \alpha_{min}$	$\to \alpha_{min} = -\frac{\ln P_+}{\ln r}$ **	$f \to 0$
$-\infty$	Lower limit $D_{-\infty}$	$\sim -q \cdot \alpha_{max}$	$\to \alpha_{max} = -\frac{\ln P_-}{\ln r}$ **	$f \to 0$

*The measure M has entropy $I = -\lim I(r)/\lim r = f_I$, which is the fractal dimension of the set of concentration for the measure M (here $I(r)$ is the entropy of partition of measure M over boxes of size (r)).
**Here P_+ and P_- are the largest and the smallest probabilities in boxes of size r.

and the generalized dimension D_q equals the metric dimension $D_B = D_H$ for all q. Specifically, for an homogeneous object all dimensions D_q are equal to its topological dimension, *i.e.*,

$$D_q \equiv D_H = d_T;$$

and for self-similar fractals the definition in Eq. (4) gives

$$d_T \leq D_q \equiv D_0 = D_B \leq d$$

for all values of q. A related dimension function $D(\gamma)$ was introduced by Badii and Politi [15]. It was shown that Rényi dimensions are related to the sequence of mass exponents $\tau(q)$, to the dimension $F(\alpha)$ curve, and to the dimension function $D(\gamma)$ in general ways that is useful in applications (see Refs. [4,12]). The Rényi dimension D_q, special values and limits of sequence of mass exponents $\tau(q)$, and $f(\alpha)$ curve for a multifractal supported by a set with fractal dimension D_F, for different moments of order q are given in Table I. These concepts form the basis of mathematical formalism of the theory of multifractals.

3. THEORY OF ELASTICITY OF FRACTALS

In the majority of investigations of elasticity of fractals the elastic behavior of self-similar

TABLE II. Comparison of Poisson's ratio ν, calculated using analytical relationship, with the computer values, based on two-dimentional elastic random networks and with experimental values of aerogel SiO_2.

Properties	Two-dimentional random network of size $L \times L$ near the percolation threshold (ξ_C – correlation length)		Aerogel SiO_2
	$L/\xi_C \to \infty$	$L/\xi_C \to 0$	
Connectedness of random network	Elasticity of network is determined by dangling bonds	Bonds, determining elasticity of network are multiduplicated	Fractal cluster
Fractal dimension of elastic backbone D_F	Dimension of geodesic line $D_\tau = 1.1 \pm 0.02$ [22]	Random walk dimension $D_w = 0.67$ [22]	D_F, measured by small angle neutron scattering and molecular adsorption 2.3 ± 0.1 [44]
$\nu = D_F/(d-1) - 1$	0.1 ± 0.01	-0.33	0.15 ± 0.05
Poisson's ratio (results of numerical simulation and experimental data)	0.08 ± 0.04 [20]	$-1/3$ [20]	0.12 ± 0.08 [44]

structures is studied by computer simulation methods [19–22]. It is usual to consider two limiting cases of isotropic and central forces governing elasticity. It is shown in Ref. [22] that problems of elasticity of statistically self-similar networks considered in these limits belong to two different classes of universality. Moreover, numerical experiments [19,20] show that elastic random percolation networks on a plane ($d = 2$), whose dimensionality is $L < 0.2\xi_C$ (ξ_C is the correlation radius), have characteristic negative values of the Poisson ratio, ν, whereas for networks of size $L > 0.2\xi_C$ this ratio is positive. The limiting values of $\nu(L/\xi_C)$ are universal in the limits $L/\xi_C \to 0$ and $L/\xi_C \to \infty$ (see Table II), *i.e.*, they are independent of the ratio of the local elasticity parameters used in numerical calculations of the elastic properties of random percolation networks.

The second approach to simulation of elasticity of multifractals is the classical concept of the entropic nature of the elastic behavior of multifractal polymer networks [41,42].

Finally, there have been frequent references in the literature of the analogy between elastic behavior of multifractal polymer networks and random springs, whose size exceed certain characteristic scale of length L_F [43,44]. This analogy has been used by Webman [15] to propose a heuristic picture of elastic deformation of fractals, according to which application of force F causes deformation of fractal only over distances exceeding the characteristic scaling length L_F, which depends on the force F. Therefore, an external force F acting on an elastically deformed fractal creates a new characteristic scaling length [15]. Elsewhere [29–31] we used this picture to develop thermodynamical theory

of elasticity of multifractals and explain certain features of the elastic deformation of self-similar structures. Below we consider this theory.

It is well known that classical theory of elasticity of solid is based on two experimentally established facts [43]:

1) Hooke's law, according to which relative strain $\epsilon_{\|}$ is proportional to the acting stress $\sigma_{\|}$:

$$\epsilon_{\|} = \sigma_{\|}/E.$$

2) Poisson's law, postulating the effect of transverse strains

$$\epsilon_{\perp}^{i} = -\nu\epsilon_{\|} \quad (\sigma_{\|}^{i} = 0).$$

In developing the theory of elasticity of fractals we have also started with two postulates:

1) When an elastically isotropic multifractal is deformed by an external force F, a unique new characteristic scaling length L_F appears and the force obeys the following relation:

$$F = \frac{\partial U}{\partial L_F} - T\frac{\partial S}{L_F}, \tag{5}$$

where U is the internal energy and S is the entropy.

2) In the reversible deformation case, an elastically isotropic multifractal retains self-similarity, *i.e.*, the law describing the change in density ρ as a result of elastic deformation is similar to the law describing the change in ρ due to geometric change in the dimensionalities of a multifractal structure:

$$\frac{\rho(F_1)}{\rho(F_2)} = \left(\frac{\partial L_{F_1}}{\partial L_{F_2}}\right)^{-\alpha} = \lambda_F^{-\alpha}, \quad \alpha = d - D_{\mathrm{F}}, \tag{6}$$

where D_{F} is the metric dimension of structure.

In the case of uniaxial deformation it follows from Eq. (6) that the change in the dimensionality of a fractal along the direction of the force F, which is $\lambda_x = L_x/l_x$, is accompanied by a change in its transverse dimensionality in orthogonal directions of the surrounding d-space $\lambda_i = L_i/l_i$, where $i = x, y, \ldots$. These are related to $\lambda_x = \lambda_F$ by the following relations:

$$\lambda_i = \lambda_{\perp} = \lambda_x^{-\nu_F} = \lambda_F^{-\nu_F}, \quad i = 2, 3, \ldots, d, \tag{7}$$

where $\alpha = 1 - (d-1)\nu_F$, so that

$$\nu_F = -\frac{\ln \lambda_{\perp}}{\ln \lambda_F} = \frac{D_{\mathrm{F}}}{d-1} - 1. \tag{8}$$

Equations (7) and (8) are satisfied in the general case of n–dimensional deformation of a multifractal structure in a d–space. For example, in the case of biaxial deformation of a fractal ($1 \leq D_{\mathrm{F}} \leq 3$) in three-dimensional space ($d = 3$) it follows from Eq. (6) that

$$\lambda_z = \lambda_F^{-\nu_F}, \quad \lambda_F = (\lambda_x \lambda_y)^{1/(1-\nu_F)}, \tag{9}$$

where ν_F and D_{F} are still related by Eq. (8), which is satisfied also in the case of triaxial deformation of a fractal in three-dimensional space when

$$\lambda_F = (\lambda_1 \lambda_2 \lambda_3)^{1/\alpha}, \quad \alpha = d - D_{\mathrm{F}} = 1 - 2\nu_F, \tag{10}$$

and the law (6) is valid; here, $\rho \propto \lambda_F^{-\alpha}$. Note, that the coefficient of transverse deformations of multifractals $\nu_F = \ln \lambda_\perp / \ln \lambda_F$ is equal to the Poisson's ratio $\nu = -[(\lambda_i^2 - 1)/(\lambda_j^2 - 1)]^{1/2}$ only in the limit of infinitely small strains $\epsilon = |\lambda^2 - 1|^{1/2} \ll 1$.

Therefore, it follows that when the conditions (5) and (6) are valid, the transverse deformation factor of an elastically isotropic multifractal is defined uniquely by its metric dimension of Eq. (8). It is clear from Table II, that the value of ν calculated from Eq. (8) agree well with the results obtained in Refs. (19] and [20] by numerical modeling of the elasticity by percolation networks on a plane ($d = 2$) and with the experimental value of ν for aerogel SiO_2 reported in Ref. [44] and obtained by study of propagation of longitudinal and transverse elastic waves.

It follows from Eq. (9) that the condition of incompressibility ($\alpha \equiv 0$) is satisfied for multifractals, whose metric dimension is the same as the dimension of the surrounding euclidean space ($D_{\mathrm{F}} = d$), from which it follows that

$$\nu_{\max} = \frac{1}{d-1}. \tag{11}$$

We can easily see that if $d = 2$ and 3, the equality (12) is identical to the conditions of incompressibility of two- ($\nu_{\max} = 1$) and three-dimensional ($\nu_{\max} = 0.5$) elastically isotropic solids derived in the classical theory of elasticity (see Ref. [43]).

Using linear relationship between entropy and information [45], and the definition of the information dimension D_{I} of multifractals, we can represent the change in the entropy as a result of elastic deformation of multifractal in d-dimensional space with law (6) valid, by the following expression:

$$\Delta S = -B \sum_{i=1}^{d} \lambda_i^{D_{\mathrm{I}}} - d, \tag{12}$$

where the parameter B is independent of λ_i and can be determined, for example, using the approach of Ref. [46].

According to the definition of correlation dimension D_{C}, the change in the internal energy $U(\lambda_i)$ of multifractal during the reversible deformation can be written as follows:

$$\Delta U = -C(\lambda_F^{\alpha_{\mathrm{C}}} - 1), \quad \alpha_{\mathrm{C}} = d - D_{\mathrm{C}}, \tag{13}$$

where C is constant. Substituting (8) and (12), (13) into (5) and using thermodynamic equation [30]

$$D_{\mathrm{I}}(d - D_{\mathrm{F}}) = D_{\mathrm{C}}(d - D_{\mathrm{C}})$$

with the condition $F = 0$ for $\lambda_i = \lambda_F = 1$, we have

$$F = C\left\{D_{\mathrm{I}}\lambda_1^{D_{\mathrm{I}}-1} - D_{\mathrm{I}}[D_{\mathrm{F}} - (d-1)]\lambda_1^{-D_{\mathrm{I}}[D_{\mathrm{F}}-(d-1)]-1} - D_{\mathrm{C}}(d - D_{\mathrm{C}})\lambda_1^{d-D_{\mathrm{C}}-1}\right\}. \quad (14)$$

In the case of regular fractals, we have

$$D_{\mathrm{F}} = D_{\mathrm{I}} = D_{\mathrm{C}} = D_{\mathrm{H}}, \quad (14)$$

where D_{H} is Hausdorff-Besicovitch dimension. Then it follows from Eqs. (8) and (14) that

$$F = C\left\{\lambda_1^{D_{\mathrm{F}}-1} - [D_{\mathrm{F}} - (d-1)]\lambda_1^{-\nu D_{\mathrm{F}}-1} - (d - D_{\mathrm{F}})\lambda_1^{\alpha-1}\right\}. \quad (16)$$

The true stress σ_1 is related to F by

$$\sigma_{11} = F\lambda_1^{1-\alpha} = C\left\{\lambda_1^{2D_{\mathrm{F}}-1} - [D_{\mathrm{F}} - (d-1)]\lambda_1^{D_{\mathrm{F}}(1+\nu_F)-d} - (d - D_{\mathrm{F}})\right\}, \quad (17)$$

which allows for the change in the area of the $(d-1)$–dimensional cross section of a deformed fractal in a plane (hyperplane) orthogonal to the direction of force F.

Similarly, using Eqs. (5)–(15), we can obtain the relation $\sigma_{ii}(\lambda_{jj})$ for n–axial deformation of an n–dimensional fractal in the d space. We can easily see that in the limit of infinitesimally small strains $|\epsilon_i| = |\lambda_i^2 - 1|^{1/2} \ll 1$ these relationships can be represented by the classical form of generalized Hooke's law for d–dimensional solid:

$$\epsilon_{ii} = \frac{1}{E}\left(\sigma_{ii} - \nu\sum_{j=1}^{d-1}\sigma_{jj}\right). \quad (18)$$

This allows us to find the relationships between the elastic moduli (*i.e.*, Young's modulus E, shear modulus G, bulk modulus B, longitudinal elastic modulus C_{11}) of fractals, which are derived by analogy to the derivation of corresponding relationships in the theory of the elasticity of solids:

$$G = E\frac{d-1}{2D_{\mathrm{F}}}, \quad B = \frac{E}{d(d - D_{\mathrm{F}})}, \quad (19)$$

$$C_{11} = 2G\frac{1 + (d-2)(d - D_{\mathrm{F}})}{(d-1)(d - D_{\mathrm{F}})}. \quad (20)$$

Substituting Eq. (8) in the system (19), (20) for $d = 2$ and 3 we can see that these expressions are identical to those for two- and three-dimensional elastically isotropic solids (see Ref. [43]).

Notice that, in contrast to the transverse strains in solids, which according to Poisson's law occur when $\sigma_\perp = 0$, the transverse deformation of fractals, which ensure that the law (6) is satisfied, occur under the action of stresses that appear due to the fine structure of a fractal, which can be demonstrated in a clear manner by considering the process of elastic deformation of the Koch curves ($D_T = 1$, $1 < D_F < 2$, $d = 2$).

4. FRACTAL THEORY OF THE RUBBER ELASTICITY OF POLYMERS

Experimental data on the reversible deformation of elastomers are usually interpreted in terms of classical theory of rubber elasticity (see, for example Refs. [41,42]). Classical theory of rubber elasticity was developed in the 40's independently by a number of workers (see Refs. 42]). The main simplification of the classical theory is the assumption that the subchains of a polymer chain (*i.e.* chains between two neighboring crosslinks) can take on with equal probability an arbitrary conformation comparable with the given distance between the ends of the subchain, *i.e.*, between the crosslinks that limit the subchain. The elasticity of polymers chain is in this case of purely entropic nature. It is easy to understand, considering that the network is deformed, that the distance between crosslinks are correspondingly altered thus decreasing the set of possible conformations for the aggregate of subchains. Now it is obvious, that the success of classical theory does not, in itself, imply that the chains of polymer networks obey Gaussian statistics, which lies at the basis of this typical mean-field theory. It is well known [46] that Gaussian statistics is characteristic only for networks prepared by the coalescence of concentrated solution of chains when they are compressed or weakly ($\lambda < 1.2$) extended. Such structures are characterized by information dimension $D_I = 2$ and the general relationship of Eq. (12) reduces to the classical form

$$\Delta S = -B\left(\sum_{i=1}^{d}\lambda_i^2 - d\right),$$

so that if the $\nu = 0.5$ (the second assumption of classical theory of rubber elasticity [41,42]), we have a classical expression for $F_1(\lambda_1)$ in the form

$$F_1 = \frac{E}{3}(\lambda_1 - \lambda_1^{-2}), \tag{21}$$

where E is the Young modulus proportional to the temperature for which the expression in terms of the parameters of polymer structure has been subjected to repeated refinements [42,46]. Relation (21) agrees well enough with experimental data in the compression region ($\lambda < 1$), but at $\lambda > 1$ considerable deviations appear right away [42]. If $1 < \lambda_1 < 2$, the graph of relation (21) usually lies above and for $\lambda_1 > 2$ well below the experimental curve $F(\lambda_1)$, which in the range $\lambda_1 > 4$ is usually described by the asymptote $F_1 \propto \lambda_1^2$ (see Fig. 1).

Traditionally, the refinement of the relation (21) is made by phenomenological modifications of the entropic theory, the main progress in which is summarized in Refs. [42,47].

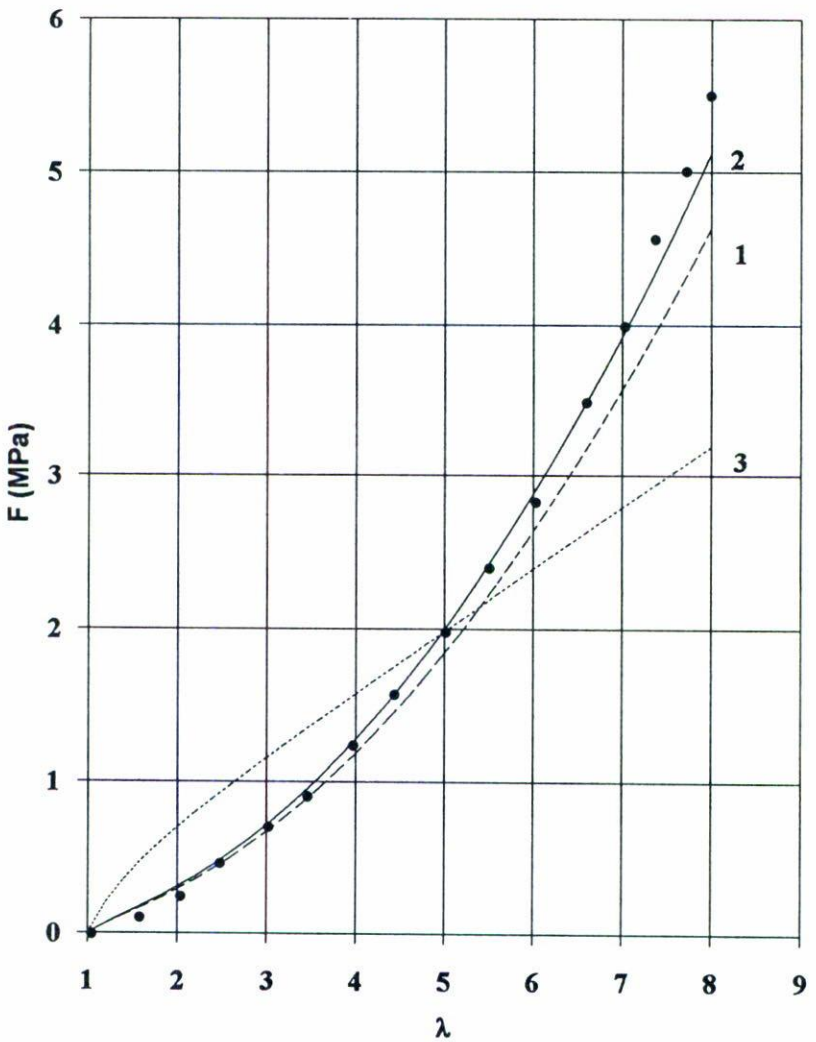

FIGURE 1. Function $F(\lambda_1)$ for uniaxial stress of rubber: 1) Calculation using Eq. (25) for $E = 0.36$ MPa ($\nu = 0.5$); 2) calculation using Eq. (24) for $E = 0.34$ MPa and $\nu = 0.48$; 3) calculation using classical formula (21) for $E = 0.4$ MPa ($\nu \equiv 0.5$). (Points represent experimental data from Ref. [48]).

The required precision in matching the calculations to experiments is attained by the use of additional etching parameters, which are essentially of the fitting type [42]. In this approach of Ref. [48], which is based on the use of empirical relation of the elastic potential on the invariants of strain and temperature, which ensures any desired precision of approximation to the experimental data if sufficient number of the fitting-type matching parameters is selected. Apart from the large number, and not always clear physical meaning of the matching parameters [41,42,47], the main shortcoming of both such phenomenological modifications of the entropic theory and the empirical models of the elastic potential is the need to use different values of the same elastic parameters such as Young modulus to describe the experimental data obtained under different loading

conditions and also to describe the same data, but within the frameworks of different modifications of the entropic theory or of the elastic potential [47]. This is due to the indeterminacy of the absolute values of the elastic parameters and of the relationships between them, which do not satisfy the expressions of Eqs. (19), (20) even in the limit of infinitesimally small strains.

Below we consider problems of rubber elasticity of the elastomers. Elastomers, in particular polymers, are known to have random self-similar multifractal structure [11-15]. It is natural, therefore, to describe rubber elasticity by using the results of the theory of elasticity of multifractals discussed above.

Since in the case of self-similar structures we always have $D_C \leq D_I$, it follows from Eqs. (12) and (13) that these shortcomings of the entropic theory of rubber elasticity of polymers are due to the conflict between the two main assumptions of the classical theory:

1) the Gaussian statistics of polymer chains, which is valid in the case of structures whose information dimension is $D_I = 2$;

2) the incompressibility of elastomers, which according to Eq. (6) is satisfied when $D_F = d = 3 = D_I$.

The former assumption is valid in the case of long polymer chains and networks formed by cross linking of concentrated solution of chains ($D_I = 1$, $D_F = 2$, $md = 3$) (see Refs. [14] and [15]) for which the Poisson ratio is —according to Eq. (8)— zero ($\nu = 0$), which implies constancy of the transverse dimensionalities in the surrounding space when a chain is subjected to uniaxial deformation. This can be easily described by considering elongation of strongly twisted nondeformable filament ($d_T = 1$, $D_H = 2$, $d = 3$). Substituting $D_F = 2$ and $\nu = 0$ into Eq. (16), we have

$$F = E(\lambda_1 - 1), \tag{22}$$

which is identical with the classical result for a long polymer chain [41,42].

Networks which swell in a good solvent are strongly non-Gaussian [46], so that the dimension of the structure of real polymers formed by interpenetrating blobs [15,41,42] is within the range $2 < D_H < 3$, and in general we have

$$1 \leq D_q \leq D_{q-1} \leq D_2 = D_C \leq D_1 = D_I \leq D_0 = D_F \leq 3. \tag{23}$$

Assuming in the first approximation that $D_C = D_I = D_F$ and substituting Eq. (8) into Eq. (16), we obtain the relationship between the nominal stress F and the strain factor λ_1 in the case of uniaxial tension (compression) of an elastomer:

$$F = \frac{E}{1 + 6\nu + 4\nu^3}\left\{\lambda_1^{1+2\nu} - 2\nu\lambda_1^{-1-2\nu(1+\nu)} - (1 - 2\nu)\lambda_1^{-2\nu}\right\}. \tag{24}$$

This expression differs from the classical expression (21) even in the limit of incompressibly deformed material, when Eq. (24) reduces to

$$F = \frac{E}{4.5}(\lambda_1^2 - \lambda_1^{-2.5}), \tag{25}$$

obeying the asymptotic expression $F \propto \lambda_1^2$, when $\lambda_1 \gg 1$, which agrees well with experiments.

It is clear from the graphs in Fig. 1 that the calculation based on Eq. (24) is in good agreement with the experimental data right up to $\lambda_1 = 7$, when cracks appear in the deformed material and the effective value of the transverse strain factor of Eq. (24) becomes larger then 0.5.

5. FRACTAL FRACTURE MECHANICS

A solid body responds to extreme loading by undergoing large deformation and/or fracture. This phenomenon *i.e.*, loss of contact between parts of the body, is a problem that our civilization has faced for as long as there have been manmade structures. The problem actually is worse today than in the previous centuries, because more can go wrong in our complex technological society. At the same time, the nature of processes which determine crack formation and growth in materials is far from being understood. The failure of materials under stress is a complex process involving a broad range of physical, chemical and sometimes biological processes. It was found that fracture surfaces are of fractal character in a wide range of spatial scales [11,23–25]. Moreover, it was shown that the fractal dimension can be regarded as a measure of fracture toughness in solids [25]. Now, quantitative analysis of fracture surfaces has become an important feature in the process of obtaining better knowledge of the microstructural processes involved during the deformation and rupture of materials. Some simple models of fractal geometry of failure and crack growth were proposed in works of P. Meaking *et al.* [26], H. Herrmann [27], M. López-Sancho *et al.* [37], A. Balankin [50], and other authors.

Most real failure processes of practical importance exhibit rich phenomenology extending over a wide range of length scales from the atomic level to the overall size of the sample or structure. Because most failure processes involve complex interaction between large number of processes, it has in most cases been difficult to develop understanding on fundamental level. Nevertheless, considerable advances have been made towards developing satisfactory understanding of mechanical failure processes on phenomenological and statistical basis. Even in ideal homogeneous materials, a complex nonlocal stress-strain field develops as the material begins to fail and an understanding of the evolution of the stress-strain field is an important ingredient in developing better understanding of material failure [4]. The self-similar character of evolution of defects at various scales allows to use new methods of the analysis in terms of fractal theory.

The causes and laws of formation of the fractal geometry of fracture of initially homogeneous solids have been studied by A. Balankin [31–35]. He proposed a quantum-statistical approach to solid mechanics in which fractal nature of fracture follows from fundamental principles as a result of the collective excitations of the atoms in crystal lattice [4]. It was shown that from the topological standpoint, the difference between ductile and brittle fracture is that in brittle fracture, the crack front (crack has a smooth surface), whose metric dimension lies in the range from 1 to 2, is fractal (at stresses $\sigma > \sigma_C$ classical Griffith's crack with a smooth front and surface propagates), and in ductile fracture the surface of cracks whose metric dimension lies in the range from 2 to 3 is fractal. This

difference is attributable to the difference in the kinetics of self-organization of dissipative structures coming from the inherent presence in materials of defects of different dimension such as point defects, dislocations, disclinations, etc. Some conclusions concerning interpretations of solution to the problems of linear theory of near crack edge were obtained on the basis of fractal concept in the works of A. Balankin *et al.* (see Ref. [4]). It was shown that the phenomenon of fractal topology of fracture of solids is a consequence of Poisson's effect of transverse strains. Some new effects in the phenomena of fracture have been predicted by using methods of fractal mathematics.

Now it is obvious that fractal concepts and multifractal analysis are very useful in understanding nature of the fracture phenomena. However, since, obviously, it is still too early to speak about the structure of fractal fracture mechanics, it seems preferable to develop partial theories, describing the behavior of certain classes of phenomena by using fractal concepts. Below we consider the foundations of the fractal approach to fracture mechanics. The most important experimental and theoretical results of fracture analysis which were obtained by using fractal mathematics are presented.

5.1 Fractal dynamics of fracture of model elastic lattices

Various fracture models have been suggested and in these the edge, the surface, or the distribution of cracks were found to be self-similar [25-29,35–39]. A fundamental relation between fractal geometry an the self-similarity of the fracture processes was pointed out in Refs. [31,34].

Fracture of planar lattices was consider in works [15,19,20,50]. Their dynamics has been described by equations of elasticity of continuous media

$$(\lambda+\mu)\partial_i\left(\sum_j^d \partial_j\mu_j\right)+\mu\left(\sum_j^d \partial_j^2\right)\mu_i=0, \tag{26}$$

where μ_i are the components of the displacement field and ∂_i is the partial derivative with respect to the i–th component of the position vector r. The description of the crack propagation after breaking the first bond yields the rule according to which the probability of breaking of the bond between adjacent sites i and j is proportional to the m–th power of the stress acting on it,

$$P_{ij}\propto\sigma_{ij}^m. \tag{27}$$

It was established in Refs. [15,19,20,50] that the fractal dimension of self-similar configurations of cracks is independent of the nature of loading which is modeled by specifying the relevant boundary conditions, and is governed by the values of Poisson's ratio (see Fig. 2a) and probability exponent m (see Fig. 2b and Table III).

We can see from the graph in Fig. 2a that the dependence of the fractal dimension of cracks $D_{\mathrm{F}}(\nu)$ on Poisson's ratio ν is linear for $m=1$ and it is governed by the dimension of the field of inhomogeneous strains $D_{\mathrm{F}}=(d-1)(1+\nu)=1+\nu$ (see Refs. [4,31]). We can easily see that the last relationship is identical with Eq. (8), describing the relationship

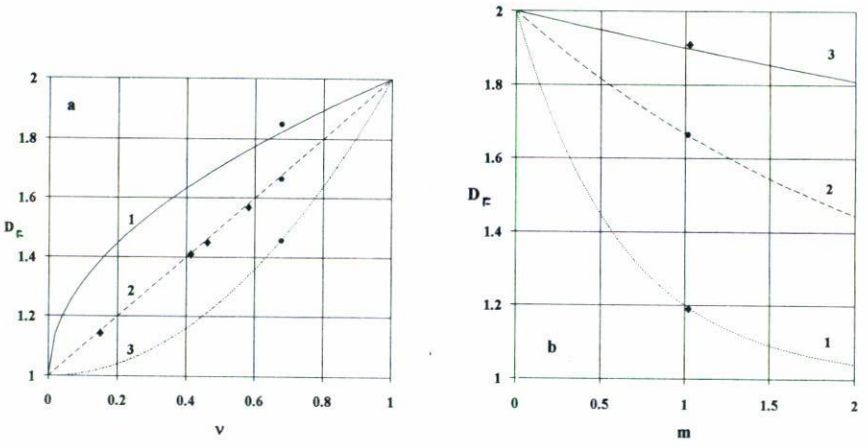

FIGURE 2. Fractal dimension of crack configurations as a function of parameters of the stochastic model (26), (27) for fracture of elastic lattice: a) $D_F(\nu)$ calculated assuming that $m = 0.5$ (curve 1), $m = 1$ (curve 2), $m = 2$ (curve 3); b) $D_F(m)$ calculated assuming that $\nu = 0.2$ (curve 1), $\nu = 2/3$ (curve 2), and $\nu = 0.9$ (curve 3). (Continuous curves are calculated using first Eq. (31), and the points are the results of computer simulations using Eqs. (26), (27) from Refs. [23,24] (♦) and [32] (•).)

between metric dimension and coefficient of transverse deformations of the multifractals. This result is a reflection of the feasibility or representing the equations of the linear theory of elasticity at small values of ϵ_{ij} in the form of Eq. (5) and (6), where the role of L_F is played by the characteristic inhomogeneity scale of the strain field.

In view of shear rigidity of solids, we can expect heterogeneous fluctuations of density (both spontaneous —quantum and thermal— and those induced by an external agent, by a change in boundary conditions, and by bond breaking, as in the model described by Eqs. (26) and (27)) to be always accompanied by the appearance of shear stresses; since the minimum scale of stable inhomogeneous fluctuations of density considerably exceeds the interatomic spacing, the spatial distribution of heterogeneous fluctuations of density and shear deformations in a deformed solid is scale-invariant. Therefore, the distribution and the configurations of cracks in adiabatic fracture should have self-similar structure. This is confirmed by the results of numerous investigations [19–20,50] and is illustrated by the data shown in Fig. 2.

It should be noted that modeling of critical fluctuations in the case of a one-dimensional harmonic or anharmonic crystal [51], which is characterized by $\nu = 0$ because there is no effective transverse strain, yields a smooth ($D_H = d - 1 = 1$) fracture surface, whereas the model of noninteracting anharmonic oscillators placed in a thermostat [52] corresponds to the opposite limit $\nu = \nu_M = 1/(d-1)$, which ensures constancy of the volume of a deformed medium, leading to a homogeneity ($D_H \propto d$) of the distribution of microcracks over the volume of the sample.

TABLE III. Comparison of fractal dimensions of cracks calculated using analytical equation $d_f = 1 + \nu^m$ with those computer based on two-dimentional network ($d = 2$) with $\lambda = \mu$ ($\nu = 2/3$).

The value of m–parameter of bond-breaking probability exponent $p_i \sim \epsilon_i^m$	Typical crack (broken bonds) generated by means of the stochastic fracture model	Effective fractal dimensions obtained from the crack growth models		
		Results of numerical simulation for different boundary conditions ($\nu = 2/3$) [23]		Analytical value $1 + \nu^m$
		Dilatational strain ϵ_1	Shear strain ϵ_S	
$\ll 1$	Dispersion fracture of elastic lattice	2.00	2.00	2.00
0.5	Dispersion fracture of elastic lattice	1.9 ± 0.1	1.9 ± 0.01	1.82
1.0	Multiple fracture of elastic lattice	1.66 ± 0.05	1.65 ± 0.05	1.67
2.0	Growth of fractal crack	1.45 ± 0.05	1.40 ± 0.05	1.44
∞	Propagation of the smooth crack	1.00	1.00	1.00

In the dilational model of fracture [53] the fluctuation volume l_d^3, which occurs in Zhurkov's expression for the lifetime-under-load τ, is governed by the critical dimension of dilation l_d. Obviously, l_d is the natural lower limit L_0 of the self-similarity of fracture surface. Comparison of an estimate of l_d [53] with the values of L_0 obtained in fractographic investigations [23,49] shows that, with satisfactory degree of precision, we can assume that $L_0 = l_d$.

In the course of fracture the effective value of the transverse strain factor ν_{eff} changes because of the accumulation of damage and because of relaxation of stresses on rough crack faces and during plastic deformation. This is the reason for the formulation of multifractal self-similar crack configuration and their surfaces, and for the distribution of their sizes. The fractal dimension of the main crack (depending on the mechanism of fracture, we can speak either of the size of the surface or of the configuration) is governed by the effective value ν_{eff} and in the case of fracture of three-dimensional solids it is given

by

$$D_F = 2(l + \nu_{eff}), \tag{28}$$

where ν_{eff} is a functional of the fracture process [54].

Turning back to the stochastic model of fracture described by Eqs. (26) and (27), we shall consider the process of formation of cracks in the presence of uniaxial strain ($\epsilon_{\|} = \text{const}$, $\epsilon_{\perp} = 0$) in a model of elastic lattice under the action of a stress $\sigma_{\|}$. We shall assume that propagation of crack can be described as a sequence of mappings of the state of lattice in the phase space [4,30]. Since in the stochastic model of fracture the cracks are known to have an irregular rough surface (configuration), it follows that the mapping, describing propagation of crack, is of the compressive type. This happens due to the dissipation of energy of the elastic strains on a rough surface of the crack. This energy dissipation by relaxation of stresses at broken bonds has been established in the model described by Eqs. (26) and (27) by numerical modeling [15,19,20]. Theoretical relaxation of stresses on a rough (although not self-similar) crack surface has been discussed in detail in Ref. [55].

It was shown (see Ref. [43]) that, in the course of crack propagation under self-similar conditions, the fraction of energy of the elastic strains stored by a solid and lost in relaxation of stresses on self-similar crack surface, is

$$\eta = l - \kappa^{-\beta}, \quad \beta = D_F - (d - l), \tag{29}$$

i.e.,

$$D_F = (d - 1) - \frac{\ln(l - \eta)}{\ln \kappa}, \tag{30}$$

where $\kappa = L_{i+1}/L_i$ is the self-similarity parameter representing the hierarchy of the spatial scales L_i of the structural levels of the fracture process [4–6].

In the case of uniaxial deformation of an elastic lattice we can expect transverse stresses $\sigma_{\perp}^{i} = \nu\sigma_{\|}$, where $i = 1, 2, \ldots, (d - 1)$. It follows from the distribution (27) that, when a crack propagates, breaking of bonds under the action of stresses $\langle\sigma_{ij}\rangle_{\perp}$ occurs on the average $(d - l)\nu^m$ times less than under the action of stresses $\langle\sigma_{ij}\rangle_{\|}$. Here, the symbol $\langle\ldots\rangle$ denotes averaging over the states of the lattice. Averaging is essential because of the redistribution of the stresses at the remaining unbroken bonds taken place in the course of crack propagation. Using the expression for the fractal dimension of a phase path in terms of the characteristic numbers Λ^+ and Λ^- of compressive mapping between the states of the lattice in the "coordinate + energy" space, and bearing in mind that

$$\Lambda^+\Lambda^- \propto (1 - \epsilon_{\|})(1 + \nu^m\epsilon_{\|})^{d-1} \propto (1 - \epsilon_{\|})^{\alpha},$$

where $\alpha = 1 - (d - 1)\nu^m$, we obtain the following expression for the fractal dimension of self-similar configuration ($d = 2$) or for the surface ($d = 3$) of crack formed in accordance with the model described by Eqs. (26) and (27):

$$D_F = 1 + \nu^m, \quad D_F = 2(1 + \nu^m). \tag{31}$$

These results are valid for $d = 2$ and 3, respectively. It is clear from the data presented in Fig. 2 and in Table III that the calculation carried out on the basis of (31) is in a good agreement with the results obtained by numerical modeling of the fracture dynamics.

The propagation of fractal Griffith crack was considered in Refs. [31,54]. It was shown that near the tip of fractal Griffith crack, stresses are described by the following asymptotic expression:

$$\sigma \propto R^{-a}, \quad a = \frac{d - D_{\mathrm{F}}}{2}, \tag{32}$$

where R is the distance from the crack tip and D_{F} is fractal dimension of a self-similar crack surface of Eq. (32) or the dimension of the fractal line at the front of a smooth crack equal to $D_{\mathrm{F}} = (d-1)(1+\nu_{\mathrm{eff}}) - 1$.

We have considered self-similar Griffith crack in the model described by Eqs. (26) and (27) in order to determine ν_{eff} and D_{F} of Eq. (28). In the $d = 2$ case the number of bonds in a crack of size F obeys $M \propto R^2$, whereas the number of bonds in a crack of size R is proportional, according to Eq. (27), to $K = MP(R)$, where, bearing in mind Eq. (27), we have $P \propto \sigma^m \propto R^{-ma}$, *i.e.*

$$K \propto R^{2-ma}. \tag{33}$$

On the other hand, by definition we have $K \sim R^{D_{\mathrm{F}}}$, so that comparison of Eqs. (32), (33), and (31), shows that in the model of Eqs. (26), (27) fractal Griffith crack propagates if

$$m = 2, \quad i.e., \quad \nu_{\mathrm{eff}} = \nu^2. \tag{34}$$

This allows us to express the dependence of the stress intensity factor K_I on R obtained in Eq. (32) for $d = 3$ in the form

$$K_I \propto \sigma R^{(1-2\nu^2)/2}, \quad K_I \propto \sigma R^{(1-\nu^2)/2}. \tag{35}$$

The first of the relationships in Eqs. (35) corresponds to the propagation of crack with self-similarity and the second corresponds to the propagation of smooth crack, the front of which is fractal curve. Notice, that the relations (35) agree well with the results of experimental investigations reported in Ref. [25].

5.2 Fracture of materials with multifractal microstructure

The failure of materials and structures under applied load is a subject of both practical importance and conceptual difficulty. Microscopic failure play fundamental role in many systems of industrial importance ranging from aircraft structures and pressurized nuclear reactors to cracks in underground oil reservoirs and in ceramics and fiber composites. Analysis of fracture of strongly heterogeneous materials (aerogels, composite, quasialloys, amorphous and porous materials, rocks, geophysical media, etc.) is usually complicated by many factors: heterogeneity and anisotropy of mechanical properties, redistribution

of stresses linked to the generation and growth of macroscopic discontinuities, etc. All these factors have different effect on fracture processes and correct description of failure transition from microscopic to macroscopic level is possible only on the basis of adequate procedure, also taking into the account the stochastic nature of mechanical properties of a real material. Usual treatments based on continuum elasticity theory do not provide simple tools for discussing essential nonlinearities of this problem.

Many physical systems are being manufacture which have fractal or multifractal structure in the wide range of space scales. Three examples of physical systems for which this discussion is of interest are: 1) random composite material, 2) colloidal aggregates such as gold and silica aggregates, and 3) geophysical media. These structures are made up to units that are considerably larger than atomic size and remain stable in the configurations in which they are prepared. Other materials to which the theory discussed below could be applied are microscopically disordered network materials such as gels, polymers, and glasses. The majority of them can be regarded as an ensemble of fractals of different dimensions characterized by different weights.

When a solid body fractures, crack propagation over distance ΔL is ensured by the release of elastic energy $U(\Delta L)$, which is spent on the formation of fracture surface. When smooth ($d_T = 2$) crack propagates in three-dimensional ($d_T = 3$) linear-elastic medium, the asymptotic forms of the distribution of stresses $\sigma_{ij}(r)$, and displacements of crack edges $u_i(r)$, as a functions of distance r to its moving tip are given by [10]

$$\sigma_{ij} = \frac{K_I}{\sqrt{r}}\varphi(\theta), \quad u_i = \frac{K_I\sqrt{4}}{E} f(\theta), \tag{36}$$

where K_I is the stress intensity factor. The specific energy, therefore, is

$$U_0 = \frac{U(\Delta L)}{\Delta L} = \text{const.}$$

A fundamentally different situation arises during fracture of materials with multifractal microstructure. We describe crack propagation in multifractal structure within the framework of nonlinear fracture mechanics, utilizing J and Γ invariant integrals (see Ref. [10]). After almost literal repetition of the operations of Ref. [10] we obtain the asymptotic forms

$$\sigma_{ij} \sim K_F r^{-\alpha}, \quad u_i \sim \frac{K_F}{E} r^{\beta}, \tag{37}$$

where

$$\alpha = n\frac{d_\mathrm{F} - D_\mathrm{F}}{n+1}, \quad \beta = \frac{1 + n(D_\mathrm{F} + 1 - d_\mathrm{F})}{n+1}, \tag{38}$$

and n is the exponent of unit strains $\epsilon_{ij} \sim I^{1/n}$ as a function of stress

$$I = \left[(\sigma_{ij} - 1/2\sigma_{kk}\delta_{ij}(\sigma_{ij} - 1/3\sigma_{kk}\delta_{ij})\right]^{1/2}. \tag{39}$$

It is easy to see that in the case $n = 1$, in the limit

$$d_F = d = 3, \qquad D_F = d_T = 2$$

the asymptotic forms (37), (38), and (36) are identical. In the case $d_F = d = 3$, $D_F = d_T = 2$, but for $n \neq 1$, from (37) and (38), we obtain the well known result of Ref. [10] for a smooth crack in nonlinear elastic medium. When $n = 1$ and $d_F = 3$, but $D_F > 2$, Eqs. (37) and (38) become asymptotic expressions obtained in Ref. [56] for fractal crack in linear elastic solid. We note that (37) and (38) suggest that brittle (quasibrittle) fracture of materials with multifractal structure is possible only if the metric dimension of fracture surface D_F is less or equal to the metric dimension of multifractal structure d_F.

Using probability fracture criteria (27), and literally repeating the operations described above, we obtain

$$D_F = 2(1 + \nu^m), \tag{40}$$

which coincides with the expression (28) for the fractal dimension of the surface of a crack in solid. In particular, $m = 2$, corresponds to Griffith fractal crack, within the limits of self-similarity $L_0 < \Delta L < L_M$. Therefore, the specific density of elastic energy released when the crack length increases abruptly by ΔL increases:

$$U_F \sim (\Delta L)^{2\nu(1-\nu)}. \tag{41}$$

The comparison between the prediction by using Eq. (41) and the experimental data, from Ref. [57], is given in Fig. 3. We can see that theoretical calculations are in a good agreement with experimental evidence. The increase in $U_F(\Delta L)$ must be taken into account when predicting the functionality of actual structures made of materials with multifractal microstructure and when simulating geodynamic phenomena.

5.3 Fractal kinetics of ductile and brittle fracture of solids

Fracture of solids belong to the class of processes in which complex behavior at the microscopic level is behind the macroscopic effects. Behavior of deformed solid subjected to mechanical action in governed by the formation and evolution of dissipative structures, which provide optimal conditions for dissipation of energy flowing into the body from the outside. Many factors play relevant role: grain, boundaries, microcracks, particles and impurities, temperature, etc., leading to rich phenomenology ranging from cleavage to ductile fracture.

Traditionally, the analysis of processes that control failure of solids on microlevel has been confined to the consideration of models that take into account only paired interatomic bonds (see Ref. [10]). At the same time, strong correlation of the relative position of atoms at distances L_0 that significantly exceed the interatomic distances r_{ij}, *i.e.*, the correlation that ensures shear stability of solids, is characteristic of states of condensed matter. Therefore, rheological behavior is determined by the dynamics of collective excitations governed by an external factor [4], and a failure is a collective, essentially nonequilibrium

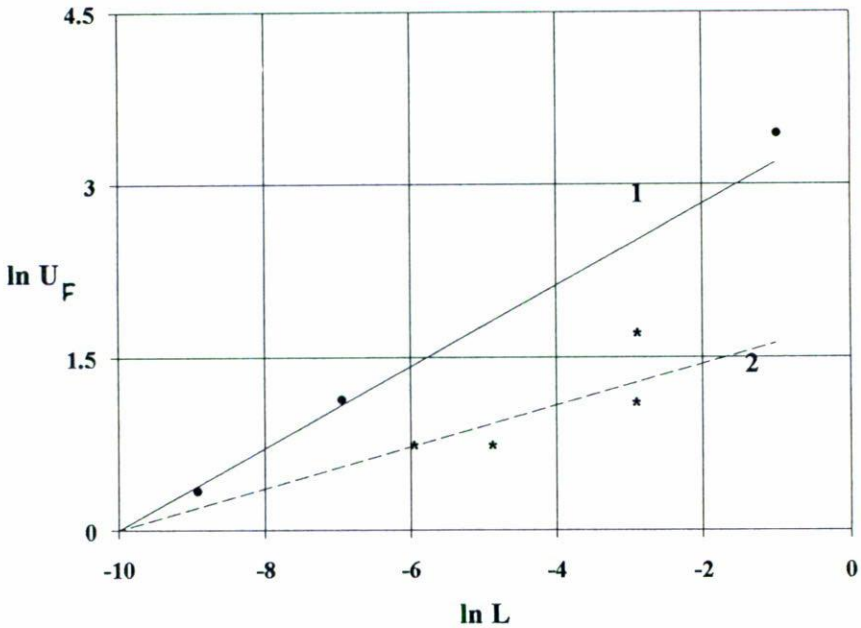

FIGURE 3. Dependence of surface fractal energy U_F on the characteristic scale of fracture. Solid lines are the results of calculation using Eq. (41) for $m = 2$ and $\nu = 0.23$ (curve 1) and $\nu = 0.1$ (curve 2). Points represent experimental data from Ref. [57].

process, whose kinetics is governed by self-organization of dissipative structures that ensure optimal (for specified loading) level of dissipation of energy of the external action [4]. Consistent allowance for collective effects in fracture kinetics, including effects at the atomic level, is possible within the framework of quantum-statistical approach which is being developed [7], for synergetics of deformed solid.

As a consequence of shear rigidity of condensed media, significant difference appears in the characteristic relaxation times for the energy τ_ϵ, and impulse $\tau_p \ll \tau_\epsilon$ of atoms and structural elements of the deformed medium [4,7]. Therefore, in the inelastic deformed solid, mechanical energy accumulates in self-localized, highly nonequilibrium regions that form an open subsystem (which exchanges energy and matter with regions of the body that are in a quasiequilibrium state) in which the energy of elastic deformations is dissipated. It is precisely this effect that gives rise to the localization of plastic strains and fracture regions in solids. It has been shown elsewhere [31,50] that as a result of the effect of transverse strains, self-organizing dissipative structures during irreversible deformation of the media with shear rigidity have scale-invariant multifractal structure. The fractal dimension of dissipative structure, which determines the rate of energy dissipation upon irreversible deformation of a solid, corresponds to the effective value of the transverse strain coefficient ν_{eff}, which is the functional of the process of irreversible deformation. Statistical self-similarity of configurations of cracks and the multifractal nature of the fracture surfaces of solids, which are a sort of counterparts to dissipative structures that

are superposed on the original structure of the material, serve as graphic reflection of the scale invariance of dissipative structures [35]. It has been found that well-defined values of the fractal dimension of crack profiles and fracture surface are characteristic for different samples of given material that has been put through analogous thermomechanical treatment [4,34].

In the process of irreversible deformation of metals, the energy accumulated in self-localized, highly nonequilibrium regions in the form of potential energy of elastic strains dissipates through stress relaxation during plastic deformation. It is consumed on the formation of micro-, meso-, and macrodefects, and consequently is being released on the rough surface of the cracks formed. If the critical density of the potential energy of dilatation (corresponding to the bifurcation point of the original structure of the solid being deformed [49]) $W_{cv} = \sigma_c^2/2E$ or form changing $W_{cd} = \tau_c/2G$ (where E and G are Young modulus and shear modulus τ_c and σ_c are the critical shear stress and the critical stress of microseparation), builds up, self-organization of the dissipative structure that ensures subcritical growth of a main crack occurs. Fracture during subcritical growth of cracks is related, in general, to the cooperation between two competing mechanisms governed by different types of dissipative structures that are responsible for crack formation through microshear or microseparation [35,49]. Every event of crack advancement is associated with the formation of critical nucleus by the mechanism of microshear (if $W_d \gg W_{cd}$) or microseparation (if $W_v \gg W_{cv}$). The nuclei of cracks formed by microshear are associated with the attainment, in the slip plane, of the critical density of dislocations, and those formed by microseparation are associated with the attainment of critical density of disclinations in an element of the volume that has undergone the extreme plastic deformation. If failure is controlled by microshear associated with a low-energy-intensity pileup of dislocations in the slip plane, local brittle or quasibrittle fracture initiated by translational instability occurs. In this case, shear or cleavage faces appear revealed on the fracture surface. But if microseparation associated with high-energy-intensity pileups of disclinations is the controlling micromechanism of fracture, then ductile fracture initiated by rotational instability occurs. A fractographic feature of local ductile fracture is the presence of ragged microrelief (under cyclic and static loadings) [49].

From topological standpoint, the difference between ductile and brittle fracture is that in brittle fracture crack front, whose metric dimension lies in the range

$$1 < D_{\mathrm{F}} \leq 2, \tag{42}$$

is fractal (at stresses $\sigma \geq \sigma_c$ a classical Griffith crack with a smooth front, whose metric dimension equal to topological dimension of the line $d_T = 1$ propagates), and in ductile fracture the surface of cracks whose dimension is

$$2 < D_{\mathrm{F}} \leq 3 \tag{43}$$

is fractal. This difference is attributable to the difference in the kinetics of self-organization of dissipative structures from defects of different metric dimension.

To determine the peculiarities of the kinetics of clusterization of elementary defects responsible for the topological difference between brittle (quasibrittle) fracture and ductile

(quasiductile) fracture, let us examine the Smolukhovskii equation with a multiplicative kernel $\Phi(i,j) \sim (i,j)^{\omega}$:

$$\frac{dC_k}{dt} = \sum_{j=1}^{k} \Phi(j, k-j) C_j C_{k-j} - C_k \sum_{j}^{k} \Phi(j,k) C_j, \tag{44}$$

where C_j is concentration of clusters of j elementary defects, and the parameter ω is the fraction dimension D_{f} of the dissipative structure:

$$\omega = D_{\mathrm{f}}^{-1} = \frac{2+\theta}{2D_{\mathrm{F}}}, \tag{45}$$

where D_{F} is metric dimension of elementary defects, and θ is the exponent of the anomalous diffusion coefficient on the fractal $D \sim r^{-\theta}$ (see Refs. [12–15]).

The metric dimension of dislocation pileups in the slip plane, as of elementary defects during failure through microshear, obviously does not exceed the topological dimension of the plane, *i.e.*, $d_{\mathrm{H}} \leq 2$. Therefore, in this case $\omega \geq 0.5$, since it is always in the case that $\theta \geq 0$ (for percolation clusters, $D_{\mathrm{F}} = 4/3$ and $\omega = 3/4 > 1/2$). It is known [15] that in the case $\omega > 0.5$ the asymptotic solution of Eq. (44) describes the growth of the only "surviving" cluster, which may be identified with the main crack in the case in question. The dependence of growth rate of the main crack on its size R

$$V = \frac{dR}{dt} \sim R^{Z}, \quad Z = 1 + (2\omega - 1)D_{\mathrm{C}} = 1 + (2+\theta)\frac{D_{\mathrm{C}}}{d_H} - D_{\mathrm{C}}, \tag{46}$$

has the form of Paris equation $V \sim K_I^n \sim R^{na}$ (see Refs. [10,49]), where $a = 0.5(2 - D_{\mathrm{F}})$ is the exponent that defines the asymptotic behavior of the stress intensity factor $K_I \sim R^{-a}$ for cracks with fractal front, and D_{C} is the correlation dimension of the cluster (main crack). Setting $Z = na$, we have

$$n = 2\left[\frac{D_{\mathrm{C}}(2+\theta)}{d_{\mathrm{H}}(2-d_F)} - \frac{D_{\mathrm{F}} - 1}{2 - D_{\mathrm{F}}}\right]. \tag{47}$$

It thus follows in the case $D_{\mathrm{F}} = D_{\mathrm{C}} < 2$, that $n > 2$ (this is in agreement with the result of numerous experimental studies); here, if $D_{\mathrm{F}} = d_{\mathrm{H}} = 2[1 - (1+\theta)/(n-2)]$, then,

$$n = 2\left[1 + \frac{1+\theta}{2-d_H}\right], \quad m = n\left[1 + \frac{\theta}{2-d_H}\right], \tag{48}$$

where m is the exponent in the Weibull distribution (see Ref. [49]).

In general case where $D_{\mathrm{F}} = D_{\mathrm{C}}$, the dimension of the crack front formed during brittle and quasibrittle fracture is

$$D_{\mathrm{F}} = \frac{d_{\mathrm{H}}(n-1)}{2 + \theta + (0.5n - 1)d_{\mathrm{H}}}. \tag{49}$$

The metric dimension of fracture surface is $d_{\mathrm{H}}^S = 2$ in brittle and $d_{\mathrm{H}}^S = D_{\mathrm{F}} + 1 > 2$ in quasibrittle fracture.

The dimension of defects responsible for rotational instability (disclinations, etc.) that leads to fracture via the microseparation mechanism is $d_{\mathrm{H}} > 2$. If the stronger condition $d_{\mathrm{H}} > 2 + \theta$, which ensures that $\omega < 0.5$ (*i.e.*, $D_{\mathrm{F}} > 2$), is satisfied here, then according to Eq. (44), there is simultaneous increase in the set of equal clusters (cracks) that form the main crack in accordance with percolation mechanism [8,34]. The dimension of the ductile fracture surface is determined by the fraction of energy dissipated as a result of elastic stress relaxation during plastic deformation and on the rough surface of the cracks (η). Ductile (quasiductile) fracture due to rotational instability of the lattice is accompanied by a cascading of energy transfer from elastic deformations on larger scales L_{i+1} to smaller scales L_i, down to the microscale L_0, where the residual energy of elastic deformations goes to the formation of new discontinuity surface (the process is similar to cascade breakup of vortices during the turbulence of fluid flows). If the fraction of energy of elastic deformations consumed in dissipative processes upon the transition from one structural level to another is independent of L_i (*i.e.*, if $\eta = \mathrm{const}$), then the law of energy conservation

$$(1 - \eta) N_i W_{cv} \, \Delta R_i = N_{i+1} W_{cv} \, \Delta R_{i+1},$$

where N_i is the number of fragments of the crack of the i–th scale L_i and ΔR_i is the increase in the linear dimensionality of a crack, yields the relation

$$d_F^S = 2 - \frac{\ln(1 - \eta)}{\ln \kappa} \geq 2, \quad \kappa = L_{i+1}/L_i. \tag{50}$$

Here $\kappa = \mathrm{const}$ is the self-similarity parameter, which defines the hierarchy of spatial scales of the structural levels of failure. Since $d_F^S \leq 3$, we have $\eta \leq 1 - \kappa^{-1}$. At large η, fracture is impossible.

In quasielastic fracture, the primary mechanism of dissipation is the relaxation of elastic stresses on the rough fractal crack face. Here the η–th part of energy of elastic deformations is dissipated (the remainder goes to the formation of new surfaces during crack formation):

$$\eta = 1 - \kappa^{-\beta}, \quad \beta = d_F^S - (d - 1) = d_F - 1 > 0.$$

During propagation of classical Griffith's crack, we have $\beta = 0$ and $\eta = 0$ in agreement with the classical result.

Mechanisms of fracture of solids imitated in the stochastic model (26), (27), of fracture of elastic lattice for various values of m–parameter and topologically equivalent classes of the kinetic and percolation models of fracture of solids are listed in Table IV.

REFERENCES

1. I.A. Kunin, *Elastic Media with Microstructure*, Springer-Verlag, New York (1982).
2. T.W. Chow, F.K. Ko (eds.), *Textile Structural Composites*, Elsevier, New York (1989).

TABLE IV. Mechanism of fracture of solids imitated for various values of m–parameters stochastical model of fracture of elastic lattice and topologically equivalent classes of percolation and kinetic models of fracture.

The value of m–parameter of bond-breaking probability exponent $p_i \sim \epsilon_i^m$	Imitated mechanism of fracture of solids	Class of percolation models that is topologically equivalent to the stochastic model of elastic lattice fracture	The value of ω–parameter of kinetic model of fracture of solids
$\ll 1$	Ductile fracture	Anisotropic-correlated percolation	$< 3/8$
0.5	Quasiductile fracture (cracks with fractal surfaces)	Anisotropic-correlated percolation	$3/8 < \omega < 1/2$
1.0	Free fracture	Chaotic percolation	$> 1/2$
2.0	Quasibrittle fracture (crack with the fractal front)	Anisotropic percolation	$> 1/2$
∞	Brittle fracture (Griffith's crack)	Anisotropic percolation	—

3. S. Torquato, *Appl. Mech. Rev.* **44** (1991) 37.
4. A.S. Balankin, *Synergetics of Deformed Solids*, Department of Defense USSR, Press, Moscow (1991).
5. M.A. Sadovsky, V.F. Pisarenko, *Seismic Process in Block Media*, Nauka, Moscow (1991).
6. V.E. Panin (editor), *Structural Levels of Plastic Deformation and Fracture of Solids*, Nauka, Novosibirsk (1990).
7. A.S. Balankin, *Sov. Tech. Phys. Lett.* **15** (1989) 878.
8. A.S. Balankin, *Sov. Tech. Phys. Lett.* **16** (1990) 248.
9. E. Coserat., F. Cosserat, *Theories of Corps Deformables*, Hermann, Paris (1909).
10. G. Cherepanov, *Mechanics of Brittle Fracture*, McGraw Hill, New York (1979).
11. B.B. Mandelbrott, *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York (1982).
12. J. Feder, *Fractals*, Plenum Press, New York (1988).
13. K. Falconer, *Fractal Geometry*, John Wiley & Sons, New York, 1990.
14. M. Fleischmann, D.J. Tildesley, R.S. Ball, *Fractals in the Natural Sciences*, Princeton University Press, Princeton (1989).
15. L. Pietronero and E. Tosatti (editors), *Fractals in Physics*, North-Holland, Amsterdam (1986).
16. T. Tel, *Zeit. Naturforsch.* **A43** (1988) 1154.
17. M.J. Feigenbaum, M.N. Jensen, I. Procaccia, *Phys. Rev. Lett.* **57** (1986) 1503.
18. M. Schroder, *Fractals, Chaos, Power Laws*, W.H. Freeman and Company, New York (1991).
19. D.J. Bergman, E. Duering, *Phys. Rev.* **B34** (1988) 8199.
20. E. Duering, D.J. Bergman, *Phys. Rev.* **B37** (1988) 9460.
21. Y. Kantor, *Physica* **D38** (1989) 215.

22. I.M. Sokolov, *Sov. Phys. Usp.* **29** (1986) 924.
23. C.W. Lung, Z.Q. Mu, *Phys. Rev.* **B38** (1988) 11781.
24. C.W. Lung, S.Z. Zhang, *Physica* **D38** (1989) 242.
25. Z.Q. Mu, C.W. Lung, *J. Phys. D: Appl. Phys.* **21** (1988) 848.
26. P. Meakin, G. Li, L.M. Sander, E. Louis, F. Guinea, *J. Phys. A: Math. Gen.* **22** (1989).
27. H.J. Herrman, *Physica* **D38** (1989) 192.
28. E. Louis, F. Guinea, *Physica* **D38** (1989) 235.
29. A.S. Balankin, *Sov. Tech. Phys. Lett.* **17** (1991) 632.
30. A.S. Balankin, A.L. Bugrimov, *Polymer Science* **34** (1992) 889.
31. A.S. Balankin, *Sov. Phys. Solid. State* **34** (1992) 658.
32. A.S. Balankin, *Sov. Tech. Phys. Lett.* **17** (1991) 229.
33. A.S. Balankin, A.L. Bugrimov, *Polymer Science* **34** (1992) 246.
34. A.S. Balankin, *Sov. Phys. Dokh.* **37** (1992) 379.
35. A.S. Balankin, V.S. Ivanova, V.P. Breusov, *Sov. Phys. Dokl.* **37** (1992) 105.
36. E. Louis F. Guinea, *Europhysics Lett.* **3** (1989) 871.
37. M.P. Lopez-Sancho, F. Guinea, E. Louis, *J. Phys. A: Math. Gen.* **21** (1988) L1079.
38. T.A. Written, L.M. Sander, *Phys. Rev. Lett.* **47** (1981) 1400.
39. A.S. Balankin, *Sov. Tech. Phys. Lett.* **17** (1991) 391.
40. D. Vavris, V.B. Ryabov, *Sov. Phys. Tech. Phys.* **34** (1989) 918.
41. P. De Gennes, *Scaling Concepts in Polymer Physics*, Cornel University Press, Ithaca (1979).
42. J.H. Weiner, *Statistical Mechanics of Elasticity*, Wiley, New York (1983).
43. L.D. Landau, E.M. Lifshiz, *Theory of Elasticity*, Pergamon Press, Oxford (1982).
44. M. Gronauer, A. Kadur, J. Fricke, in: *Aerogels, Proc. 1st Intern. Symposium, Wurzburg, Germany, 1985*, Springer Verlag, Berlin (1986).
45. G. Cherbit (ed.), *Fractals, Non-integral Dimensions and applications*, John Wiley & Sons, New York (1990).
46. S.V. Panyukov, *Sov. Phys. JETP* **71** (1990) 372.
47. F.F. Ternovskii, A.R. Khokhlov, *Sov. Phys. JETP* **63** (1986) 728.
48. L. Treloar, *The Physics of Rubber Elasticity*, Clarendon Press, Oxford (1975).
49. V.S. Ivanova, A.A. Shanyavsky, *Quantitative Fractography*, [in Russian], Metallurgiya, Moscow (1988).
50. A.S. Balankin *Sov. Phys. Tech. Lett.* **17** (1991) 531.
51. G. Leibfried, *Gittertheorie der Mechanischen und Thermischen Eigenschaften der Kristalle*, Springer Verlag, Berlin (1955).
52. S.I. Zakharov, *Sov. Phys. JETP* **60** (1984) 346.
53. S.N. Zhurkov, *Sov. Phys. Solid State* **25** (1983) 1797.
54. A.S. Balankin, V.S. Ivanova, *Sov. Tech. Phys. Lett.* **17** (1991) 12.
55. N.A. Lavrov, L.I. Slepyan, *Sov. Phys. Dokl.* **36** (1991) 344.
56. A.B. Mosolov, *Sov. Phys. Tech. Phys.* **36** (1991) 754.
57. L.S. Rozenfield, *Met. Scr.* (1990) 1354.