On self-dual gravity structures on ground ring manifolds in two-dimensional string theory

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Abstract. We generalize the geometric structures generated by Witten's ground ring. It is shown that these generalized structures involve in a natural way some geometric constructions from Self-dual gravity [1,12]. The formal twistor construction on full quantum ground ring manifold is also given.

Resumen. Generalizamos las estructuras geométricas generadas por el "ground ring" de Witten. Se demuestra que nuestras estructuras generalizadas involucran en forma natural algunas construcciones conocidas en gravedad autodual [1,12]. También se da la construcción tuistorial formal sobre la variedad cuántica "ground ring".

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1. Introduction

In recent years, the search for a physical principle to be represented by some mathematical structure of string theory has become essential. In this direction, a most interesting proposal states that this theory should be background independent. This idea has been discussed in a number of papers by Witten and Zwiebach [2,3]. A Lagrangian which is invariant under changes of backgrounds (just the Yang-Mills Lagrangian is invariant under gauge transformations) is proposed in a 2-dimensional toy model. This work deals with an element of "the space of all 2–dimensional field theories" [2].

Quantum gravity in two dimensions is a system of integrable models in which some quantities are computed which are very difficult to find in four dimensions. Liouville theory of 2d gravity and discrete matrix models are examples of these models with central charge $c \leq 1$. These models can be characterized for the case $c = 1$ with usual decoupling between "matter" and ghost as the background,

$$\text{2D quantum gravity} \otimes \text{CFT} \otimes \text{ghost}.$$

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It is very important to study the physical states for these backgrounds (including their symmetries) in order to understand the structure of vacuum in 2–dimensional string theory [4,5]. An infinite number of discrete states has been discovered with spin zero and ghost number equal to one.

In 1991 E. Witten proved that states with spin zero and ghost number zero have in a natural way the mathematical structure of a commutative ring with generators (in the uncompactified SU(2) point) $\mathcal{O}_{u,n}$ with $u = s - 1 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and $n \in \{u, u-1, \ldots, -u\}$ [4] (for details see Appendix). Here we have the generators

$$\mathcal{O}_{0,0} \equiv 1, \tag{1.1}$$

for $s = 1$, $u = 0$, $n = 0$ and

$$\begin{aligned} x \equiv \mathcal{O}_{\frac{1}{2},\frac{1}{2}} &= \left(cb + \tfrac{i}{2}(\partial X - i\partial\phi)\right) \exp[i(X + i\phi)/2], \\ y \equiv \mathcal{O}_{\frac{1}{2},-\frac{1}{2}} &= \left(cb - \tfrac{i}{2}(\partial X + i\partial\phi)\right) \exp[-i(X - i\phi)/2], \end{aligned} \tag{1.2}$$

for $s = \frac{3}{2}$, $u = \frac{1}{2}$, $n = \frac{1}{2}, -\frac{1}{2}$. Under the product given by the operator product expansion, the states generate a structure called chiral ground ring. The states encode all relevant information about the symmetries of the theory, and furthermore they give rise to recursion relations among the tachyon bulk scattering amplitudes perturbing the ground ring [6,7]. The symmetries involved here are of two types:

a) The group SDiff($\mathcal{A}$) of diffeomorphisms preserving the area of a flat two-dimensional ring manifold. The coordinate functions of $\mathcal{A}$ are precisely the chiral ground ring generators, $\mathcal{A} = \{x, y\}$. In other words, if ω is a volume form on $\mathcal{A}$ *i.e.* $\omega \in \Omega(\mathcal{A})$, $\omega = dx \wedge dy$ we have that if $\phi \in$ SDiff($\mathcal{A}$), then $\phi^*\omega = \omega$.

b) In the case of non-chiral ground ring (or the full quantum ground ring) we have a combination of left and right movers represented by the ground rings, $C(\mathcal{A}_L)$ and $C(\mathcal{A}_R)$ respectively, where $\mathcal{A}_L = \{x, y\}$ and $\mathcal{A}_R = \{\bar{x}, \bar{y}\}$. This combination is needed to form the spin $(0,0)$ quantum field operators $\mathcal{V}_{u,n,n'} = \mathcal{O}_{u,n} \cdot \bar{\mathcal{O}}_{u,n'}$ and can be considered to be a tensorial product of rings $C(\mathcal{W}) = C(\mathcal{A}_L) \otimes C(\mathcal{A}_R)$ with generators x, y, $\bar{x}$, $\bar{y}$. Again these generators are the coordinate functions on certain flat four-dimensional manifold $\mathcal{W}$. In a like manner there is a symmetry group $\mathcal{H}$ acting on $\mathcal{W}$; this action consists in identifying the generators of the form $x, y, \bar{x}, \bar{y} \to tx, ty, t^{-1}\bar{x}, t^{-1}\bar{y}$. The group $\mathcal{H}$ is generated by the vector field $S = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \bar{x}\frac{\partial}{\partial \bar{x}} - \bar{y}\frac{\partial}{\partial \bar{y}}$. The quotient space $\mathcal{W}/\mathcal{H} = \mathcal{Q}$ has the topology of a 3–dimensional cone. The symmetry group that generates the states in 2–dimensional string theory is just the group of symplectic diffeomorphisms preserving the volume of this cone. The coordinates of the cone have a natural interpretation in the context of $c = 1$ matrix models. [Note that the group of diffeomorphisms preserving a 3–dimensional volume has been considered by Takasaki [8] in the context of infinite hierarchies, within Plebański's approach to 4d self-dual gravity [9]]. Very recently Lian and Zuckerman [10] have shown that for every string background the BRST cohomology has an algebraic structure known as Gerstenhaber algebra and they suggest that the homotopy Lie algebras of the Gerstenhaber algebra and the algebra constructed by Witten and Zwiebach [5] are closely related.

There exists a nice geometric differential approach to this set of things on the real flat 4-dimensional ring manifold $\mathcal{W}$ as well as on the 3-dimensional algebraic variety $\mathcal{Q}$. In this paper we intend to make a natural generalization of the differential geometric methods in 2d string theory by considering, instead of $\mathcal{W}$, a Ricci half-flat (*i.e.*, $R_{\dot{A}\dot{B}} = 0$) Kähler 4-ring manifold $\mathcal{M}$, with a chart given by the local complex coordinate functions $\{x^\mu\} = \{x, y, \bar{x}, \bar{y}\}$. Then an open set of $\mathcal{M}$ will thus look like the complexification of $\mathcal{W}$, *i.e.* $\mathcal{W}^{\mathcal{C}}$. (Notice that we are using the same letter to designe the general manifold and the $\mathcal{C}^4$-plane). $\mathcal{W}^{\mathcal{C}}$ is the product of two complex 2-ring manifolds: $\mathcal{W}^{\mathcal{C}} = \mathcal{A}_L^{\mathcal{C}} \times \mathcal{A}_R^{\mathcal{C}}$. It is remarkable that $\mathcal{A}_L^{\mathcal{C}}$ and $\mathcal{A}_R^{\mathcal{C}}$ are symplectic manifolds with symplectic 2-forms given by ω and $\tilde{\omega}$, respectively. This fact causes that, automatically, many of the constructions given in Refs. [1,9,11,12] lead to the self-dual gravity structures in two dimensional string theory.

In Sect. 2 we briefly review some basic results of the Witten-Zwiebach theory in terms that can be useful for the search of the self-dual gravity in string theory.

Section 3 is devoted to study self-dual gravity [9] in the context of 2d string theory. We claim that the Witten-Zwiebach theory finds a natural generalization within the $\mathcal{H}$ and $\mathcal{H}$-$\mathcal{H}$ spaces theory [13,14]. It is also demonstrated that our construction appears to be exactly that of Witten-Zwiebach locally.

In Sect. 4, we prove the existence of self-dual gravity structures on the full quantum ground ring manifold in a different way as that given by Ghoshal *et al.* This new method gives a deeper understanding of their results [15]. Using the construction of Ref. [1] we re-derive the first heavenly equation from the natural symplectic structure on the chiral ground ring manifold. Then, we make some comments about the curved twistor construction on the ground ring manifold. We first identify the chiral ground ring manifold with the twistor surface, and then show for the ground ring manifold that any local information (relevant to 2d string theory) can be completely represented on some twistor space. Finally, in Section 5 some further implications of this work are considered.

2. Generalities and Ground Ring Manifolds

2.1. Generalities

It is well known that at the SU(2)-radius the BRST cohomology classes for spin zero and ghost number zero are characterized by the operators $\mathcal{O}_{u,n}$, with $\mathcal{O}_{0,0} \equiv 1$, $x \equiv \mathcal{O}_{\frac{1}{2},\frac{1}{2}}$ and $y \equiv \mathcal{O}_{\frac{1}{2},-\frac{1}{2}}$ [4,5]. These operators generate the chiral ground ring $C(\mathcal{A}_L)$. The operators $\mathcal{O}_{u,n} = x^{u+n} \cdot y^{u-n}$ are precisely the polynomial functions on the x–y plane with area form $\omega = dx \wedge dy$. Thus, the pair $(\mathcal{A}_L, \omega)$ is a two-dimensional symplectic manifold with the symplectic two-form ω. A similar procedure done for the right side of $\mathcal{A}_L$ leads to $(\mathcal{A}_R, \tilde{\omega})$, where $\tilde{\omega} = d\bar{x} \wedge d\bar{y}$.

The spin zero and ghost number one states are

$$Y_{s,n}^{+} = cV_{s,n} \cdot \exp\left[\sqrt{2}\phi \mp s\phi\sqrt{2}\right], \tag{2.1}$$

with $s \geq 0$. These states are precisely the polynomial vector fields on the x–y plane that generate the area-preserving diffeomorphisms. In terms of the symplectic form ω we have

$$Y^{+i}_{u+1,n} = \omega^{ij}\partial_j \mathcal{O}_{u,n} = cW^{+i}_{u,n}. \tag{2.2}$$

The complementary vector fields are $a\mathcal{O}_{u,n}$ with $a\mathcal{O}_{u,n}(0) = \frac{1}{2\pi i}\oint \frac{dz}{z} a(z)\mathcal{O}_{u,n}(0)$, where a is defined in Ref. [5]. Then, the ghost number two states $aY^{+}_{s,n}$ $(s = u+1)$ are the polynomial bivector fields $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on the x–y plane.

If we denote the tangent bundle of the x–y plane by $T\mathcal{A}$, with $\mathcal{A} = \mathcal{A}_L = \mathcal{A}_R$, then the corresponding discrete states are precisely the sections of the μ–th order exterior algebra

$$\pi^\mu : \Lambda^\mu T\mathcal{A} \to \mathcal{A} \tag{2.3}$$

for $\mu = 0, 1, 2$; we have $\Lambda^0 T\mathcal{A} = \mathcal{C}$, $\Lambda^1 T\mathcal{A} = T\mathcal{A}$, etc. Here $\mu = 0$ corresponds to the polynomial functions $\mathcal{O}_{u,n}$; $\mu = 1$ corresponds either to $Y^{+}_{s,n}$ or to $a\mathcal{O}_{u,n}$, and $\mu = 2$ does to $aY^{+}_{s,n}$. Of course, there exists the dual version of this construction where the exterior algebra is precisely the Grassmann-Cartan algebra $\Omega(T^*\mathcal{A})$ (for details see Ref. [5]).

The exterior derivative corresponds to the b_0 operator which commutes with $\hat{\mathcal{O}} = \frac{1}{2\pi i}\oint \frac{dw}{w}\mathcal{O}$; *i.e.*, the operator $[b_0, \mathcal{O}] = 0$ is BRST invariant. Only the zero forms (or bivectors in the dual version) contribute to the central extension $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = \iota$ of the polynomial area-preserving diffeomorphisms of the plane SDiff($\mathcal{A}$).

If $\alpha \in \Omega(T^*\mathcal{A})$, then

$$b_0\alpha = d\alpha + \delta_{j=0}\alpha(0)\iota. \tag{2.4}$$

As we deal with a flat and contractible ring (which is very much like as a topological space), a closed one-form λ on the plane x–y is necessarily exact *i.e.*, $\lambda_{u,n} = d\mathcal{O}_{u,n}$ for some $\mathcal{O}_{u,n}$. This one-form λ corresponds to the vector field

$$Y^{+i}_{u+1,n}\frac{\partial}{\partial x^i} = -(\partial_i \mathcal{O}_{u,n})\omega^{ij}\frac{\partial}{\partial x^j}, \tag{2.5}$$

which is an area-preserving vector field derived from the hamiltonian function $\mathcal{O}_{u,n}$.

2.2. Ground Ring Manifold

Given $\mathcal{A}_L$ and $\mathcal{A}_R$ we define *the full quantum ground ring manifold* $\mathcal{W}$ to be the product

$$\mathcal{W} = \mathcal{A}_L \times \mathcal{A}_R. \tag{2.6}$$

Which can be shown graphically

$$\begin{array}{ccc} & \mathcal{A}_L \times \mathcal{A}_R & \\ \rho \swarrow & & \searrow \tilde{\rho} \\ \mathcal{A}_L & & \mathcal{A}_R \end{array} \tag{2.7}$$

where ρ and $\tilde{\rho}$ are the respective projections.

To construct quantum field operators we have to combine left and right movers. Thus, for the spin $(1,0)$ and spin $(0,1)$ currents we get

$$\begin{aligned} \mathcal{J}_{s,n,n'} &= W^{+}_{s,n} \cdot \bar{\mathcal{O}}_{s-1,n'} \\ \bar{\mathcal{J}}_{s,n,n'} &= \mathcal{O}_{s-1,n} \cdot \bar{W}^{+}_{s,n'}. \end{aligned} \tag{2.8}$$

These operators act on the chiral ground ring operators according to the rule

$$\begin{aligned} \mathcal{J}(\mathcal{O}(P)) &= \frac{1}{2\pi i} \oint_C \mathcal{J}(z) \cdot \mathcal{O}(P), \\ \bar{\mathcal{J}}(\bar{\mathcal{O}}(P)) &= \frac{1}{2\pi i} \oint_C \bar{\mathcal{J}}(z) \cdot \bar{\mathcal{O}}(P), \end{aligned} \tag{2.9}$$

and are precisely derivations of the full quantum ground ring.

It is an easy matter to see that due to the fact that both $(\mathcal{A}_L, \omega)$ and $(\tilde{\mathcal{A}}_R, \tilde{\omega})$ are symplectic manifolds and that $\mathcal{W} = \mathcal{A}_L \times \mathcal{A}_R$, the pair $(\mathcal{W}, \rho^\star\omega - \tilde{\rho}^\star\tilde{\omega})$ is also a symplectic manifold.

3. Self-dual Gravity Structures in 2D String Theory

Consider the effective topological action for all discrete states of the $c = 1$ string at the SU(2) point. This action reads [5]

$$\int_{\mathcal{C}^2} \nu F \wedge F, \tag{3.1}$$

where ν is a scalar and F is defined by the U(1)–gauge field A on $\mathcal{C}^2$ as follows: $F = dA$. The U(1) principal bundle p: $\mathcal{W} \to \mathcal{Q} = \mathcal{W}/\mathcal{H}$ produces a dimensional reduction of the action (3.1):

$$S^\star \left(\int_{\mathcal{C}^2/\Gamma} \nu F \wedge F \right) = \int_{\mathcal{C}^2/(\Gamma \otimes \mathrm{U}(1))} \nu \cdot du \wedge da, \tag{3.2}$$

where S is a section of the above bundle, $\mathcal{C}^2/\Gamma$ represents the Kleinian singularities for the relevant subgroup Γ, u is a function on the 3–dimensional cone $\mathcal{Q}$ and a is the pull back of an abelian gauge field from $\mathcal{Q}$. In Ref. [15] it is shown that the action (3.1) leads to the equation of motion, which appears to be precisely the first heavenly equation

$$\Omega_0 \wedge \Omega_0 + 2\omega \wedge \tilde{\omega} = 0, \tag{3.3a}$$

with $\Omega_0 = \partial\bar{\partial}\Omega$, or equivalently

$$\Omega_{x\bar{x}}\Omega_{y\bar{y}} - \Omega_{x\bar{y}}\Omega_{y\bar{x}} = 1. \tag{3.3b}$$

This shows an unexpected presence of self-dual gravity structures in the string context.

One finds easily [15] that the element of the full quantum ground ring,

$$\Omega = x\bar{x} + y\bar{y}, \tag{3.4}$$

satisfies the first heavenly equation (3.3b). Moreover if one interprets Ω to be the Kähler function, then the corresponding metric is flat.

On the other hand we can see that the element of the full quantum ground ring $\Omega_{u,n,n'}(x,y,\bar{x},\bar{y}) = \mathcal{O}_{u,n} \cdot \bar{\mathcal{O}}_{u,n'}$ does not satisfy the first heavenly equation as it is, but a "modified" one given by Q-Han Park in Ref. [16]:

$$\gamma(\Omega_{x\bar{x}}\Omega_{y\bar{y}} - \Omega_{x\bar{y}}\Omega_{\bar{x}y}) + (1-\gamma)\Omega_{xy} = W(x,y), \tag{3.5}$$

with $\gamma = 1$ and $W = 0$. This case corresponds precisely to the topological model (W-Z term only). That is,

$$\Omega_{x\bar{x}}\Omega_{y\bar{y}} - \Omega_{x\bar{y}}\Omega_{y\bar{x}} = 0. \tag{3.6}$$

4. Self-dual Gravity Structures on Ground Ring Manifolds

In the previous section we have obtained the first heavenly equation of the self-dual gravity from the action [15]

$$\int_{C^4} \nu \cdot F \wedge F.$$

In this section, we re-derive the same structure in a different way making emphasis only on the fact that the full quantum ground ring manifold $\mathcal{W}$ is merely the product manifold $\mathcal{A}_L \times \mathcal{A}_R$, and that the chiral ground ring manifolds are symplectic manifolds with symplectic two-forms $\omega = dx \wedge dy$ and $\tilde{\omega} = d\bar{x} \wedge d\bar{y}$ respectively. For this end we use the construction of Refs. [1,12].

Consider the flat chiral complexified ground ring manifolds $\mathcal{A}_L^{\mathcal{C}}$ and $\mathcal{A}_R^{\mathcal{C}}$ and the complexified full ground ring manifold $\mathcal{W}^{\mathcal{C}} = \mathcal{A}_L^{\mathcal{C}} \times \mathcal{A}_R^{\mathcal{C}}$. Since $(\mathcal{A}_L^{\mathcal{C}}, \omega)$ and $(\mathcal{A}_R^{\mathcal{C}}, \tilde{\omega})$ are symplectic manifolds it is very easy to show that $\mathcal{W}^{\mathcal{C}}$ is also a symplectic manifold with symplectic form $\rho^*\omega - \tilde{\rho}^*\tilde{\omega}$ (ρ: $\mathcal{W}^{\mathcal{C}} \to \mathcal{A}_L^{\mathcal{C}}$ and $\tilde{\rho}$: $\mathcal{W}^{\mathcal{C}} \to \mathcal{A}_R^{\mathcal{C}}$ are the respective projections).

Now, let $T^r\mathcal{A}_L^{\mathcal{C}}$, $T^r\mathcal{A}_R^{\mathcal{C}}$ and $T^r\mathcal{W}^{\mathcal{C}}$ be the r-th order holomorphic tangent bundles of $\mathcal{A}_L^{\mathcal{C}}$, $\mathcal{A}_R^{\mathcal{C}}$ and $\mathcal{W}^{\mathcal{C}}$, respectively. Then we have the following sequence of projections for $\mathcal{A}_L^{\mathcal{C}}$:

$$\to T^r\mathcal{A}_L^{\mathcal{C}} \to T^{r-1}\mathcal{A}_L^{\mathcal{C}} \to \cdots \to T^1\mathcal{A}_L^{\mathcal{C}} \to T^0\mathcal{A}_L^{\mathcal{C}} \cong \mathcal{A}_L^{\mathcal{C}}; \tag{4.1}$$

and similarly for $\mathcal{A}_R^{\mathcal{C}}$ and $\mathcal{W}^{\mathcal{C}}$.

Following Refs. [1,12] one can define functions $\mathcal{O}_{u,n}^{(\lambda)}$, $\bar{\mathcal{O}}_{u,n'}^{(\lambda')}$ and $\mathcal{V}_{u,n,n'}^{(\lambda)}$; vector fields $Y_{s,n}^{+(\lambda)}$, $\bar{Y}_{s,n}^{+(\lambda')}$ as well as $\mathcal{J}_{u,n,n'}^{(\lambda)}$, $\bar{\mathcal{J}}_{u,n,n'}^{(\lambda')}$, and differential forms $\omega^{(\lambda)}$, $\tilde{\omega}^{(\lambda)}$, (with $\lambda =$

$0, 1, 2, \ldots, r$) on the r–order holomorphic tangent (or cotangent) bundles $T^r\mathcal{A}_L^{\mathcal{C}}$, $T^r\mathcal{A}_R^{\mathcal{C}}$ and $T^r\mathcal{W}^{\mathcal{C}}$. For example,

$$\begin{aligned}\mathcal{O}_{u,n}^{(\lambda)}(j_r \circ \psi(0)) &= \frac{1}{\lambda!}\left.\frac{d^\lambda(\mathcal{O}_{u,n}\circ\psi)}{dt^\lambda}\right|_{t=0}\\ \mathcal{V}_{u,n,n'}^{(\lambda)}(j_r \circ \psi(0)) &= \frac{1}{\lambda!}\left.\frac{d^\lambda(\mathcal{V}_{u,n,n'}\circ\psi)}{dt^\lambda}\right|_{t=0},\end{aligned} \tag{4.2}$$

where $j_r(\psi)$ is the r–jet of the holomorphic curve ψ. Then,

$$\mathcal{O}_{u,n}^{(\lambda)}(j_r \circ \psi(0)) \cdot \mathcal{O}_{u,n'}^{(\lambda')}(j_r \circ \psi(z)) = \frac{1}{\lambda!}\left.\frac{\partial^\lambda(\mathcal{O}_{u,n}\circ\psi)}{\partial s^\lambda}\right|_{s=0} \cdot \frac{1}{\lambda'!}\left.\frac{\partial^{\lambda'}(\mathcal{O}_{u,n'}\circ\psi)}{\partial t^{\lambda'}}\right|_{t=z}. \tag{4.3}$$

From Theorem 2 of Ref. [1] one can see that $(T^r\mathcal{A}_L^{\mathcal{C}}, \omega^{(\lambda)})$ and $(T^r\mathcal{A}_R^{\mathcal{C}}, \tilde{\omega}^{(\lambda)})$ are symplectic manifolds. Therefore, we can define another symplectic manifold $T^r\mathcal{W}^{\mathcal{C}}$ with symplectic two form $\rho^*\omega^{(r)} - \tilde{\rho}^*\tilde{\omega}^{(r)}$. It is very easy to establish the bundle diffeomorphism

$$q: T^r\mathcal{W}^{\mathcal{C}} \to \rho^* T^r\mathcal{A}_L^{\mathcal{C}} \oplus \tilde{\rho}^* T^r\mathcal{A}_R^{\mathcal{C}}. \tag{4.4}$$

The following diagram summarizes our construction:

$$\begin{array}{lccc} T^2\mathcal{W}^{\mathcal{C}} & \rightarrow & \rho^*T^2\mathcal{A}_L^{\mathcal{C}} \oplus \tilde{\rho}^*T^2\mathcal{A}_R^{\mathcal{C}} & \\ \pi_1^2 \downarrow & & \downarrow \rho_1^2 & \searrow \tilde{\rho}_1^2 \\ T\mathcal{W}^{\mathcal{C}} & & \rho^*T^2\mathcal{A}_L^{\mathcal{C}} & \rightarrow \tilde{\rho}T^2\mathcal{A}_R^{\mathcal{C}} \\ \pi \downarrow & & & \\ \mathcal{W}^{\mathcal{C}} & & & \end{array} \tag{4.5}$$

In order to show the existence of self-dual gravity structures on the full quantum ground ring manifold we restrict ourselves to the case $r = 2$. As it is mentioned in Ref. [12] $(\rho^*T^2\mathcal{A}_L^{\mathcal{C}}, \rho^*\omega^{(2)} - \tilde{\rho}^*\tilde{\omega}^{(0)})$ and $(\rho^*T^2\mathcal{A}_R^{\mathcal{C}}, \rho^*\omega^{(0)} - \tilde{\rho}^*\tilde{\omega}^{(2)})$ are also symplectic manifolds. Here

$$\begin{aligned}\omega^{(0)} &= \omega = dx \wedge dy,\\ \tilde{\omega}^{(0)} &= \tilde{\omega} = d\bar{x} \wedge d\bar{y},\\ \omega^{(2)} &= dx \wedge dy^{(2)} + dx^{(1)} \wedge dy^{(1)} + dx^{(2)} \wedge dx,\\ \tilde{\omega}^{(2)} &= d\bar{x} \wedge d\bar{y}^{(2)} + d\bar{x}^{(1)} \wedge d\bar{y}^{(1)} + d\bar{x}^{(2)} \wedge d\bar{x}.\end{aligned} \tag{4.6}$$

We need only to consider the manifold $\rho^*T^2\mathcal{A}_L^{\mathcal{C}}$.

To show the existence of a self-dual gravity structure in the 2d string theory we use the arguments of Refs. [1,12].

Consider the sequences

$$\begin{aligned}&\mathcal{N} \xrightarrow{i} T^2\mathcal{W}^{\mathcal{C}} \xrightarrow{\pi_1^2} T\mathcal{W}^{\mathcal{C}} \xrightarrow{\pi^1} \mathcal{W}^{\mathcal{C}},\\ &\mathcal{N} \xrightarrow{i} T^2\mathcal{W}^{\mathcal{C}} \xrightarrow{\rho_1^2} \rho^*T^2\mathcal{A}_L^{\mathcal{C}} \xrightarrow{\rho^1} \mathcal{W}^{\mathcal{C}},\end{aligned} \tag{4.7}$$

where $\mathcal{N}$ is a horizontal lagrangian submanifold of both $T\mathcal{W}^{\mathcal{C}}$ and $\rho^*T^2\mathcal{A}_L^{\mathcal{C}}$.

Let $\sigma: \mathcal{S} \to T^2\mathcal{W}^{\mathcal{C}}$, $\mathcal{S} \subset \mathcal{W}^{\mathcal{C}}$, be a holomorphic section such that $i(\mathcal{N}) = \sigma(\mathcal{S})$, where $i: \mathcal{N} \to T^2\mathcal{W}^{\mathcal{C}}$ is an injection.

Let $\mathcal{L}(\mathcal{S})$ be the set of all holomorphic sections of $T^2\mathcal{W}^{\mathcal{C}} \to \mathcal{W}^{\mathcal{C}}$ such that $\pi_1^2 \circ i(\mathcal{N})$ and $\rho^2 \circ i(\mathcal{N})$ are horizontal lagrangian submanifolds of $T\mathcal{W}^{\mathcal{C}}$ and $\rho^*T^2\mathcal{A}_L^{\mathcal{C}}$, respectively. Thus $\mathcal{L}(\mathcal{S})$ is defined by the equations

$$\begin{aligned}&\sigma_1^*(\omega^{(1)} - \tilde{\omega}^{(1)}) = 0, \quad \text{on} \quad \mathcal{S} \subset \mathcal{W}^{\mathcal{C}}\\ &\sigma_2^*(\omega^{(2)} - \tilde{\omega}^{(0)}) = 0, \quad \text{on} \quad \mathcal{S} \subset \mathcal{W}^{\mathcal{C}}\end{aligned} \tag{4.8}$$

[notice that we have omitted the pull-back ρ^* and $\tilde{\rho}^*$ in these formulas], where $\omega^{(1)} = dx \wedge dy^{(1)} + dx^{(1)} \wedge dy$, $\tilde{\omega}^{(1)} = d\bar{x} \wedge d\bar{y}^{(1)} + d\bar{x}^{(1)} \wedge d\bar{y}$, $\sigma_1 = \pi_1^2 \circ \sigma$ and $\sigma_2 = \rho^2 \circ \sigma$.

Now the problem arises of how $\sigma \in \mathcal{L}(\mathcal{S})$ determines a self-dual gravity structure on the open sets $\mathcal{S}$ of the full quantum ground ring manifold $\mathcal{W}^{\mathcal{C}}$.

The solution of this problem is given by the following theorem.

THEOREM [1,12]. Let $\sigma: \mathcal{S} \to T^2\mathcal{W}^{\mathcal{C}}$, $\mathcal{S} \subset \mathcal{W}^{\mathcal{C}}$, be a holomorphic section. The triplet $(\omega, \tilde{\omega}, \Omega_0) = (\sigma^*\omega^{(0)}, \sigma^*\tilde{\omega}^{(0)}, \sigma^*\omega^{(1)})$ defines a self-dual structure on $\mathcal{S}$ if and only if there exist a choice of a holomorphic section σ such that $\sigma^*(\omega^{(1)} - \tilde{\omega}^{(1)}) = 0$ and $\sigma^*(\omega^{(2)} - \tilde{\omega}^{(0)}) = 0$ (for the proof see Ref. [1,12]).

Thus, the desired self-dual gravity structures arise in a natural manner from the mathematical structure of the quantum states in 2d string theory.

Taking $\Omega_0 = \sigma_1^*[dx \wedge dy^{(1)} + dx^{(1)} \wedge dy]$ we arrive at the first heavenly equation,

$$\Omega_0 \wedge \Omega_0 + 2\omega \wedge \tilde{\omega} = 0.$$

This can be extended to the cases with $r \geq 3$. Then, by using the projective limit one can formulate the problem in terms of the infinite-dimensional tangent bundle $T^\infty\mathcal{W}^{\mathcal{C}} = \rho^*T^\infty\mathcal{A}_L^{\mathcal{C}} \oplus \tilde{\rho}^*T^\infty\mathcal{A}_R^{\mathcal{C}}$ (for details see Refs. [1,12]). It can be proved that given $(\mathcal{W}^{\mathcal{C}}, \rho^*\omega - \tilde{\rho}^*\tilde{\omega})$ a symplectic manifold $(T^\infty\mathcal{A}_L^{\mathcal{C}}, \omega_2(t))$ turns out to be a formal symplectic manifold, where $\omega_2(t) = \sum_{k=0}^{\infty} \pi_k^*\omega^{(k)}t^k$; $t \in \mathcal{C}$, and $\pi_k: T^\infty\mathcal{A}_L^{\mathcal{C}} \to T^k\mathcal{A}_L^{\mathcal{C}}$ is the natural projection.

By the Proposition 2 of Ref. [1] we observe that $(T^\infty\mathcal{W}^{\mathcal{C}}, \omega(t))$ is a formal symplectic manifold with

$$\omega(t) = t^{-1}\rho^*\omega_2(t) - t\tilde{\omega}_2(t^{-1}) \tag{4.9}$$

where $t \in \mathcal{C}^* \equiv \mathcal{C} - \{0\}$ and $T^\infty \mathcal{W}^\mathcal{C} = T^\infty \mathcal{A}_L^\mathcal{C} \times T^\infty \mathcal{A}_R^\mathcal{C} = T^\infty(\mathcal{A}_L^\mathcal{C} \times \mathcal{A}_R^\mathcal{C}) = \rho^* T^\infty \mathcal{A}_L^\mathcal{C} \times \tilde{\rho}^* T^\infty \mathcal{A}_R^\mathcal{C}$.

4.1. *Curved twistor construction on full quantum ground ring manifolds*

Consider the formal symplectic manifold $(T^\infty \mathcal{W}^\mathcal{C}, \omega(t))$. Since $\mathcal{A}_L^\mathcal{C}$ and $\mathcal{A}_R^\mathcal{C}$ are diffeomorphic, we have $T^\infty \mathcal{W}^\mathcal{C} = T^\infty \mathcal{A}_L^\mathcal{C} \times T^\infty \mathcal{A}_R^\mathcal{C}$. Define the holomorphic maps

$$\hat{\mathcal{D}} = (D, I) \colon T^\infty \mathcal{A}_L^\mathcal{C} \times \mathcal{C}^* \to T^\infty \mathcal{A}_L^\mathcal{C} \times \mathcal{C}^*,$$

where $I(t) = t^{-1}$ and the graph of the diffeomorphism D, $\operatorname{gr} D$, can be identified with some local section $\operatorname{gr} D = \sigma' \colon \mathcal{S} \to T^\infty \mathcal{W}^\mathcal{C}$ such that $\sigma'^* \omega(t) = 0$. From Eq. (4.9), this last relation holds if and only if

$$D^* \omega_2(t^{-1}) = t^{-2} \omega_2(t). \tag{4.10}$$

Consider now a local section σ'' of the formal tangent bundle $T^\infty \mathcal{W}^\mathcal{C} \to \mathcal{W}^\mathcal{C}$ on a open set $\mathcal{S} \subset \mathcal{W}^\mathcal{C}$ such that $\sigma''^* \omega(t) = 0$. For $t \in \mathcal{C}^*$

$$\sigma'' = \left(\Psi^A(t), \tilde{\Psi}^B(t^{-1})\right). \tag{4.11}$$

Assume that $\Psi^A(t)$ and $\tilde{\Psi}^B(t^{-1})$ converge in some open discs $\mathcal{U}_0$ and $\mathcal{U}_\infty$ ($0 \in \mathcal{U}_0$ and $\infty \in \mathcal{U}_\infty$) respectively, such that $\mathcal{U}_0 \cap \mathcal{U}_\infty \neq \phi$. Consequently, the functions $\Psi^A \colon t \mapsto \Psi^A(t)$ and $\tilde{\Psi}^B \colon s \mapsto \tilde{\Psi}^B(s)$ define local holomorphic sections of the twistor space $\mathcal{T}$. Due to the condition (4.10) defining the self-dual structure on the quantum ground ring manifold $\mathcal{W}^\mathcal{C}$ we get the transition functions for a global holomorphic section $\Psi \in \tilde{\Gamma}(\mathcal{T})$. Thus one can recuperate the Penrose twistor construction [17]. Of course the inverse process is also possible (see Refs. [1,12]).

5. Final remarks

In this work we have looked for self-dual gravity structures in 2d string theory. This was motivated mainly by the works of Witten and Zwiebach [5] as well as Ghoshal *et al.* [15]. These papers provide a "physical" approach that shows an unexpected presence of self-dual gravity structures in string theory. Goshal *et al.* [15] have found these structures by looking for the solutions to the equations of motion derived from the action (3.1). One class of solutions in particular implies the existence of self-dual gravity structures. Here we have proposed the purely geometric ("mathematical") approach based on the symplectic geometry of the chiral ground ring manifold. As we have shown, this geometry leads directly to self-dual structures in 2d string theory. Now, the natural question arises about what is the relation between the "physical" and the "mathematical" approaches. If Ghoshal *et al.* conjecture [15], which states that the dynamics of the states in 2d string theory is given by the self-dual gravity structure, is true, our geometric approach might be more convenient.

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APPENDIX. THE CHIRAL GROUND RING STRUCTURE

In this appendix, we review the basic arguments which lead to the chiral ground ring structure, as proposed in [4]. This structure arises in 2d string theory. The bosonic string theory with 2d target spacetime involves Liouville gravity coupled to some conformal field theory (CFT) with central charge $c = 1$. In the absence of a cosmological constant, Liouville gravity and the CFT decouple resulting the world-sheet Lagrangian

$$\mathcal{L} = \frac{1}{8\pi}\int_\Sigma d^2x\,\sqrt{h}\,(h^{ij}\partial_i X\,\partial_j X + h^{ij}\partial_i\phi\,\partial_j\phi) - \frac{1}{2\pi\sqrt{2}}\int_\Sigma d^2x\,\sqrt{h}\cdot\phi R^{(2)}; \tag{A1}$$

here Σ is a Riemann surface, h the world-sheet metric, X a bosonic field, ϕ a Liouville field and $R^{(2)}$ is the Ricci scalar.

The Lagrangian (A1) contains an infinite number of discrete states in addition to the tachyonic state $V_p = \exp(ipX)$ being X a free field. The presence of the additional states appears for the first time in the study of $c = 1$ matrix models.

Introducing a SU(2) symmetry on the states of (A1), these discrete states arise in a natural way. Concretely, using the theory compactified at the SU(2) radius, the momenta p of X take the discrete values $p = n/\sqrt{2}$, $n \in \mathcal{Z}$.

One can see that the conformal operator $\exp(isX/\sqrt{2})$ belongs to a multiplet of a SU(2) representation corresponding to the highest weight. The other members of this multiplets are the operators $V_{s,n}$ such that $V_{s,s} = \exp(isX/\sqrt{2})$ and $V_{s,-s} = \exp(-isX/\sqrt{2})$. The operator with $s = |n|$ corresponds precisely to the tachyon operator. (some states remain for $|n| < s$)

Now, introducing the ghost fields b and c of spin 2 and -1, respectively, one can construct the spin 0 BRST invariant primary fields of ghost number 1:

$$Y^{\pm}_{s,n} = cW^{\pm}_{s,n}, \tag{A2}$$

from the spin 1 fields

$$W^{\pm}_{s,n} = V_{s,n}\exp(\sqrt{2}\phi \mp s\phi\sqrt{2}). \tag{A3}$$

The operators $W^{\pm}_{s,n}$ have momentum $(n, i(-1 \pm s))\cdot\sqrt{2}$.

The composite states $Y^{\pm}_{s,n}$ with $|n| < s$ have partners at an adjoining value of the ghost number 0 or 2. Here we would like to consider only those of ghost number 0. These will be partners of the positive part of (A2) $Y^{+}_{s,n}$. Redefining $s = u + 1$, these partner states are $\mathcal{O}_{u,n}$ and have momentum $(n, iu)\cdot\sqrt{2}$. The u's take the values $0, \frac{1}{2}, 1$, etc. and $n = u, u-1, \ldots, -u$.

The correspondence between states with ghost number 1 and spin 1 and those with ghost number 0 and spin 0 is

$$\begin{aligned} &Y^+_{1,0} = c\partial X \Leftrightarrow \mathcal{O}_{0,0} = 1, \\ &\qquad Y^+_{\frac{3}{2},\frac{1}{2}} \Leftrightarrow x \equiv \mathcal{O}_{\frac{1}{2},\frac{1}{2}} = (cb + \tfrac{i}{2}(\partial X - i\partial\phi))\exp[i(X + i\phi)/2], \\ &\qquad Y^+_{\frac{3}{2},-\frac{1}{2}} \Leftrightarrow y \equiv \mathcal{O}_{\frac{1}{2},-\frac{1}{2}} = (cb - \tfrac{i}{2}(\partial X + i\partial\phi))\exp[-i(X - i\phi)/2]. \end{aligned} \tag{A4}$$

These states can be constructed using the BRST analysis.

Combining the operators (A3) and (A4) one can define the *quantum field operators* of spin $(1,0)$ and $(0,1)$ to be, respectively,

$$\begin{aligned} \mathcal{J}_{s,n,n'} &= W^+_{s,n} \cdot \bar{\mathcal{O}}_{s-1,n'}, \\ \bar{\mathcal{J}}_{s,n,n'} &= \mathcal{O}_{s-1,n} \cdot \bar{W}^+_{s,n'}, \end{aligned} \tag{A5}$$

where the bar represents complex conjugation.

These operators generate a Lie algebra of symmetries, namely the Lie algebra of *volume preserving diffeomorphisms* of a 3 dimensional algebraic variety defined by

$$a_1a_2 - a_3a_4 = 0, \tag{A6}$$

where $a_1 = x\bar{x}$, $a_2 = y\bar{y}$, $a_3 = x\bar{y}$ and $a_4 = \bar{x}y$.

The *chiral ground ring* $\mathcal{A}_L$ structure come from the pair $\{x, y\}$ (similarly for the right part taking $\{\bar{x}, \bar{y}\}$). $\mathcal{A}_L$ defines a ring structure under the usual operator product expansion [4]. The chiral symmetries of the Lagrangian (A1) form the group of diffeomorphisms preserving the area of the plane generated by $\{x, y\}$. This group is denoted by $\mathrm{SDiff}(\mathcal{A}_L)$.

In this way, the existence of both the set of states (A4) and the chiral ground ring structure implies the existence of a W_∞–symmetry in 2d string theory.

Further, the discussion in Ref. [5] shows that the above results can be put into a geometrical setting. There, it is also shows that the pair $(\mathcal{A}_L, \omega)$ is a symplectic manifold, with symplectic 2–form $\omega = dx \wedge dy$.

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