Approximated photon distribution function for quantized thermocoherent electromagnetic field

E.V. KURMYSHEV*
Centro de Investigación en Física, Universidad de Sonora
Apartado postal 5-088, 83190 Hermosillo, Sonora, México

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ABSTRACT. An approximation to the photon distribution function, which describes statistics of the mixed thermal-coherent radiation, is derived. This approximation is efficient for analytic calculations in numerous problems of interaction between quantum electromagnetic fields and matter, specially when the steepest descent method is used.

RESUMEN. Se deriva una aproximación a la función de distribución de fotones, la cual describe la estadística para la radiación térmica-coherente. Esta aproximación es eficiente para cálculos analíticos en numerosos problemas de interacción entre el campo electromagnético cuántico y la materia, especialmente cuando se usa el método de fase estacionaria.

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1. INTRODUCTION

It is hardly possible to find a domain of physics where coherence is not applied. Such a fascinating phenomenon as collapses and revivals of atomic population in the system of independent two level atoms interacting with quantized cavity field was first described in the Jaynes-Cummings model (JCM) for a cavity field initially prepared in the coherent state [1,2] (see also discussion in Ref. [3–7]). The phenomenon has recently started to be observed experimentally [8,9]. One of the interesting questions in the Jaynes-Cummings and in the Dicke [10] models of quantum field interaction with atoms is how the incoherence of the system, particularly of the electromagnetic field, due to the thermalization produces collapses and revivals [11–23].

To be more precise we remind here some estimations in order to have clear heuristic ideas of how the account of thermality of the field becomes important to compete with the coherence in quantum phenomena. A priori one can expect significant influence of thermality when the photon distribution function of coherent components of the cavity field has appreciable overlap with that of thermal constituent, *i.e.*, the mean value of thermalized photons N_{t} becomes comparable with that of coherent ones N_{c}, the latter being of the order 1–100 photons. For the room temperature $T = 300$ K and for the light of mode frequency $\nu = 5\times10^{14}$ Hz one obtains $N_{\mathrm{t}} = (e^{x}-1)^{-1} = \exp(-80)$, a negligible value, where $x = h\nu/k_{\mathrm{B}}T$, h is the Planck constant and k_{B} is the Boltzman constant. So,

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*On sabbatical leave from Central Bureau for the Design of Unique Devices, 15 Butlerov Str., 117342, Moscow, Russia.

thermalization of the cavity quantum field appears to be relatively unimportant for visible frequency band and for room (and even much higher) temperature. But the situation will radically change if we consider far infrared or microwave radiation. Even in the case of microwaves of the frequency $\nu = 3 \times 10^{11}$ Hz at room temperature $T = 300$ K one has $N_{\rm t} \simeq x^{-1} \simeq 20$ photons. This estimation clearly demonstrates that there exist nonzero bands of frequencies and an interval of temperatures when one has to take into account and evaluate competitive process of coherence versus thermalization.

The experiments in quantum optics are, finally, ones on photon counting. Statistics (photon number distribution function) of quantum electromagnetic field is what we usually use to examine the evolution of quantum systems, which describe interaction between matter and quantized fields, for example, in such extremely popular models as those of Jaynes-Cummings and Dicke.

Recently studied questions of the existence and correct description of the revivals, even in the basic and so much investigated JC model, faced with certain mathematical difficulties to get correct theoretical predictions, when the initial field is a Bose-Einstein thermal one [11–14,19,20–23] or when a cavity field initially prepared in a coherent state is subjected to thermalization [12,19]. The most promising techniques, which involve integral transforms, are finally reduced to exploiting the steepest descent method to get final results in explicit form. This is one of the important reasons to have a photon distribution function (PDF) in the form which is the most suitable for the stationary phase method application.

The two of the most important states of a single mode quantized field are a coherent (more general a squeezed) state and a thermal equilibrium of the electromagnetic field at temperature T. However, a perfectly lossless cavity can not of course be built in practice. That means that a mixed thermocoherent cavity field is of great importance for a realistic theory of interaction of atoms with a cavity field (see, for example, [12,19]). In this paper we consider a particular but important, for the use of analytic calculations, question about functionally simple and compact approximation of PDF, which is fairly precise in a wide range of parameters, for quantum mixed thermocoherent electromagnetic field.

2. DERIVATION OF THE APPROXIMATED PDF

It is known [12,24,25] that the diagonal elements of the density matrix in the Fock (occupation number) representation for the single mode quantized thermocoherent electromagnetic field is given by the following expression:

$$\begin{aligned} \mathrm{P}(n) &= \frac{1}{1+N_{\rm t}} \exp\left(-\frac{N_{\rm c}}{1+N_{\rm t}}\right) \left[\frac{N_{\rm t}}{1+N_{\rm t}}\right]^n L_n\left(-\frac{N_{\rm c}}{N_{\rm t}(1+N_{\rm t})}\right) \\ &= \frac{1}{1+N_{\rm t}} \exp\left(-\frac{N_{\rm c}}{1+N_{\rm t}}\right) \left[\frac{N_{\rm t}}{1+N_{\rm t}}\right]^n \sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} \left(\frac{N_{\rm c}}{N_{\rm t}(1+N_{\rm t})}\right)^k ; \end{aligned} \qquad (2.1)$$

where $N_{\rm c}$ and $N_{\rm t}$ may be loosely interpreted as the mean values of coherent and thermal photon numbers, correspondingly; L_n is the Laguerre polynomial of n–th order, here

$L_n(0) = 1$. The calculations of the mean photon number and the deviation for the distribution function, Eq. (2.1), lead to the values

$$\langle a^+ a\rangle \equiv \bar{n} = N_c + N_t, \tag{2.2}$$

$$\left(\langle(a^+a)^2\rangle - \langle a^+a\rangle^2\right)^{1/2} \equiv \sigma = \left[N_c(2N_t+1) + N_t^2 + N_t\right]^{1/2}, \tag{2.3}$$

where a^+ and a are photon creation and annihilation operators. At zero temperature (or at $N_t \to 0$) the Eq. (2.1) tends to the Poisson distribution with the mean value equal to N_c:

$$P_c(n) = e^{-N_c}\frac{N_c^n}{n!}. \tag{2.4}$$

At "high" temperature (or at $N_c \to 0$) the Eq. (2.1) gives the thermal (black-body) photon number distribution with the mean number of photons equal to $N_t = [(1-e^{-x})^{-1} - 1] = Z - 1$:

$$P_t(n) = \frac{1}{1+N_t}\left(\frac{N_t}{1+N_t}\right)^n, \tag{2.5}$$

where Z is the partition function of a single mode field in thermal equilibrium, and, as usual, $x = \hbar\omega/k_B T$, with the temperature T and with the single mode field frequency ω.

It can be shown that the distribution, Eq. (2.1), has a peak, which is centered near $n \simeq N_c$, even for the case of N_c not too large and N_t not too small . Hence, for the saddle point technique application, we are interested in a region near $n \simeq N_c$, which contributes significantly. For example, in the case of quite large $\bar{n}$ the main contribution to the Jaynes-Cummings sum (or to the appropriate integral) comes from some region close to $n \simeq N_c$. This consideration gives us the reason to use the asymptotic expansion of Laguerre's polynomials for large order n [26–28],

$$L_n(u) \to (\pi)^{-1/2} e^{u/2} u^{-1/4} n^{-1/4} \cos\left[2(nu)^{1/2} - \pi/4\right], \tag{2.6}$$

to find out the suitable approximated photon number distribution. Substitution of the Laguerre polynomial into the Eq. (2.1) as its asymptotic expression Eq. (2.6) with the argument $u \equiv N_c/(N_t(1+N_t))$ leads to the following sequence of relations:

$$\begin{aligned} L_n(-u) &\to (\pi)^{-1/2} e^{-u/2} e^{-i\pi/4} u^{-1/4} n^{-1/4} \cos\left[i2(nu)^{1/2} - \pi/4\right] \\ &= (\pi)^{-1/2} e^{-u/2} e^{-i\pi/4} u^{-1/4} n^{-1/4} \\ &\quad \times 2^{-1/2}\left[\cosh(2(nu)^{1/2}) + i\sinh(2(nu)^{1/2})\right] \\ &\to 2^{-1}(\pi)^{-1/2} e^{-u/2} u^{-1/4} n^{-1/4} \exp(2(nu)^{1/2}), \end{aligned} \tag{2.7}$$

where $u \equiv N_c/(N_t(1+N_t)) > 0$ and the Laguerre polynomial's order n is supposed to be large (Eq. (2.7) is valid asymptotically). Straightforward substitution of the approximated expression Eq. (2.7) for Laguerre's polynomial into the Eq. (2.1) gives the function

$$
\begin{aligned}
P(n) &= K n^{-1/4} \exp\bigl(n \ln v + 2(nu)^{1/2}\bigr),\\
K &= \frac{1}{1+N_t} \exp\left(-\frac{N_c}{1+N_t}\right) 2^{-1}(\pi)^{-1/2} e^{-u/2} u^{-1/4}\\
&= 2^{-1}(\pi)^{-1/2} \frac{1}{1+N_t} \exp\left(-\frac{N_c}{N_t}\right) u^{-1/4},\\
v &= \frac{N_t}{1+N_t}, \qquad u = \frac{N_c}{N_t(1+N_t)},
\end{aligned}
\tag{2.8}
$$

where N_c and N_t are the parameters of PDF. However, the last expression Eq. (2.8) is only in qualitative agreement with the original PDF [Eq. (2.1)].

To achieve not only qualitative but quantitative agreement with the exact distribution one can parameterize, in general, an expression of the kind of Eq. (2.8) and then equate the zeroth, first and second order moments calculated with the exact distribution [see Eqs. (2.2), (2.3)] and with the approximate parameterized one. But this way is hardly performed analytically .

The simplified version, which deals with the 0–th order moment equality, is in fact a normalization procedure and leads to the following expressions (C is a normalization constant):

$$
\begin{aligned}
1 &= \sum_{n=0}^{\infty} C P(n) = C K \sum_{n=0}^{\infty} n^{-1/4} \exp\bigl(n \ln v + 2(nu)^{1/2}\bigr)\\
&\simeq C K \int_0^{\infty} n^{-1/4} \exp\bigl(n \ln v + 2(nu)^{1/2}\bigr)\, dn.
\end{aligned}
\tag{2.9}
$$

The last integral being evaluated with the steepest descent method gives us the normalization constant

$$
C \simeq \bigl(2\sqrt{\pi}\bigr)^{-1} K^{-1} (-\ln v)^{1/2} \exp(u/\ln v),
$$

and finally leads to the approximated photon distribution function (APDF) as follows:

$$
\begin{aligned}
\tilde{P}(n) &= \bigl(2\sqrt{\pi}\bigr)^{-1} \mu \exp(-u/\mu) u^{-1/4} n^{-1/4} v^n \exp\left[2(nu)^{1/2}\right],\\
v &= \frac{N_t}{1+N_t} < 1, \qquad u = \frac{N_c}{N_t(1+N_t)},\\
\mu &= -\ln v.
\end{aligned}
\tag{2.10}
$$

The Eq. (2.10) is a good approximation for the thermocoherent photon number distribution Eq. (2.1) in a wide range of parameters N_c and N_t (see computer simulations below). However, it can be hardly used for the limit cases of very large or very small values or u, especially in the vicinity of $n \simeq +0$. Both limit cases have to be investigated more carefully.

Quite a simple analytical form (exponential with respect to the argument n) of the distribution $\tilde{P}(n)$ [Eq. (2.10)], enable us to use it for analytical investigation of the atomic inversion time evolution in the Jaynes-Cummings and Dicke models.

3. Computer simulation and comparison of the approximated PDF and the exact one

Some asymptotic formulae and approximate equalities have been used above for derivation of $\tilde{P}(n)$ [Eq. (2.10)], but the range of the applicability of the latter can hardly be evaluated analytically [26–29]. For this reason, in order to give more evidences and to see restrictions of the approximation, we made a computer simulation in a wide and interesting range of parameters of PDF. Here we give a comparative analysis of the four groups of computer simulations.

In the first series we keep constant the mean number of thermal photon $N_t = 1$ and change the number of coherent photons $N_c = 5$, 10, 15, 20, 30, 40.

Figures 1a–f represent comparative plots of the exact PDF $\mathrm{P}(n)$ (thick solid line) and the approximated one $\tilde{P}(n)$ (thin solid line) for the parameters N_t and N_c given above. The accordance of the curves is rather convincing, the only small quantitative discrepancy observed is a shift to the right of the peak of the approximated PDF with respect to that of the exact one. In this range of the parameters PDF keeps qualitative resemblance with the Poisson distribution function [Eq. (2.4)] for it has a clearly seen crest.

In the second series, we keep constant the mean number of coherent photons $N_c = 10$ and change the number of thermal photons $N_t = 1, 3, 5, 10, 30$ (see Figs. 2a-d, 1b). The quantitative accordance of the approximated PDF $\tilde{P}(n)$ (thin solid line) with that of the exact one $\mathrm{P}(n)$ (thick solid line) is rather good, with a small right hand shift of the APDF with respect to the exact one.

However, the APDF has certain problems in the limit case $N_t \to 0$. Computer analysis of the series $\{N_c = 5$ and $N_t = 0.03, 0.05, 0.1, 0.2, 0.3, 0.5, 1, 3, 5, 10\}$ and $\{N_c = 3$ and $N_t = 0.01, 0.05, 0.1, 0.3, 0.5, 1, 3\}$ has clearly demonstrated that the approximation [Eq. (2.10)], cannot be considered as the satisfactory one for the very small values of N_t such that $\{N_t \leq 0.1 N_c$ and $N_c \leq 5\}$. One needs another approximation in this range of parameters N_t and N_c.

4. Concluding remarks

The APDF works well in the range of moderate values of parameters N_c and N_t: the values of N_c are quite arbitrary, and $N_t \geq 0.1\, N_c$ if $N_c \leq 5$. This is the transient and the most difficult range for the analytical study of the interaction of quantum electromagnetic

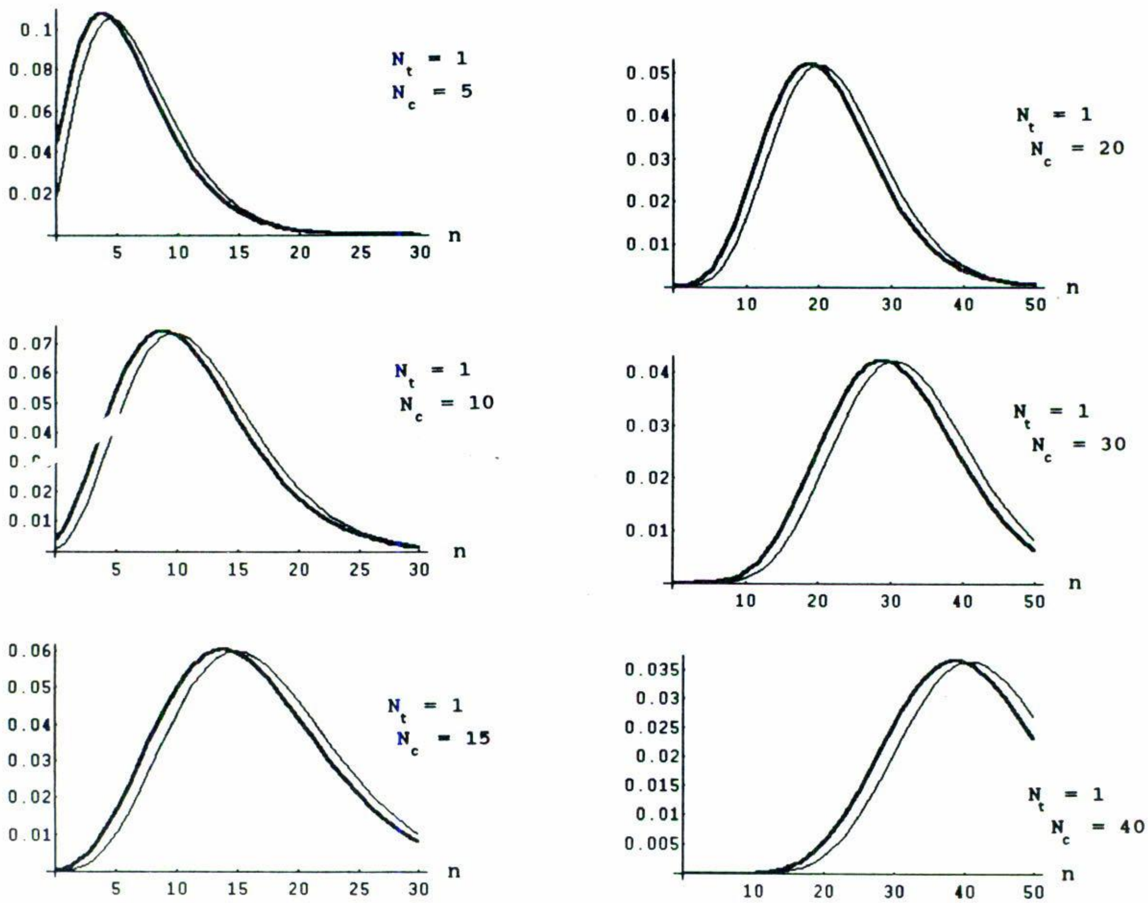

FIGURE 1. Comparative plots of the exact PDF P(n) (thick solid line) and the approximated PDF $\tilde{P}(n)$ (thin solid line) for the given $N_t = 1$ and different $N_c = 5$, 10, 15, 20, 30, 40.

field with matter. Moreover, it has quite good accordance with exact PDF for the limit case of strong thermalization of the radiation. The only significant shift of the crest of approximated PDF with respect to that of the exact PDF is observed in the limit $N_t \to 0$, while keeping the strong resemblance in the form. However, for this case one does not really neeed to use thermocoherent PDF [Eq. (2.1)], but can directly use Poisson PDF [Eq. (2.4)], for the coherent field, which quite adequate and simple for applications.

Even if we mainly referred to the Jaynes-Cummings model to discuss applicability of APDF, it has a much broader variety of applications to the problems of quantum field interaction with matter, since in order to have analytical results in a closed form one often has to use steepest descent method. Moreover, one can readily extract approximation to the Laguerre polynomials $L_n(u)$ by means of comparison of the Eq. (2.1) with the

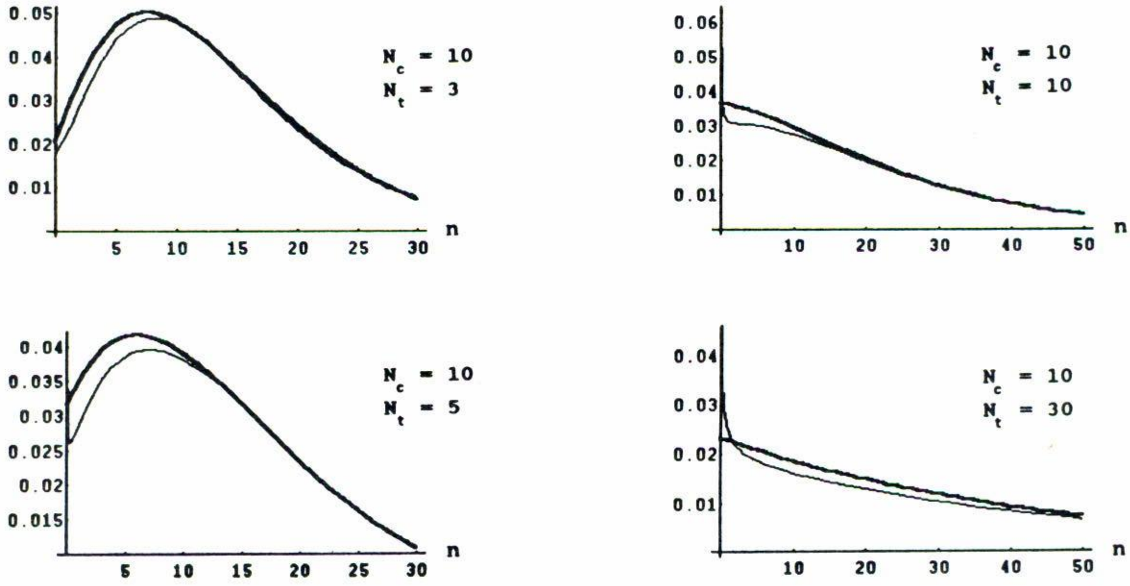

FIGURE 2. Comparative plots of the exact PDF P(n) (thick solid line) and the approximated PDF $\tilde{P}(n)$ (thin solid line) for the set of parameters $N_c = 10$ and $N_t = 3, 5, 10, 30$.

Eq. (2.10). This approximation of the Laguerre polynomials is valid for values of the argument $u = N_c/(N_t(1 + N_t))$ in the interval determined by the restrictions on the parameters N_c and N_t for the APDF to be valid.

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