Laws and exact solutions in classical special relativistic dynamics of N–particle systems*

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ABSTRACT. After reviewing briefly the relevant features of Newtonian dynamics, several possible formulations of special relativistic dynamics are described and the similarities and differences of the two theories are noted. For various types of interactions, including electrodynamic ones, examples of exact solutions of the equations of motion for two particles are given which illustrate features that can not occur in Newtonian dynamics. The possible physical relevance of such solutions is discussed.

RESUMEN. Después de presentar brevemente las características relevantes de la dinámica newtoniana, se describen varias formulaciones posibles de la dinámica en relatividad especial y se destacan las similitudes y diferencias entre ambas teorías. Se presentan ejemplos de soluciones exactas para las ecuaciones de movimiento de dos partículas con diversos tipos de interacciones, incluyendo las electrodinámicas, las cuales ilustran características que no pueden ocurrir en la dinámica newtoniana. Se discute la posible relevancia física de dichas soluciones.

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1. INTRODUCTION

It is of course fundamental for the space-time concepts of the special theory of relativity that the Galilei transformation must be replaced by the Lorentz one. It was realized by Einstein from the beginning (and by Poincaré even before the publication of Einstein's fundamental paper) that while this was consistent with Maxwell's electrodynamics, it did require a modification of Newton's second law of motion as well as of the description of particle interactions. However, almost all textbooks and surveys of relativity theory only discuss the consequences of this modification for the one-body problem, *i.e.*, for the motion of a single particle in an external field, and almost no work was done on the n–body problem for many decades, mainly due to the almost universal belief that relativistic interactions could only be described as being transmitted by a field rather than as direct-particle ones as in Newtonian dynamics. Similarly, it was thought that the causality requirements of relativity theory excluded a particle dynamics with Newtonian causality, a belief not challenged until 1960 [1]. However, much work has been done on relativistic particle dynamics in the last few decades.

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This paper is concerned with the laws and exact solutions for the motion of n–particle systems in classical special relativity; unfortunately, absolutely nothing is known for these problems in general relativity. But before discussing the special-relativistic problem it is essential to point out some of the features of Newtonian dynamics which are well known, but which are not necessarily maintained in the special theory and thus merit particular attention.

2. Characteristics of the Newtonian dynamics of point particles from a relativist's perspective

In Newtonian mechanics, a point particle is fully characterized by its (constant) mass m; its motion is determined by *Newton's Second Law*

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}. \tag{1}$$

If $\mathbf{F}$ vanishes, the particle moves with constant velocity.

In the following we shall be mainly concerned with the mechanics of a closed system of n point particles interacting via two-body action-at-distance forces. For such a system there exist various descriptions, of which the three main ones are sketched below.

2.1. Newtonian formalism

Newton's equations of motion, valid in any inertial system, are

$$m_i\frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j\neq i} F_{ij}\frac{\mathbf{r}_{ij}}{r_{ij}}, \quad \mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j, \quad r_{ij} \equiv |\mathbf{r}_{ij}|, \quad i,j = 1,\ldots,n, \tag{2}$$

where all quantities are evaluated at the same time t.

For forces which depend only on the mutual separations, one can introduce potentials

$$F_{ij} = F_{ij}(r_{ij}) = -\frac{\partial V}{\partial r_{ij}}, \quad V = \tfrac{1}{2}\sum_{k\neq l}\sum V_{kl}(r_{kl}). \tag{3}$$

2.2. Lagrangean formalism

For such a conservative system, Eqs. (2) can be derived from a variational principle

$$\delta I = 0, \qquad I = \int L(x^i, v^i, t)\, dt, \qquad v^i \equiv \frac{dx^i}{dt}, \tag{4}$$

where

$$L = T - V,$$
$$T = \tfrac{1}{2}\sum_{j=1}^{n} m_j v_j^2, \qquad v_j \equiv \frac{dr_j}{dt}, \tag{5}$$

and Eqs. (2) result from Lagrange's equations

$$L_i \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} = 0. \tag{6}$$

2.3. Hamiltonian formalism

These equations can also be derived from Hamilton's Principle, with the Hamiltonian

$$H \equiv \sum_j p_j \cdot v_j - L, \qquad p_j^i \equiv \frac{\partial L}{\partial v_j^i}. \tag{7}$$

The features of Newtonian mechanics for this conservative system which are of importance for the later discussion are

a) The equations of motion are invariant under the 10-parameter (proper, orthochronous) Galilei group [2].

While it is possible to formulate equations differing from Newton's with the same invariance properties [3], these will not be discussed here.

b) The Newtonian, Lagrangean, and Hamiltonian formulations are equivalent.

c) The motion of the n particles is determined by the laws of motion, the force laws, and the $6n$ initial conditions

$$\mathbf{r}_i(t_0) = \mathbf{r}_{i0}, \quad \mathbf{v}_i(t_0) = \mathbf{v}_{i0}, \quad i = 1 \ldots n. \tag{8}$$

The system is said to possess $3n$ degrees of freedom.

d) There always exist ten conservation laws [3,4]

$$\begin{aligned}
&\frac{dE}{dt} = 0, \qquad E \equiv \sum_j \mathbf{p}_j \cdot \mathbf{v}_j - L, \quad p_j^i \equiv \frac{\partial L}{\partial v_j^i},\\
&\frac{d\mathbf{P}}{dt} = 0, \qquad \mathbf{P} \equiv \sum_j \mathbf{p}_j,\\
&\frac{d\mathbf{J}}{dt} = 0, \qquad \mathbf{J} \equiv \sum_j \mathbf{r}_j \times \mathbf{p}_j,\\
&\frac{d\mathbf{M}}{dt} = 0, \qquad \mathbf{M} \equiv \sum_j m_j \mathbf{r}_j - \mathbf{P}t.
\end{aligned} \tag{9}$$

These, by Noether's Theorem, are a consequence of the invariance (up to a divergence) of the Lagrangean (4) under the Galilei group [5].

It should be noted that one can define a unique center-of mass coordinate $\mathbf{R}$ through

$$\frac{d\mathbf{M}}{dt} = 0, \qquad \mathbf{M} \equiv M\mathbf{R} - \mathbf{P}t, \qquad M\mathbf{R} \equiv \sum_j m_j \mathbf{r}_j, \qquad M \equiv \sum_j m_j, \tag{10}$$

e) The two-body problem can always be reduced to a one-body problem and therefore is always integrable.

f) For $n > 2$, except for particular force laws (such as harmonic oscillators), the *only* known analytic solutions are those with *homographic motion*: The configuration formed by the n bodies at a given time t moves in such way that it remains similar to itself.

The first such solution was given by Lagrange in 1772; an incomplete classification of the possible types of motion was given in Ref. [6] and a complete one in Ref. [7].

The main problem to be discussed here *is the extent to which these features remain valid, or must be modified, in the special theory of relativity.* Some of the modifications can be discussed in general, but others, some of the most striking ones, can at present only be shown through particular examples.

3. Forms of special-relativistic dynamics

Obviously, feature a) has to be modified to

A) The equations of motion must be (explicitly or implicitly) invariant under the 10-parameter (proper, orthochronous) Poincaré group.

In the following only explicitly Poincaré-invariant formalism will be discussed in any detail. We shall consider the Minkowskian four-space with coordinates x^μ ($\mu = 0, 1, 2, 3$), where $x^0 = t$ and x^1, x^2, x^3 are the usual Cartesian coordinates, and a metric $\eta_{\mu\nu}$, where

$$\eta_{00} = 1, \qquad \eta_{11} = \eta_{22} = \eta_{33} = -c^{-2}, \qquad \eta_{\mu\nu} = 0 \text{ if } \mu \neq \nu. \tag{11}$$

The particles are described by coordinates $z_i^\mu(\tau_i)$, where the τ_i's are the proper times defined by

$$d\tau_i \equiv (\eta_{\mu\nu}\, dz_i^\mu\, dz_i^\nu)^{1/2}, \qquad i = 1, \ldots, n; \tag{12}$$

here and in the following, summation over repeated over repeated Greek indices is understood. Whenever Latin indices are used, they (and their summation) run from 1 to 3.

The four-velocities and four-accelerations

$$v_i^\mu \equiv \frac{dz_i^\mu}{d\tau_i}, \qquad a_i^\mu \equiv \frac{d^2 z_i^\mu}{d\tau_i^2}, \tag{13}$$

satisfy

$$v_i^\mu v_{i\mu} = 1, \qquad v_i^\mu a_{i\mu} = 0. \tag{14}$$

The following two-body invariants can be constructed from the positions and velocities of the particles [3,8]:

$$\begin{aligned} s_{ij}^2 &\equiv (z_i^\varrho - z_j^\varrho)(z_{i\varrho} - z_{j\varrho}), \qquad \omega_{ij} \equiv v_i^\varrho v_{j\varrho}, \\ \kappa_i^j &\equiv (z_j^\varrho - z_i^\varrho)v_{i\varrho}, \qquad i,j = 1,\ldots n. \end{aligned} \tag{15}$$

Other invariants can be constructed involving higher derivatives of the positions and more than two particles, but no such generalizations will be considered here.

Analogous to the situation in Newtonian dynamics, various formalisms are possible.

3.1. *"Newtonian" formalism*

The obvious generalization of Newton's Second Law (1) is

$$m_i a_i^\mu = F_i^\mu. \tag{16}$$

Since from Eq. (14) the four-force F must be perpendicular to the four-velocity, it must be of the form [1]

$$\begin{aligned} F_i^\mu = \sum_{j \neq i} \Big\{ &(z_i^\mu - z_j^\mu + \kappa_i^j v_i^\mu)\,{}_1f_{ij} + (v_j^\mu - \omega_{ij} v_i^\mu)\,{}_2f_{ij} \\ &+ a_i^\mu\,{}_3f_{ij} + (a_j^\mu - a_j^\varrho v_{i\varrho} v_i^\mu)\,{}_4f_{ij} \Big\}_{g_{ij}=0}, \end{aligned} \tag{17}$$

if we restrict ourselves to two-body forces, and with the further restriction discussed before, the f_{ij}'s will be taken as arbitrary functions of the two-body invariants (15). Then the equations of motion most resembling the Newtonian from (2) following from Eqs. (16) and (17) are

$$m_i a_i^\mu = \sum_{j \neq i} \Big\{ (z_i^\mu - z_j^\mu + \kappa_i^j v_i^\mu) f_{ij}(s_{ij}^2) \Big\}_{g_{ij}=0}, \tag{18}$$

which still depend on the velocities, however. Furthermore, unlike the situation in Newtonian dynamics, one can not simply evaluate all quantities at the same time, since simultaneity is not a Poincaré-invariant concept. Therefore the proper times and must be related by Poincaré-invariant constraints associating one point $z_i(\tau_i)$ on the i–th world line with one or more points $z_j(\tau_j)$ on the j–th one (including the possibility of a weighted integral over the j–th world line), and vice versa; these constraints are expressed by the functions g_{ij} and g_{ji} of the two-body invariants (15). They can be arbitrarily prescribed (*Action-at-a-distance theories*) or determined from some partial differential equations

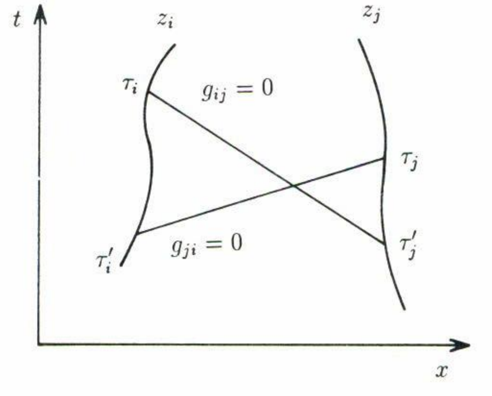

FIGURE 1. Two world lines i and j with points associated through relativistic functions g_{ij} and g_{ji}.

(*Field theories*). The force terms F_i derived from field theories may include *self-action terms* F_{ii}. In the simplest case each of the constraints may only associate a single pair of points as shown in Fig. 1.

Unfortunately the two pairs of points can only coincide in very exceptional circumstances. An example is provided by Fokker's time-asymmetric electrodynamics of two charges, one acting on the second via retarded fields, the second acting on the first via advanced ones [9].

Examples of exact solutions which can be obtained either in field theories or in action-at-a-distance theories will be discussed in detail later.

3.2. "Lagrangean" formulation

Such laws do follow, however, if the equations of motion follow from a variational principle

$$\delta(I_1 + I_2) = 0, \tag{19}$$

with

$$\begin{aligned} I_1 &\equiv -\sum_{i<j}\sum g_i g_j \iint_{-\infty}^{\infty} U_{ij}(s_{ij}, \omega_{ij}, \kappa_i, \kappa_j)\, d\tau_i\, d\tau_j, \\ I_2 &\equiv -\sum_j \int_{-\infty}^{\infty} M_j(\tau_j)(v_{j\rho} v_j^{\rho})^{\frac{1}{2}}\, d\tau_j, \end{aligned} \tag{20}$$

where the M_j are Lagrangean multipliers which have to be determined from the equations resulting from the variation (19) [3]. This is a generalization of Fokker's variational principle of electrodynamics for time-symmetric interactions [10] and can describe time-like, space-like, or light-like interactions depending on the invariants (15); it can not describe purely retarded interactions, however. (Space-like interactions depending on the s_{ij}'s alone were also considered in Refs. [11]).

For conciseness in later use if will be convenient to define "adjunct" potentials V_j by

$$V_j(z_j^\rho, v_j^\rho) \equiv \sum_{k<j} g_k \int_{-\infty}^{\infty} U_{kj}\, d\tau_k + \sum_{k>j} g_k \int_{-\infty}^{\infty} U_{jk}\, d\tau_k \tag{21}$$

in terms of the "kernels" U_{ij} of the variational principle (19), which are not assumed to be necessarily symmetric in i and j.

The resulting equations of motion are

$$\frac{dP_{j\rho}}{d\tau_j} = g_j \frac{\partial V_j}{\partial z_j^\rho} - g_j \frac{d}{d\tau_j}\left[\frac{\partial V_j}{\partial v_j^\rho} + v_{j\rho}\left(V_j - v_j^\sigma \frac{\partial V_j}{\partial v_j^\sigma}\right)\right]. \quad P_j^\rho \equiv m_j v_j^\rho, \quad j = 1, \ldots, n. \tag{22}$$

There exist *many* distinct theories with space- as well as time- or light-like interactions which are relativistic generalizations of the *same* Newtonian force laws [12].

The special case of theories possessing an *adjunct field theory* can be obtained from the variational principle (19) provided the adjunct potential (21) can be separated into a sum of terms which are products of two factors, one of which involves only the coordinates z_j of particle j. This implies that the kernel U_{ij} consists of a sum of terms each of which contains ω_{ij} only in the form ω_{ij}^ℓ, where ℓ is an integer. Such theories are discussed further in Refs. [3] and [13]; they include Fokker's time-symmetric electrodynamics ($\ell = 1$) and the variational principles of scalar or vector mesodynamics ($\ell = 0$ or 1) [14].

In close analogy of Newtonian theory

D) There always exist ten conservation laws.

Their existence is assured from Noether's Theorem by the invariance of the interaction term (19) under the 10-parameter Poincaré group [3,5,15,16]. The invariance under the four time and space translations yields the laws of conservation of energy and momentum:

$$\frac{\partial P_\mu}{\partial \tau_j} = 0,$$

$$P_\mu(\tau_1 \ldots \tau_n) \equiv \sum_j \left\{P_{j\mu}(\tau_j) + g_j\left[\frac{\partial V_j}{\partial v_j^\mu} + v_{j\mu}\left(V_j - v_j^\sigma \frac{\partial V_j}{\partial v_j^\sigma}\right)\right]\right\}$$
$$+ \sum\sum_{i<j} g_i g_j \left(\int_{\tau_i}^{\infty}\int_{-\infty}^{\tau_j} - \int_{-\infty}^{\tau_i}\int_{\tau_j}^{\infty}\right) \frac{\partial U_{ij}}{\partial s_{ij}^\mu}\, d\tau_i\, d\tau_j. \tag{23}$$

Similarly, invariance under the 6-parameter subgroup of rotations yields [3,12]

$$\frac{\partial L^{\mu\nu}}{\partial \tau_j} = 0,$$

$$\begin{aligned} L^{\mu\nu}(\tau_1 \ldots \tau_n) \equiv \sum_j \Bigg\{ & \left\{ P_j^\mu + g_j \left[\frac{\partial V_j}{\partial v_j^\rho} + v_{j\rho} \left(V_j - v_j^\sigma \frac{\partial V_j}{\partial v_j^\sigma} \right) \right] H^{\rho\mu} \right\} z_j^\nu \\ & - \left\{ P_j^\nu + g_j \left[\frac{\partial V_j}{\partial v_j^\rho} + v_{j\rho} \left(V_j - v_j^\sigma \frac{\partial V_j}{\partial v_j^\sigma} \right) \right] H^{\rho\nu} \right\} z_j^\mu \Bigg\} \\ & + \sum_{i<j}\sum \tfrac{1}{2} g_i g_j \left(\int_{\tau_i}^{\infty} \int_{-\infty}^{\tau_j} - \int_{-\infty}^{\tau_i} \int_{\tau_j}^{\infty} \right) d\tau_i \, d\tau_j \\ & \left[\frac{\partial U_{ij}}{\partial s_{ij}^\rho} \left\{ H^{\rho\mu}(z_i + z_j) - H^{\rho\nu}(z_i + z_j) \right\} \right. \\ & \left. + \frac{\partial U_{ij}}{\partial v_i^\rho} (H^{\rho\mu} v_i^\nu - H^{\rho\nu} v_i^\mu) - \frac{\partial U_{ij}}{\partial v_j^\rho} (H^{\rho\mu} v_j^\nu - H^{\rho\nu} v_j^\mu) \right], \end{aligned} \tag{24}$$

where the spatial components L^{mn} are the components of the angular momentum $\mathbf{J}$ and the components L^{m0} are those of the vector $\mathbf{M}$ of the center-of-mass theorem analogous to that of Eq. (9). For certain forms of U_{ij} additional conservation laws can exist. However, unlike the Newtonian situation, where the conservation laws involve only particle variables evaluated at the same time t, the laws (24) depend on the n proper times of the particles and involve integrals over their n world lines.

If the interactions are such that an adjunct field theory exists whose field quantities obey partial differential equations analogous to those of the standard field theory, one can obtain ten "local" conservation laws, involving both the particles and the adjunct fields, which have the same structure as those familiar from field theory. However, they too involve integrals over the world lines.

Many of the equations and results discussed here can be generalized to encompass interactions which involve transfer of electric charge (in addition to energy, momentum, and angular momentum) between particles [13,17], and particles with an intrinsic angular momentum, but can not be considered here.

Unlike the Newtonian case, in the "Newtonian" and "Lagrangean" formalisms no general statement analogous to c) can be made; at present, the problem of the specification of appropriate initial conditions can only be discussed through examples of exact solutions, which will be taken up later. Their study, however, is much more difficult than in Newtonian theory since in general

E) The two-body problem can not be reduced to a one-body problem due to the non-instanteneity of the interactions. This difficulty is related to the fact that the "center-of-mass theorem" contained in Eq. (24) does not allow the definition of a unique center of mass.

3.3. "Hamiltonian" formalism

It might appear that these difficulties can be overcome through a "Hamiltonian" formalism which by definition involves only quantities evaluated at the same time and for an n-particle system specifies the motion by $6n$ quantities. However, the early attempts at such a formulation led to various *No-Interaction Theorems* [18] showing that such a formulation with non-vanishing interactions is incompatible with the "world line condition" requiring that the particle coordinates have the proper transformation properties under the Poincaré group. The various attempts to circumvent this difficulty can not be described here; several of them were discussed at a recent conference [19].

However, even before the formal recognition of this difficulty, Thomas [20] gave up the concept of invariant world lines in relativistic dynamics, and Bakamjian and Thomas [21] showed how this allowed the introduction of interactions in a "center-of-mass system" in a "Hamiltonian" formalism. This, however, introduced a new problem, that of *cluster separability* [22]. The interaction of two particles did not become independent of the other $n-2$ particles if these two subsystems were widely separated. This difficulty was overcome first by Sokolov [23] and then in a more direct and elegant manner by Coester and Polizou [24]. Although this was done in a quantum mechanical formalism, the results are equally valid in a classical one. But the fact that the world line condition is not satisfied is more disturbing in a classical context than in a quantum mechanical one and therefore this approach will not be pursued here. However, this condition is satisfied in a certain approximation [25] and therefore the Hamiltonian approach is of interest for motions which do not deviate too much from Newtonian ones, although not for the study of exact relativistic solutions which are our main concern here, many of which do not even possess a Newtonian limit.

It is clear from the above that a fundamental distinction between Newtonian and relativistic dynamics is that in the latter

B) The "Newtonian", "Lagrangean", and "Hamiltonian" formalisms are *not* equivalent

in general. In particular, no known field theories (or their adjunct analogues) lead to an n-particle dynamics allowing a "Hamiltonian" formulation. The "Newtonian" one allows a "Lagrangean" formulation only in the special case that the force terms [the right hand side of Eq. (18)] are of the form of those of Eq. (22).

4. EXACT SOLUTIONS

In the following only the "Newtonian" formalism will be considered, since it includes the "Lagrangean" one as a subcase, and the "Hamiltonian" one, as discussed above, is not suitable for the study of the motions of interest here.

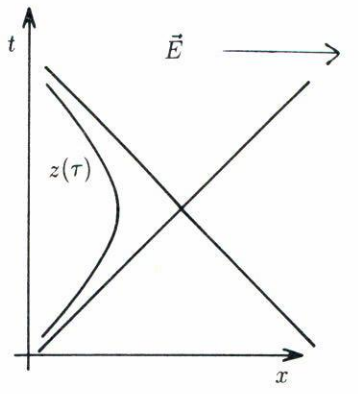

FIGURE 2. Hyperbolic motion of a charge in a constant electric field.

Before investigating motions involving two or more particles, however, we have to consider some examples of one-particle motions because some of their features are relevant for various two-particle motions. The first one is the motion of a single charge in a constant electric field E, the well-known "hyperbolic motion" shown in Fig. 2. It is actually the first exact solution obtained in relativistic dynamics, discovered by Max Born in 1909 [26] and described by

$$z_0 = \alpha \sinh(\tau/\alpha), \quad z_1 = -\alpha \cosh(\tau/\alpha), \quad z_2 = z_3 = 0 \tag{25}$$

with

$$\alpha = \frac{M}{eE}. \tag{26}$$

As was later shown by Schott [27], this is also the only solution of the "Lorentz-Dirac" equation in which the radiation reaction terms have no effect; however, for this to be true, the electric field has to extend over all space.

The next example is the motion of a point particle of mass M carrying an electric charge e and a mesic charge g, moving in an external electromagnetic field $F_{\mu\nu}$ and neutral vector meson field $G_{\mu\nu}$ whose equation of motion is [28]

$$M\dot{v}_\mu - \tfrac{2}{3}(e^2+g^2)(\ddot{v}_\mu + v_\mu \dot{v}^2) - g^2\chi^2 v^\nu \int_{-\infty}^{\tau} \frac{s_\mu v_\nu(\tau') - s_\nu v_\mu(\tau')}{s^2} J_2(\chi s)\, d\tau' = (eF_{\mu\nu} + gG_{\mu\nu})v^\nu, \tag{27}$$

if the field of the particle is assumed to be purely retarded. If it is assumed to be half-retarded, half advanced, one has instead

$$M\dot{v}_\mu - \tfrac{1}{2}g^2\chi^2 v^\nu \int_{-\infty}^{\infty} \frac{s_\mu v_\nu(\tau') - s_\nu v_\mu(\tau')}{s^2} J_2(\chi s)\, d\tau' = (eF_{\mu\nu} + gG_{\mu\nu})v^\nu. \tag{28}$$

Here a dot denotes a derivative with respect to the proper time τ, J is the Bessel function, χ is a constant (equal to $1/\hbar$ times the meson mass in the quantized theory), $s_\mu(\tau') = z_\mu(\tau) - z_\mu(\tau')$, and $s = (s_\rho s^\rho)^{1/2}$.

If the particle is a singularity of a neutral scalar meson field F_μ (with potential U) rather than a vector one, one has instead

$$M\dot{v}_\mu - \tfrac{1}{3}(2e^2 + g^2)(\ddot{v}_\mu + v_\mu \dot{v}^2) - \tfrac{1}{2}g^2\chi^2 v_\mu + g^2\chi^2 \int_{-\infty}^{\tau} \frac{s_\mu}{s^2} J_2(\chi s)\, d\tau'$$
$$+ g^2\chi \frac{d}{d\tau}\left[v_\mu \int_{-\infty}^{\tau} \frac{1}{s} J_1(\chi s)\, d\tau'\right] = eF_{\mu\nu}v^\nu + gF_\mu + g\frac{d}{dr}(Uv_\mu) \quad (29)$$

in the retarded case, and

$$M\dot{v}_\mu + \tfrac{1}{2}g^2\chi^2 \int_{-\infty}^{\infty} \frac{s_\mu}{s^2} J_2(\chi s)\, dr' + \tfrac{1}{2}g^2\chi \frac{d}{dr}\left[v_\mu \int_{-\infty}^{\infty} \frac{1}{s} J_1(\chi s)\, dr'\right]$$
$$= eF_{\mu\nu}v^\nu + gF_\mu + g\frac{d}{dr}(Uv_\mu) \quad (30)$$

in the time-symmetric one. If g vanishes, Eqs. (27) and (29) reduce to the "Lorentz-Dirac" equation.

In all of these cases, the particle can perform a hyperbolic motion described by Eq. (25) in a constant electric field E. However, for a singularity of a neutral vector meson field obeying Eq. (27) the amplitude must satisfy the relation [28]

$$-\frac{M}{\alpha} + g^2\chi^2 I_1(\alpha\chi)K_1(\alpha\chi) = -eE, \quad (31)$$

where I and K are the modified Bessel functions of the order indicated. Similarly, a singularity of a neutral scalar meson field obeying Eq. (29) can perform such a motion provided the amplitude α satisfies the relation

$$\frac{M}{\alpha} + g^2\chi^2 I_0(\alpha\chi)K_0(\alpha\chi) = +eE. \quad (32)$$

Unfortunately, Eqs. (31) and (32) cannot be solved analytically for α. It is convenient to introduce an "effective mass" M_e, *i.e.*, a quantity taking the role of M in the hyperbolic motion (25), by

$$\alpha = \frac{M_e}{eE}. \quad (33)$$

For weak fields, *i.e.*, small E and thus large α, M_e approaches $M - \frac{1}{2}g^2\chi$ and $M + \frac{1}{2}g^2\chi$, respectively. Its general behavior is shown in Fig. 3, derived from Eq. (31), and in Fig. 4, derived from Eq. (32), as a function of $\alpha\chi$. Thus, unlike Newtonian mechanics, the inertial

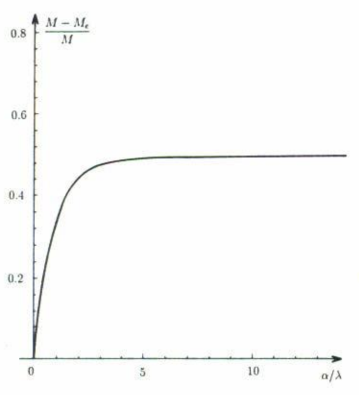

FIGURE 3. "Effective mass" behavior of a singularity of a neutral vector meson field with retarded self-interaction in hyperbolic motion.

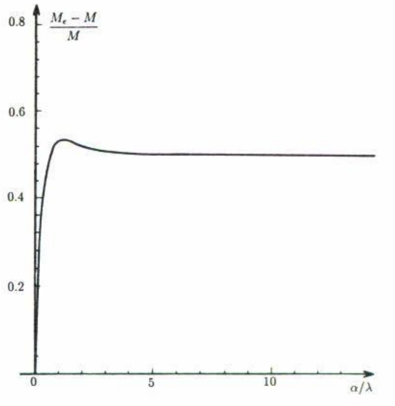

FIGURE 4. "Effective mass" behavior of a singularity of a neutral scalar meson field with retarded self-interaction in hyperbolic motion.

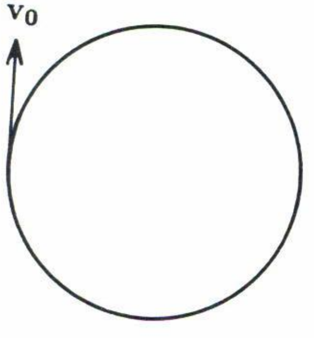

FIGURE 5. Circular motion of a singularity of a neutral vector meson field due to its retarded self-force.

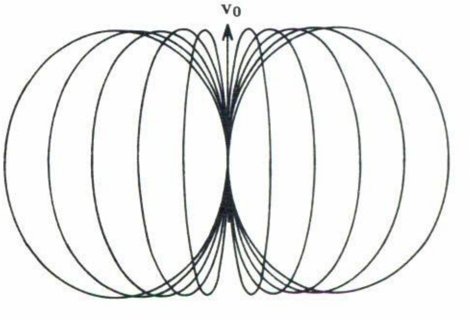

FIGURE 6. Examples of circular motion with the same initial position and velocity.

property of a mass point is not necessarily characterized by a single constant in a field theory. However, no such anomalous behavior can occur for the time-symmetric cases of Eqs. (28) and (30), or for either the retarded or the time-symmetric equations of motion of an action-at-a-distance theory of mesic interactions [14,17]. On the other hand, for the field theoretical time-symmetric case of a singularity of a neutral vector meson field an even more unusual situation can arise even in the absence of an external field [29]: The self-force implicit in Eq. (28) permits a circular motion of the particle with certain radii and velocities depending on M, g, and χ, as showing in Fig. 5. Furthermore, since the initial velocity v_0 shown there clearly is not sufficient to determine the plane of the circle, infinitely many circular motions are possible with the same v_0, as indicated in Fig. 6 [30].

This however, still does not exhaust the possible motions of the particle. As viewed form a frame of reference moving with a velocity of a magnitude equal to that of the

FIGURE 7. The motion of Fig. 5 viewed (not to sacale) from a particular frame of reference perpendicular to $\mathbf{v_0}$.

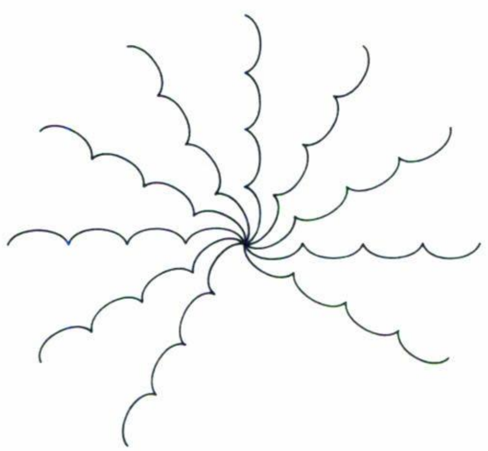

FIGURE 8. Examples of possible cycloidal motions of Fig. 7 with the same initial position and (zero) velocity.

initial velocity $\mathbf{v}_0$, but perpendicular to it, the circular motion of Fig. 5 becomes the cycloid of Fig. 7. But in this frame of reference the turning points of the cycloid have zero velocity. Thus at any such point the plane of motion is doubly undetermined. Some possible motions in one plane are indicated in Fig. 8; this plane, however, could still be rotated about any axis within that plane passing through the central point of the motions shown [30].

Thus, while in Newtonian theory a particle at rest would remain at rest, in the particular case under consideration it could perform a doubly infinite set of relativistic motions! This with initial conditions which are as "Newtonian" as they could possibly be. To select one of these motions, one would have to specify an acceleration of the magnitude appropriate for the circular motion of Fig. 5 and of an arbitrary direction. Furthermore, clearly none of these motions could be obtained by an approximation method which uses the Newtonian initial motion, *i.e.*, no motion at all, as the first approximation.

The above one-particle solutions are helpful in the study of various two-particle solutions. First, let us consider two unlike charges in a constant electric field, each performing a hyperbolic motion, as indicated in Fig. 9. But for the motions shown the two charges are *outside each other's light cones* during their entire motion. Therefore this figure also represents a solution of the electrodynamic two-body problem, whether the charges were

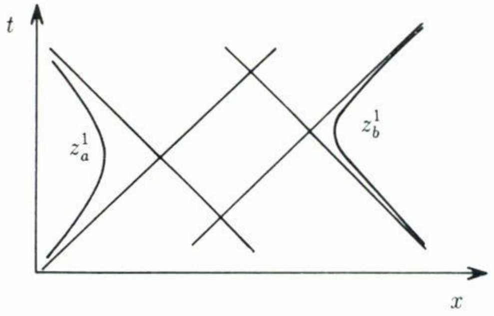

FIGURE 9. Two unlike charges in hyperbolic motion in a constant electric field.

meant to interact via retarded, advanced, or time-symmetric fields [31]. Clearly a solution of this character can not arise in Newtonian theory with its instantaneous interactions, and can not be obtained starting from a Newtonian solution as the lowest approximation. It was discovered in 1949 and is the first example of a motion having event horizons (although this term was invented much later). While the case shown in Fig. 9 represents a one-dimensional motion, clearly motions of similar character can be obtained for particles moving along parallel lines rather than along the same one.

The examples involving hyperbolic motion required the presence of an external field. An example of a one-dimensional interaction which permits motions with some properties similar to those considered, but without the need for an external field, was given by Staruszkiewicz [32]. In the notation used here, it amounts to taking I_1 in Eq. (19) as

$$I_1 = g_1 g_2 \int \int \theta \left[\left(x^0\right)^2 - \left(x^1\right)^2 \right] d\tau_1 \, d\tau_2, \tag{34}$$

where θ is the step function. Then a possible solution is shown in Fig. 10; if the particles are within each other's light cones, they interact with a constant force and therefore perform hyperbolic motions, if they are outside, there is no interaction. Thus in the example shown there is a time interval in which both particles are at rest. While Staruszkiewicz originally thought that his solution was specified by initial positions and velocities, it was pointed out that within the time interval just considered no such specification was possible; on the contrary, at times such as t_1 or t_2 not only the velocities, but also all of their derivatives vanish. Therefore the conditions at these two times are indistinguishable and thus no specification by initial conditions is possible at all. Only a knowledge of the past motion would allow the prediction of the future one.

The interaction (34) was chosen in Ref. [32] because its adjunct field theory is that of the electrodynamics of one spatial dimension. If this (physically meaningless) requirement is dropped, one can introduce four-dimensional interactions of the form [33]

$$I_1 = \sum_{i<j} \sum g_i g_j \int \int \theta(s_{ij}^2) \, d\tau_i \, d\tau_j$$

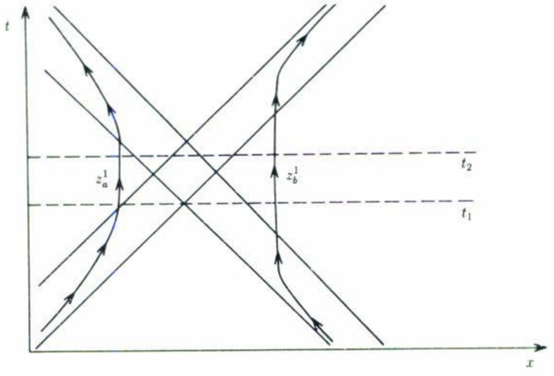

FIGURE 10. A possible motion of two particles with one-dimensional interaction described by Eq. (34) (Staruszkicwicz).

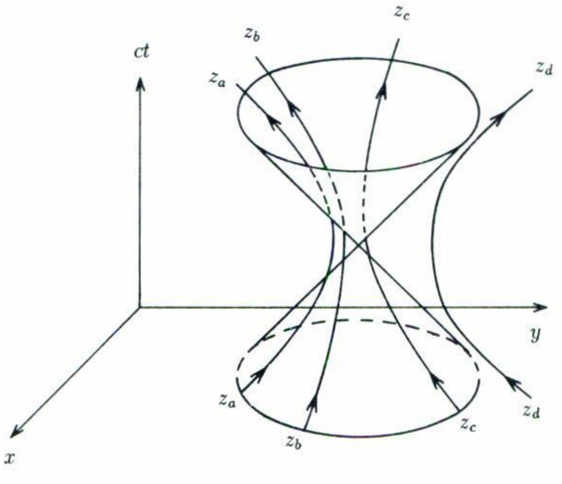

FIGURE 11. A possible motion of four particles with three-dimensional interaction described by Eq. (35).

or

$$I_1 = \sum_{i<j}\sum g_i g_j \int\int \theta(s_{ij}^2)\omega_{ij}\, d\tau_i\, d\tau_j, \qquad s_{ij}^2 = (z_i^\rho - z_j^\rho)(z_{i\rho} - z_{j\rho}), \tag{35}$$

which have properties similar to that of (34). One can then have a motion of two particles in one dimension as in Fig. 10 or of four particles in two spatial dimensions as shown in Fig. 11; further generalizations to more than two pairs of particles are possible. It should be noted that all these solutions as well as that of Staruszkiewicz are *nonanalytic*.

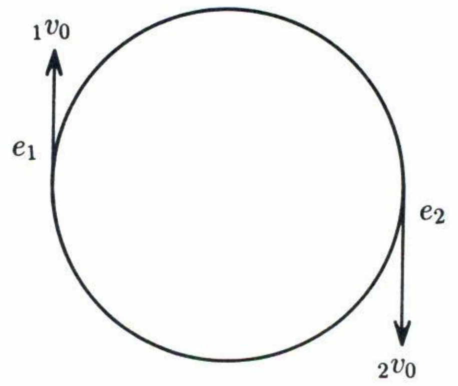

FIGURE 12. Circular motion of two unlike charges with time-symmetric interaction (Smith, Schild).

Instead of these unfamiliar interactions of unknown physical significance let us now return to the case of electrodynamics. In the following, only time-symmetric interactions will be considered to avoid the complication introduced by the possible presence of radiation reaction terms in the equations of motion. Such interactions do not pose any difficulty concerning causality here or in any other examples considered in this paper since we are only concerned with closed systems of particles [34].

The simplest two-body solution for unlike charges of equal magnitude of e/m is one where both charges move along the same circle at opposite ends of a diameter [35,36], as shown in Fig. 12. However, this is not the only possible circular motion. As shown by Chern and Havas [29], two unlike, or even two like, charges can also move in parallel circles in two planes separated by a distance d if certain relations between the masses, charges, radii, and d are satisfied. Here only the case of equal radii will be illustrated. One can have *synchronous* motion with the two charges on the same side of the circles as shown in Fig. 13 (not to scale) both in top view and in perspective, or *antisynchronous* motion with the two charges at opposite sides of the circles as shown in Fig. 14.

However, the top view of the synchronous motion is identical with that of the circular motion of a single particle of Fig. 5, and therefore, as seen in a frame of reference moving perpendicular to the initial velocities with the appropriate speed it will appear as in Fig. 7, with both charges at rest at the cusps of the cycloids. Then the figure can be rotated about an axis passing through a cusp and joining the two charges, yielding possible motions as in Fig. 8. Thus for two charges at rest an infinity of motions is possible in which they move in parallel planes at right angles to the line joining them [29]. In addition, of course, there exists a further possible motion along the line joining them just as in the Newtonian case. However, no motions as in Fig. 6 are possible except for the mirror image of Fig. 8 (not shown for clarity) since, unlike the case of a single particle, there exists a privileged plane.

The case of antisynchronous motion is more complicated and permits at most one additional circular motion, that of Fig. 12 in the plane containing the velocities of the charges of Fig. 14; conversely, the case of the circular motion of Fig. 12 may permit two additional

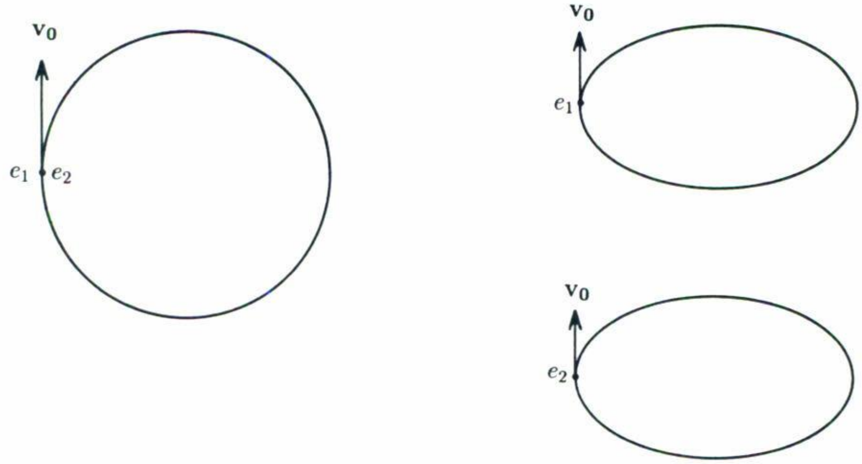

FIGURE 13. Nonplanar synchronous motion of two charges with time-symmetric interaction.

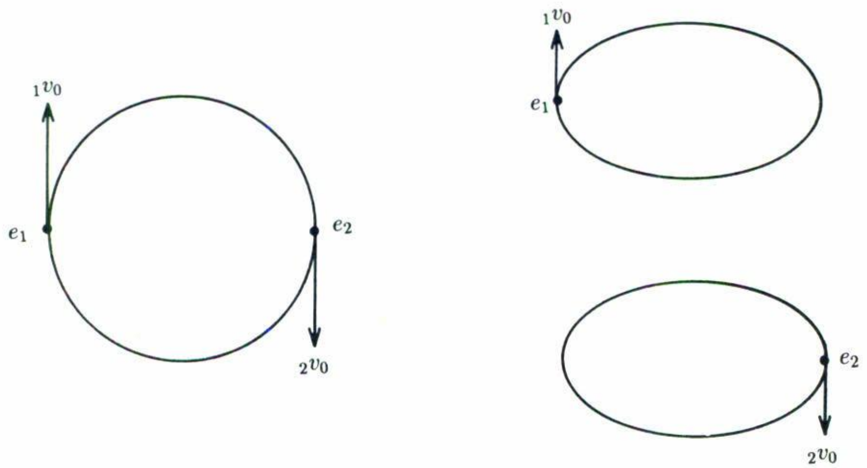

FIGURE 14. Nonplanar antisynchronous motion of two charges with time-symmetric interaction.

circular motions with the same initial condition, both of which are antisynchronous [30], as shown in Fig. 15 in perspective and in side view (not to scale).

Generalizations of both the synchronous and the antisynchronous motions to more than two particles are possible [29], but will not be considered here. All of the motions considered above show that in general

C) For given laws of motion and force laws, to determine the motion of n particles requires more than $6n$ initial conditions; there may even exist motions described by nonanalytic functions which in some range of the variables can not be specified by initial conditions at all.

These motions were chosen precisely to show that relativistic motions with properties fundamentally different from those of Newtonian dynamics are possible, many of which

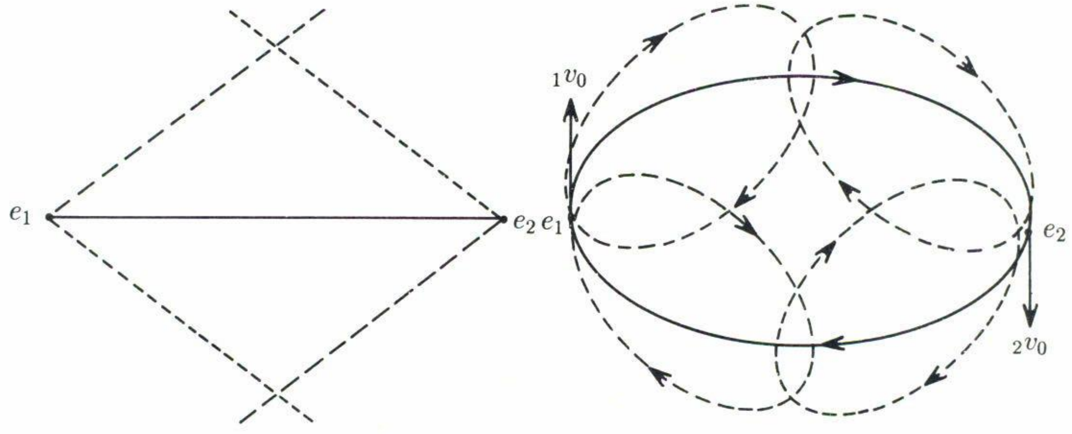

FIGURE 15. Three possible motions of two charges with time-symmetric interaction and the same initial positions and velocities (in side view and in perspective, not to scale).

do not even posses a Newtonian limit, *i.e.*, a non-trivial limit if one lets c go to zero in the expressions describing the orbits.

However, it is also possible to have a relativistic dynamics with properties which allow a variety of solutions like those of Newtonian dynamics, as shown by Havas and Plebański [1]. In the equations of motion (18) we chose for the functions g_{ij}

$$g_{ij} = \kappa_i^j. \tag{36}$$

Then Eqs. (18) reduce to

$$m_i a_i^\mu = \sum_{j \neq i} \left\{ (z_i^\mu - z_j^\mu) f_{ij}(s_{ij}^2) \right\}_{\kappa_i^j = 0}. \tag{37}$$

From the definitions (13) and (15) the vanishing of κ_i^j is equivalent to

$$t_i - t_j = \frac{1}{c^2} \left[\mathbf{r}_i(t_i) - \mathbf{r}_j(t_j) \right] \cdot \mathbf{v}_i(t_i) \tag{38}$$

in three-dimensional notation, and similarly the spatial components of Eq. (3) are

$$\frac{m_i}{\sqrt{1 - v_i^2/c^2}} \frac{d}{dt_i} \frac{\mathbf{v}_i(t_i)}{\sqrt{1 - v_i^2/c^2}} = \sum_{j \neq i} [\mathbf{r}_i(t_i) - \mathbf{r}_i(t_j)] f_{ij} [(\mathbf{r}_i - \mathbf{r}_j)^2 - c^2 (t_i - t_j)^2]. \tag{39}$$

For motions such that at all times t

$$[\mathbf{r}_i(t) - \mathbf{r}_j(t)] \cdot \mathbf{v}_i(t_i) = 0, \qquad i, j = 1, \ldots, n, \tag{40}$$

one gets from Eq. (38)

$$t_1 = t_2 = \cdots = t_n = t, \tag{41}$$

and therefore Eq. (39) reduces to

$$\frac{m_i}{\sqrt{1-v_i^2/c^2}} \frac{d}{dt} \frac{\mathbf{v}_i}{\sqrt{1-v_i^2/c^2}} = \sum_{j \neq i} \mathbf{r}_{ij} f_{ij}(r_{ij}), \tag{42}$$

where the right hand side is identical to that of Eq. (6) except for notation. Similarly, the right hand side will differ from that of Eq. (6) only by a constant factor, provided only that the motion is such that

$$v_i^2 = \text{const.} \tag{43}$$

Then Eq. (42) becomes identical to Eq. (6) with a modified, but constant, mass. Therefore any motion which satisfies Eqs. (42) and (43) is a solution of the relativistic equations of motion (37) as well as of the Newtonian ones (6) or, equivalently, any such solution of (6) is also a solution of a relativistic n–body problem, provided only that the additional restriction

$$|\mathbf{v}_i| \leq c, \qquad i = 1, \ldots, n, \tag{44}$$

holds at all times. In particular, this is true for all *homographic* motions satisfying Eqs. (42) and (43). As noted in f) above, these have been fully classified in the Newtonian case in Ref. [7]; it was also shown there that of the various Newtonian types possible four different types of solutions are also possible for the relativistic case of Eq. (42) for very general forces, and additional types of such motions can be obtained in special cases.

All the solutions of the two- and many-body problems for relativistic equations of motion other than Eqs. (42) considered above are also homographic, However, unlike the Newtonian case e), the relativistic result E) holds. Therefore, instead of f), in relativistic dynamics one has

F) For $n > 1$, the *only* known exact solutions are those with homographic motion; some of these are nonanalytic.

While the solutions just discussed of the equations of motion (42) are analogous to the Newtonian ones, this by itself does not assure that there are no other solutions with the same initial conditions. Unfortunately, the conditions (36) render these equations very difficult to integrate for motions which are not homographic. Only in the case of the motion of two particles along the same straight line has it been possible to show that under certain conditions Newtonian initial conditions are sufficient to determine the motion [1].

5. CONCLUSIONS

The particular features of Newtonian dynamics emphasized in Sect. 2 were described in the statements a) through f), and the corresponding features of special relativistic dynamics were investigated in Sects. 3 and 4 and summarized in the statements A) through F). Particular attention needs to be devoted to the fact that in special relativistic dynamics, including electrodynamics, in general the specification of initial positions and velocities is not sufficient to determine the motion and that solutions may exist which have *no Newtonian analogue and no Newtonian limit*; in such solutions, charges which in Newtonian theory will always attract or always repel each other depending on their signs, may attract *or* repel each other, or not interact at all, depending on the overall character of the motion. The fact that there may not exist a continuous transition to such motions from Newtonian ones may not be relevant physically once quantum analogues are considered because quantum theory, of course, allows discontinuous transitions. Thus one might arrive at new physical phenomena just as one arrived at the quantum phenomenon of pair creation from the apparently irrelevant classical special relativistic result that for the same momentum a particle could have either positive or negative energy, but where negative energies could be excluded since they were not accessible classically by a continuous transition from states of positive energy.

Thus special relativistic dynamics reveals many, possibly physically important, features not encompassed by Newtonian theory. The question arises naturally whether such features and possibly even more unexpected ones could also arise in the general theory of relativity. Unfortunately, we seem to be quite far from being able to attack this problem.

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