Investigación *Revista Mexicana de Física* **40**, *No. 5 (1994) 713–729*

Spinors in three dimensions. III

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Recibido el 1° de marzo de 1994; aceptado el 3 de junio de 1994

ABSTRACT. The algebraic classification of totally symmetric spinors and of trace-free totally symmetric tensors in three-dimensional spaces is considered. The geometric properties of a family of geodesics in a space with positive definite metric are expressed in terms of a one-index spinor that determines a vector field tangent to the geodesics. The usual massless free field equations in flat space-time are written as evolution equations for spinor fields in three-dimensional space.

RESUMEN. Se considera la clasificación algebraica de los espinores totalmente simétricos y de los tensores sin traza totalmente simétricos en espacios tridimensionales. Las propiedades geométricas de una familia de geodésicas en un espacio con métrica definida positiva se expresan en términos de un espinor de un índice que determina un campo vectorial tangente a las geodésicas. Las ecuaciones usuales para campos libres sin masa en espacio-tiempo plano se escriben como ecuaciones de evolución para campos espinoriales en el espacio tridimensional.

PACS: 02.40.+m; 03.50.De; 11.10.Qr

1. INTRODUCTION

This is the third of a sequence of papers devoted to the spinor formalism in three-dimensional spaces and its applications. In any three-dimensional riemannian space, a spinor calculus similar to that employed in the four-dimensional space-time of general relativity can be developed (see, *e.g.*, Refs. [1,2]). An important difference, however, is that in any three-dimensional space there exists a natural antilinear mapping of the spin space onto itself [2].

In this paper we consider the algebraic classification of totally symmetric spinors and that of trace-free totally symmetric tensors by spinor methods. We show that the geometric properties of a family of geodesics in a space with positive definite metric can be characterized in a form analogous to that found in the case of a family of null geodesics in general relativity. The massless free field equations are written in terms of SU(2) spinors and we show that the electromagnetic field determines one or two real vector fields, which are the spatial parts of the principal null directions of the electromagnetic field (see, *e.g.*, Ref. [3] and the references cited therein). In Sect. 2 we summarize the necessary background material of Refs. [1,2] and we consider the effect of complex conjugation on the spinor equivalent of a tensor. In Sect. 3, following the method employed in the spinor formalism of general relativity, the algebraic classification of totally symmetric spinors and of trace-free totally symmetric tensors is outlined. In Sect. 4, the geometric properties of a family of geodesics in a space with positive definite metric are expressed in terms of a one-index spinor associated with a vector field tangent to the geodesics and the geometrical interpretation of some of the spin-coefficients is given. In Sect. 5 the usual massless free field equations are written in space-plus-time form. In the case of the electromagnetic field it is shown that the spinor equivalents of the Poynting vector and of the Maxwell stress tensor have very simple expressions. Throughout this article, lower-case Latin indices a, b,..., range over 1, 2, 3 and capital Latin indices A, B,..., range over 1, 2.

2. PRELIMINARIES

In this section we summarize some basic facts needed for this paper. A more detailed discussion can be found in Refs. [1,2]. If $t_{ab...c}$ denote the components of an n-index three-dimensional tensor relative to an orthonormal basis, the components of its spinor equivalent are defined by

$$t_{ABCD...EF} \equiv \left(\tfrac{1}{\sqrt{2}}\sigma^a{}_{AB}\right)\left(\tfrac{1}{\sqrt{2}}\sigma^b{}_{CD}\right)\cdots\left(\tfrac{1}{\sqrt{2}}\sigma^c{}_{EF}\right)t_{ab...c}, \tag{1}$$

where the connection symbols σ_{aAB} are such that

$$\begin{aligned} \sigma_{aAB} &= \sigma_{aBA}, \\ \sigma_{aAB}\sigma_b{}^{AB} &= -2g_{ab}. \end{aligned} \tag{2}$$

The spinor indices are raised and lowered according to

$$\psi_A = \varepsilon_{AB}\psi^B, \qquad \psi^B = \varepsilon^{AB}\psi_A, \tag{3}$$

where

$$(\varepsilon_{AB}) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv (\varepsilon^{AB}), \tag{4}$$

and

$$g_{ab} = \begin{cases} \pm 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

From Eqs. (1–2) it follows that

$$t_{ab...c} = \left(-\tfrac{1}{\sqrt{2}}\sigma_a{}^{AB}\right)\left(-\tfrac{1}{\sqrt{2}}\sigma_b{}^{CD}\right)\cdots\left(-\tfrac{1}{\sqrt{2}}\sigma_c{}^{EF}\right)t_{ABCD...EF}, \tag{5}$$

$$\sigma^a{}_{AB}\sigma^b{}_{CD}g_{ab} = -(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}), \tag{6}$$

$$t_{...a...}\, s^{...a...} = -t_{...AB...}\, s^{...AB...}. \tag{7}$$

In three dimensions there are two inequivalent signatures, which can be taken as $(+++)$ and $(++-)$. Following Refs. [1,2], in the case where $(g_{ab}) = \mathrm{diag}(1,1,1)$ we choose

$$(\sigma_{1AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma_{2AB}) \equiv \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad (\sigma_{3AB}) \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{8}$$

which satisfy Eqs. (2) and, under complex conjugation,

$$\overline{\sigma_{aAB}} = -\sigma_a{}^{AB}, \tag{9}$$

while in the case where $(g_{ab}) = \mathrm{diag}(1,1,-1)$, we choose the solution of Eqs. (2) given by

$$(\sigma_{1AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma_{2AB}) \equiv \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad (\sigma_{3AB}) \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{10}$$

which obeys the relations

$$\overline{\sigma_{aAB}} = -\eta_{AR}\,\eta_{BS}\,\sigma_a{}^{RS}, \tag{11}$$

where

$$(\eta_{AB}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

The mate of a spinor ψ_A, denoted by $\hat{\psi}_A$, is defined by [2]

$$\hat{\psi}_A \equiv \begin{cases} \overline{\psi^A} & \text{if } (g_{ab}) = \mathrm{diag}(1,1,1), \\ \eta_{AB}\overline{\psi^B} & \text{if } (g_{ab}) = \mathrm{diag}(1,1,-1), \end{cases} \tag{13}$$

or, equivalently,

$$\hat{\psi}^A \equiv \begin{cases} -\overline{\psi_A} & \text{if } (g_{ab}) = \mathrm{diag}(1,1,1), \\ -\eta^{AB}\overline{\psi_B} & \text{if } (g_{ab}) = \mathrm{diag}(1,1,-1), \end{cases} \tag{14}$$

where, in accordance with the rules (3), $\eta^{AB} = \varepsilon^{CA}\varepsilon^{DB}\eta_{CD}$, *i.e.*,

$$(\eta^{AB}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Note that $\eta_{AB}\,\eta^{BC} = -\delta_A^C$. Making use of Eqs. (13–14) we obtain

$$\hat{\hat{\psi}}_A = \begin{cases} -\psi_A & \text{if } (g_{ab}) = \mathrm{diag}\ (1,1,1), \\ \psi_A & \text{if } (g_{ab}) = \mathrm{diag}(1,1,-1), \end{cases} \tag{16}$$

and

$$\overline{\alpha^A \beta_A} = \begin{cases} \hat{\alpha}^A \hat{\beta}_A & \text{if } (g_{ab}) = \text{diag}(1,1,1), \\ -\hat{\alpha}^A \hat{\beta}_A & \text{if } (g_{ab}) = \text{diag}(1,1,-1). \end{cases} \tag{17}$$

The mapping $\psi_A \mapsto \hat{\psi}_A$ is antilinear and in the case where $(g_{ab}) = \text{diag}(1,1,1)$ it is, except for a factor, the time reversal operation (see, *e.g.*, Refs. [4,5]).

If $\hat{\psi}_A = \lambda\psi_A$, then $\hat{\hat{\psi}}_A = \overline{\lambda}\hat{\psi}_A = |\lambda|^2\psi_A$ and, comparing with Eqs. (16), we see that only in the case where the metric is indefinite there exist nontrivial solutions of

$$\hat{\psi}_A = \lambda\psi_A \tag{18}$$

and necessarily $|\lambda| = 1$.

Since the connection symbols σ_{aAB} can be complex [Eqs. (8) and (10)], the spinor components $t_{AB...EF}$ given by Eq. (1) may be complex even if $t_{ab...c}$ are real. The complex conjugate of the n-index tensor $t_{ab...c}$ is defined by

$$\bar{t}_{ab...c} \equiv \overline{t_{ab...c}}. \tag{19}$$

(Since the components $t_{ab...c}$ transform by means of real matrices, the components $\overline{t_{ab...c}}$ also correspond to a tensor.) Using Eqs. (1), (9), (11) and (19) one finds that the spinor components of $\bar{t}_{ab...c}$, denoted by $\bar{t}_{AB...EF}$, are given by

$$\bar{t}_{AB...EF} = \begin{cases} (-1)^n\, \overline{t^{AB...EF}} & \text{if } (g_{ab}) = \text{diag}(1,1,1), \\ (-1)^n \eta_{AR}\,\eta_{BS} \cdots \eta_{ET}\,\eta_{FU}\, \overline{t^{RS...TU}} & \text{if } (g_{ab}) = \text{diag}(1,1,-1), \end{cases} \tag{20}$$

where n is the number of indices of $t_{ab...c}$ (*cf.* Ref. [2], Eqs. (33) and (42)). Thus, in general, the spinor components of the complex conjugate of a tensor, $\bar{t}_{AB...EF}$, do not coincide with the complex conjugates of the spinor components of that tensor, $\overline{t_{AB...EF}}$. (In fact, with σ_{aAB} given by Eqs. (8) and (10), $\bar{t}_{AB...EF}$ and $\overline{t_{AB...EF}}$ do not transform in the same manner under the spin transformations.)

3. Algebraic classification

As in the case of four-dimensional spaces, the fact that each spinor index can take only two values and that the spin transformations are given by unimodular matrices imply that the irreducible parts of an arbitrary spinor correspond to totally symmetric spinors and each totally symmetric n-index spinor $\phi_{AB...L}$ can be expressed as the symmetrized tensor product of n one-index spinors [3]

$$\phi_{AB...L} = \alpha_{(A}\beta_B \cdots \delta_{L)}, \tag{21}$$

where the parenthesis denotes symmetrization on the indices enclosed (*e.g.*, $\alpha_{(A}\beta_{B)} = \frac{1}{2}(\alpha_A\beta_B + \alpha_B\beta_A)$). This decomposition is unique except for scale factors. The existence and uniqueness of the expression (21) is a consequence of the fundamental theorem of algebra. If ξ^A is an arbitrary spinor then assuming, *e.g.*, $\xi^2 \neq 0$,

$$\begin{aligned}\phi_{AB\ldots L}\,\xi^A\xi^B\cdots\xi^L &= \phi_{11\ldots1}(\xi^1)^n + n\phi_{21\ldots1}(\xi^1)^{n-1}\xi^2 + \cdots + \phi_{22\ldots2}(\xi^2)^n\\ &= (\xi^2)^n\left\{\phi_{11\ldots1}(\xi^1/\xi^2)^n + n\phi_{21\ldots1}(\xi^1/\xi^2)^{n-1} + \cdots + \phi_{22\ldots2}\right\}\end{aligned}$$

hence, $(\xi^2)^{-n}\phi_{AB\ldots L}\,\xi^A\xi^B\cdots\xi^L$ is an nth degree polynomial in (ξ^1/ξ^2) which can be factorized in the form $\phi_{11\ldots1}(\xi^1/\xi^2 - z_1)(\xi^1/\xi^2 - z_2)\cdots(\xi^1/\xi^2 - z_n)$; therefore, $\phi_{AB\ldots L}\,\xi^A\xi^B\cdots\xi^L = \phi_{11\ldots1}(\xi^1 - z_1\xi^2)(\xi^1 - z_2\xi^2)\cdots(\xi^1 - z_n\xi^2)$, which is the product of n homogeneous first degree polynomials in ξ^A, *i.e.*,

$$\phi_{AB\ldots L}\,\xi^A\xi^B\cdots\xi^L = (\alpha_A\xi^A)(\beta_B\xi^B)\cdots(\delta_L\xi^L), \tag{22}$$

which implies Eq. (21). The spinors $\alpha_A, \beta_A, \ldots, \delta_A$, appearing in Eq. (21) are called principal spinors of $\phi_{AB\ldots L}$. Equation (22) shows that ξ_A is a principal spinor of $\phi_{AB\ldots L}$ if and only if $\phi_{AB\ldots L}\xi^A\xi^B\cdots\xi^L = 0$.

The tensor $t_{ab\ldots c}$ is trace-free and totally symmetric if and only if its spinor equivalent $t_{AB\ldots EF}$ is totally symmetric (see, *e.g.*, Ref. [6]). Thus, according to Eq. (21), if $t_{ab\ldots c}$ is an n–index trace-free, totally symmetric tensor, $t_{AB\ldots EF}$ can be expressed in the form

$$t_{AB\ldots EF} = \alpha_{(A}\beta_B\cdots\gamma_E\delta_{F)}, \tag{23}$$

and making use of Eqs. (13) and (20) it follows that

$$\bar{t}_{AB\ldots EF} = (-1)^n\,\hat{\alpha}_{(A}\hat{\beta}_B\cdots\hat{\gamma}_E\hat{\delta}_{F)}. \tag{24}$$

As in the case of the spinor formalism employed in the four-dimensional space-time of general relativity, the totally symmetric spinors of a given rank can be classified according to the multiplicity of their principal spinors. However, in the case of three-dimensional spaces, when two principal spinors are not proportional, a further subclassification can be obtained according to whether one of them is proportional to the mate of the other or not.

The simplest nontrivial case of this algebraic classification corresponds to a two-index symmetric spinor, v_{AB}, which is equivalent to a (possibly complex) vector v_a [Eqs. (1) and (5)]. The components v_{AB} can be expressed as

$$v_{AB} = \alpha_{(A}\beta_{B)}, \tag{25}$$

hence

$$v^a v_a = -v^{AB}v_{AB} = \tfrac{1}{2}(\alpha^A\beta_A)^2 \tag{26}$$

(*cf.* Eq. (7)). The vector v_a is real if and only if $\alpha_{(A}\beta_{B)} = -\hat{\alpha}_{(A}\hat{\beta}_{B)}$ [Eqs. (23–24)] which leads to the following two possibilities:

$$\text{(i)} \quad \hat{\alpha}_A = \lambda\alpha_A, \qquad \hat{\beta}_A = -\lambda^{-1}\beta_A, \tag{27}$$

which can be satisfied only if the metric is indefinite, with $|\lambda| = 1$, and

$$\text{(ii)} \quad \hat{\alpha}_A = \lambda\beta_A, \qquad \hat{\beta}_A = -\lambda^{-1}\alpha_A. \tag{28}$$

By combining Eqs. (28) one obtains,

$$\hat{\hat{\alpha}}_A = \overline{\lambda}\hat{\beta}_A = -\overline{\lambda}\lambda^{-1}\alpha_A. \tag{29}$$

When the metric is positive definite, only the case (ii) is viable and comparing Eqs. (16) and (29) we see that λ must be real. If λ is positive, from Eqs. (25) and (28) we have $v_{AB} = \alpha_{(A}\lambda^{-1}\hat{\alpha}_{B)} = \lambda^{-1/2}\alpha_{(A}\lambda^{-1/2}\hat{\alpha}_{B)}$; hence, absorbing the (real) factor $\lambda^{-1/2}$ into α_A we find that

$$v_{AB} = \alpha_{(A}\hat{\alpha}_{B)}. \tag{30}$$

If λ is negative, Eqs. (25) and (28) give $v_{AB} = -\lambda\hat{\beta}_{(A}\beta_{B)} = (-\lambda)^{-1/2}\hat{\beta}_{(A}(-\lambda)^{-1/2}\beta_{B)}$, which is also of the form (30).

On the other hand, when $(g_{ab}) = \text{diag}(1,1,-1)$, in the case (i) λ must be of the form $e^{i\theta}$; then, from Eq. (27) we have $\left(e^{i(\frac{\theta}{2}-\frac{\pi}{4})}\alpha_A\right)^{\hat{}} = e^{-i(\frac{\theta}{2}-\frac{\pi}{4})}e^{i\theta}\alpha_A = i\left(e^{i(\frac{\theta}{2}-\frac{\pi}{4})}\alpha_A\right)$ and, similarly, $\left(e^{-i(\frac{\theta}{2}-\frac{\pi}{4})}\beta_A\right)^{\hat{}} = i\left(e^{-i(\frac{\theta}{2}-\frac{\pi}{4})}\beta_A\right)$. By rewriting Eq. (25) in the form $v_{AB} = e^{i(\frac{\theta}{2}-\frac{\pi}{4})}\alpha_{(A}e^{-i(\frac{\theta}{2}-\frac{\pi}{4})}\beta_{B)}$ and absorbing the factors $e^{\pm i(\frac{\theta}{2}-\frac{\pi}{4})}$ into α_A and β_A, we find that in the case (i) v_{AB} can be expressed as

$$\text{(i)} \quad v_{AB} = \alpha_{(A}\beta_{B)}, \qquad \text{with} \quad \hat{\alpha}_A = i\alpha_A, \quad \hat{\beta}_A = i\beta_A. \tag{31}$$

Using Eqs. (17) and (26) one finds that the vectors of the form (31) are such that $v^a v_a \geq 0$.

In the case (ii), Eqs. (16) and (29) give $\lambda = \pm i|\lambda|$; then, from Eq. (28), we have $|\lambda|^{1/2}\beta_A = \pm i|\lambda|^{-1/2}\hat{\alpha}_A$. By expressing Eq. (25) in the form $v_{AB} = |\lambda|^{-1/2}\alpha_{(A}|\lambda|^{1/2}\beta_{B)}$ and absorbing the factors $|\lambda|^{\mp 1/2}$ into α_A and β_A, we conclude that in the case (ii) v_{AB} can be expressed as

$$\text{(ii)} \quad v_{AB} = \pm i\alpha_{(A}\hat{\alpha}_{B)}. \tag{32}$$

From Eqs. (17) and (26) it follows that Eq. (32) corresponds to a real vector such that $v^a v_a \leq 0$.

In the special case where $v^a v_a = 0$, Eq. (26) implies that v_{AB} must be of the form $v_{AB} = \alpha_A\alpha_B$ and v_{AB} corresponds to a real vector if and only if $\alpha_A\alpha_B = -\hat{\alpha}_A\hat{\alpha}_B$, which amounts to $\hat{\alpha}_A = \pm i\alpha_A$; therefore, the spinor equivalent of a real null vector is of the form

$$v_{AB} = \pm\alpha_A\alpha_B, \qquad \text{with} \quad \hat{\alpha}_A = i\alpha_A. \tag{33}$$

As a second example we consider a four-index totally symmetric spinor Φ_{ABCD} which is equivalent to a trace-free symmetric tensor Φ_{ab} and, according to Eq. (21), can be written as

$$\Phi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{34}$$

Making use of Eqs. (23-24) one finds that Φ_{ab} is real if and only if

$$\alpha_{(A}\beta_B\gamma_C\delta_{D)} = \hat{\alpha}_{(A}\hat{\beta}_B\hat{\gamma}_C\hat{\delta}_{D)}. \tag{35}$$

In the case with signature $(+++)$, condition (35) severely restricts the possible multiplicities in the principal spinors of Φ_{ABCD}. In fact, it is easy to see, with the help of Eq. (16), that the only possible algebraic types are

$$\Phi_{ABCD} = \pm\alpha_{(A}\hat{\alpha}_B\alpha_C\hat{\alpha}_{D)},$$
$$\Phi_{ABCD} = \alpha_{(A}\hat{\alpha}_B\beta_C\hat{\beta}_{D)}. \tag{36}$$

By contrast, when the metric is indefinite, the solutions of Eq. (35) are of the form

$$\begin{aligned}
&\Phi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}, && \text{with}\quad \hat{\alpha}_A = i\alpha_A,\ \hat{\beta}_A = i\beta_A,\ \hat{\gamma}_A = i\gamma_A,\ \hat{\delta}_A = i\delta_A,\\
&\Phi_{ABCD} = i\alpha_{(A}\beta_B\gamma_C\hat{\gamma}_{D)}, && \text{with}\quad \hat{\alpha}_A = i\alpha_A,\ \hat{\beta}_A = i\beta_A,\\
&\Phi_{ABCD} = \pm\alpha_{(A}\hat{\alpha}_B\beta_C\hat{\beta}_{D)}, && \\
&\Phi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}, && \text{with}\quad \hat{\alpha}_A = i\alpha_A,\ \hat{\beta}_A = i\beta_A,\ \hat{\gamma}_A = i\gamma_A,\\
&\Phi_{ABCD} = \pm i\alpha_{(A}\alpha_B\beta_C\hat{\beta}_{D)}, && \text{with}\quad \hat{\alpha}_A = i\alpha_A,\\
&\Phi_{ABCD} = \pm\alpha_{(A}\alpha_B\beta_C\beta_{D)}, && \text{with}\quad \hat{\alpha}_A = i\alpha_A,\ \hat{\beta}_A = i\beta_A,\\
&\Phi_{ABCD} = \pm\alpha_{(A}\alpha_B\hat{\alpha}_C\hat{\alpha}_{D)}, && \\
&\Phi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}, && \text{with}\quad \hat{\alpha}_A = i\alpha_A,\ \hat{\beta}_A = i\beta_A,\\
&\Phi_{ABCD} = \pm\alpha_A\alpha_B\alpha_C\alpha_D && \text{with}\quad \hat{\alpha}_A = i\alpha_A.
\end{aligned} \tag{37}$$

Making use of the identity

$$\alpha_A\beta_B - \alpha_B\beta_A = \alpha^R\beta_R\varepsilon_{AB} \tag{38}$$

and Eq. (6), a straightforward computation shows that if $v_{AB} \equiv \alpha_{(A}\beta_{B)}$ and $w_{AB} \equiv \gamma_{(A}\delta_{B)}$, then Eq. (34) amounts to

$$\Phi_{ABCD} = \tfrac{1}{2}(v_{AB}w_{CD} + w_{AB}v_{CD}) - \tfrac{1}{6}v^{EF}w_{EF}(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}) \tag{39}$$

or, equivalently,

$$\Phi_{ab} = v_{(a}w_{b)} - \tfrac{1}{3}v^c w_c g_{ab}, \tag{40}$$

where v_a and w_a are the vector equivalents of v_{AB} and w_{AB}, respectively.

In the case of the tensors represented by Eqs. (36), the only combinations of the principal spinors that produce real vectors are $\pm\alpha_{(A}\hat{\alpha}_{B)}$ and $\pm\beta_{(A}\hat{\beta}_{B)}$. Therefore, in a three-dimensional space with positive definite metric, at each point where it does not vanish, a trace-free symmetric tensor field Φ_{ab} determines at least one and at most two real directions. Since any two-index symmetric tensor can be expressed as the sum of a trace-free symmetric tensor and a multiple of the metric tensor, the same conclusion holds for any two-index symmetric tensor (*cf.* Sect. 5.2).

4. CONGRUENCES OF CURVES

In this section we shall restrict ourselves to spaces with a positive definite metric. Then, as shown in Eq. (30), any real vector t_a can be expressed in the form

$$t_{AB} = \alpha_{(A}\hat{\alpha}_{B)}. \tag{41}$$

Proposition. The vector field (41) is tangent to a geodesic if and only if

$$\alpha^A\hat{\alpha}^B\alpha^C\nabla_{AB}\alpha_C = 0. \tag{42}$$

Note that $\alpha^A\hat{\alpha}^B\nabla_{AB} = -t^a\nabla_a$ is real and, according to Eqs. (13–14), the complex conjugate of Eq. (42) is

$$\alpha^A\hat{\alpha}^B\hat{\alpha}^C\nabla_{AB}\hat{\alpha}_C = 0. \tag{43}$$

Proof. Using the identity (38) we see that for an arbitrary spinor ξ_A,

$$\xi_A = \frac{1}{\alpha^R\hat{\alpha}_R}\big(\hat{\alpha}_A\alpha^B\xi_B - \alpha_A\hat{\alpha}^B\xi_B\big),$$

hence,

$$\alpha^A\hat{\alpha}^B\nabla_{AB}\alpha_C = \frac{1}{\alpha^R\hat{\alpha}_R}\big(\hat{\alpha}_C\alpha^D\alpha^A\hat{\alpha}^B\nabla_{AB}\alpha_D - \alpha_C\hat{\alpha}^D\alpha^A\hat{\alpha}^B\nabla_{AB}\alpha_D\big) \tag{44}$$

and, similarly,

$$\alpha^A\hat{\alpha}^B\nabla_{AB}\hat{\alpha}_C = \frac{1}{\alpha^R\hat{\alpha}_R}\big(\hat{\alpha}_C\alpha^D\alpha^A\hat{\alpha}^B\nabla_{AB}\hat{\alpha}_D - \alpha_C\hat{\alpha}^D\alpha^A\hat{\alpha}^B\nabla_{AB}\hat{\alpha}_D\big).$$

Thus, the spinor equivalent of $t_a\nabla^a t_b$ is given by

$$
\begin{aligned}
-t_{AB}\nabla^{AB}t_{CD} &= -\alpha_A\hat{\alpha}_B\big(\alpha_{(C}\nabla^{AB}\hat{\alpha}_{D)} + \hat{\alpha}_{(D}\nabla^{AB}\alpha_{C)}\big) \\
&= -\frac{1}{\alpha^R\hat{\alpha}_R}\big\{\alpha_{(C}\hat{\alpha}_{D)}\alpha^E\alpha^A\hat{\alpha}^B\nabla_{AB}\hat{\alpha}_E - \alpha_C\alpha_D\hat{\alpha}^E\alpha^A\hat{\alpha}^B\nabla_{AB}\hat{\alpha}_E \\
&\qquad + \hat{\alpha}_D\hat{\alpha}_C\alpha^E\alpha^A\hat{\alpha}^B\nabla_{AB}\alpha_E - \hat{\alpha}_{(D}\alpha_{C)}\hat{\alpha}^E\alpha^A\hat{\alpha}^B\nabla_{AB}\alpha_E\big\} \\
&= -\frac{1}{\alpha^R\hat{\alpha}_R}\big\{t_{CD}\alpha^A\hat{\alpha}^B\nabla_{AB}\big(\alpha^E\hat{\alpha}_E\big) - \alpha_C\alpha_D\hat{\alpha}^E\alpha^A\hat{\alpha}^B\nabla_{AB}\hat{\alpha}_E \\
&\qquad + \hat{\alpha}_C\hat{\alpha}_D\alpha^E\alpha^A\hat{\alpha}^B\nabla_{AB}\alpha_E\big\},
\end{aligned}
\tag{45}
$$

which is proportional to t_{CD} if and only if Eqs. (42–43) are fulfilled.

From Eqs. (45) it also follows that (41) is tangent to an affinely parameterized geodesic if and only if, in addition to Eq. (42), $\alpha^A\hat{\alpha}_A$ is constant along the geodesic.

Given a congruence of curves (*i.e.*, a family of curves such that through each point there passes one curve in this family) we define a spinor field o_A such that

$$t_{AB} = o_{(A}\hat{o}_{B)} \tag{46}$$

are the spinor components of a tangent vector to the congruence and

$$o^A\hat{o}_A = 1. \tag{47}$$

Note that $t^a t_a = \frac{1}{2}$ [Eq. (26)] and that Eqs. (46–47) define o_A up to a factor of the form $e^{i\gamma}$. Making use of the definitions

$$
\begin{aligned}
\kappa &\equiv o^A o^B o^C\nabla_{AB}o_C, \\
\alpha &\equiv o^A\hat{o}^B o^C\nabla_{AB}o_C, \\
\beta &\equiv o^A o^B\hat{o}^C\nabla_{AB}o_C = o^A o^B o^C\nabla_{AB}\hat{o}_C, \\
\rho &\equiv o^A o^B\hat{o}^C\nabla_{AB}\hat{o}_C, \\
\varepsilon &\equiv o^A\hat{o}^B\hat{o}^C\nabla_{AB}o_C = o^A\hat{o}^B o^C\nabla_{AB}\hat{o}_C,
\end{aligned}
\tag{48}
$$

or, equivalently,

$$
\begin{aligned}
\overline{\kappa} &\equiv -\hat{o}^A\hat{o}^B\hat{o}^C\nabla_{AB}\hat{o}_C, \\
\overline{\alpha} &\equiv o^A\hat{o}^B\hat{o}^C\nabla_{AB}\hat{o}_C, \\
\overline{\beta} &\equiv \hat{o}^A\hat{o}^B o^C\nabla_{AB}\hat{o}_C = \hat{o}^A\hat{o}^B\hat{o}^C\nabla_{AB}o_C, \\
\overline{\rho} &\equiv -\hat{o}^A\hat{o}^B o^C\nabla_{AB}o_C, \\
\overline{\varepsilon} &\equiv -o^A\hat{o}^B o^C\nabla_{AB}\hat{o}_C = -o^A\hat{o}^B\hat{o}^C\nabla_{AB}o_C,
\end{aligned}
\tag{49}
$$

which amount to Eqs. (31a–c) of Ref. [1] with

$$D = -o^A \hat{o}^B \partial_{AB}, \qquad \delta = o^A o^B \partial_{AB}, \qquad \overline{\delta} = -\hat{o}^A \hat{o}^B \partial_{AB}, \tag{50}$$

from Eq. (42) we see that $D = t^a \partial_a = -t^{AB}\partial_{AB}$ is tangent to a congruence of geodesics if and only if $\alpha = 0$.

Using Eqs. (47) and (48) one finds that under the transformation $o_A \mapsto e^{i\theta/2} o_A$, where θ is a real function, which preserves conditions (46–47), the spin-coefficients (48) transform according to

$$\begin{aligned} &\kappa \mapsto e^{2i\theta}\kappa, \qquad \alpha \mapsto e^{i\theta}\alpha, \qquad \rho \mapsto \rho, \\ &\beta \mapsto e^{i\theta}(\beta - \frac{i}{2}\delta\theta), \qquad \varepsilon \mapsto \varepsilon + \frac{i}{2}D\theta, \end{aligned} \tag{51}$$

which are precisely Eqs. (39) of Ref. [1]. Therefore, choosing θ in such a way that $D\theta = 2i\varepsilon$ the new ε vanishes. In particular, if D is tangent to a congruence of geodesics, $\alpha = 0$ and we can always make $\varepsilon = 0$. Equations (44) and (48) show that α and ε vanish if and only if o_A (and hence $\hat{o}_A$) is covariantly constant along the geodesics,

$$o^A \hat{o}^B \nabla_{AB} o_C = 0.$$

This last condition implies that the triad D, δ and $\overline{\delta}$ is parallelly transported along the geodesics.

Given a system of coordinates x^μ ($\mu = 1, 2, 3$), the functions $x^\mu(u, v)$ define a one-parameter family of geodesics if for a given value of v, the curve $x^\mu(u) = x^\mu(u, v)$ is geodetic. The vector field $\zeta^\mu \equiv \partial x^\mu(u, v)/\partial v$ measures the displacement of neighboring geodesics and $t^\mu \equiv \partial x^\mu(u, v)/\partial u$ is tangent to the geodesics. Then, $t^\mu \partial\zeta^\nu/\partial x^\mu = \partial\zeta^\nu/\partial u = \partial^2 x^\nu/\partial u \partial v = \partial t^\nu/\partial v = \zeta^\mu \partial t^\nu/\partial x^\mu$ or, equivalently,

$$[t, \zeta] = 0, \tag{52}$$

where t and ζ are the differential operators (or vector fields) $t = t^\mu \partial/\partial x^\mu$, $\zeta = \zeta^\mu \partial/\partial x^\mu$. Any vector field ζ^a satisfying Eq. (52) is said to be a connecting vector of the congruence. (Equation (52) means that the Lie derivative of ζ^a with respect to t^a vanishes.)

Writing $t = D$ and $\zeta = fD + \overline{w}\delta + w\overline{\delta}$, where f is a real function and w is a complex function, making use of Eqs. (33) of Ref. [1] and the properties of the commutator (or Lie bracket) one finds that

$$\begin{aligned} [t, \zeta] = {} & (Df + 2\alpha\overline{w} + 2\overline{\alpha}w)D + (D\overline{w} + (2\varepsilon - \rho)\overline{w} - \overline{\kappa}w)\delta \\ & + (Dw + (-2\varepsilon - \overline{\rho})w - \kappa\overline{w})\overline{\delta}, \end{aligned}$$

hence, assuming $\varepsilon = 0$, ζ is a connecting vector for a congruence of geodesics with tangent vector D if and only if

$$Df = 0 \tag{53}$$

and

$$Dw = \overline{\rho}w + \kappa\overline{w}. \tag{54}$$

Equation (53) implies that if ζ is orthogonal to D at some point of a geodesic, then it is orthogonal to D along that geodesic. (Note that $D = (1/\sqrt{2})d/ds$, where s is the arc length.) In what follows we set $f = 0$; therefore, ζ is orthogonal to the congruence of geodesics everywhere and we can write

$$\zeta = x\partial_1 + y\partial_2,$$

where $\partial_1 = (\delta + \overline{\delta})/\sqrt{2}$, $\partial_2 = i(\overline{\delta} - \delta)/\sqrt{2}$ form an orthonormal basis of the normal planes to the geodesics and

$$w = \tfrac{1}{\sqrt{2}}(x + iy). \tag{55}$$

In order to find the geometrical interpretation of the functions $\Theta \equiv \operatorname{Re}\rho$, $\omega \equiv \operatorname{Im}\rho$ and κ, we consider first the case where $\kappa = 0$ and $\omega = 0$, then substituting Eq. (55) into Eq. (54) one finds that $Dx = \Theta x$, $Dy = \Theta y$, which means that as one moves along a geodesic, any connecting vector ζ orthogonal to D expands ($\Theta > 0$) or contracts ($\Theta < 0$), maintaining its orientation with respect to the axes ∂_1 and ∂_2, *i.e.*, the congruence is expanding ($\Theta > 0$) or contracting ($\Theta < 0$). In fact, using, *e.g.*, Eq. (42) of Ref. [1] it follows that $\operatorname{div} D = 2\Theta$.

When $\kappa = 0$ and $\Theta = 0$, Eqs. (54–55) give $Dx = \omega y$, $Dy = -\omega x$, which corresponds to a rigid rotation of the connecting vector relative to the axes ∂_1 and ∂_2. If $\rho = 0$ and κ is real, from Eqs. (54–55) we get $Dx = \kappa x$, $Dy = -\kappa y$, which correspond to a volume-preserving shear with principal axes ∂_1 and ∂_2. When κ is complex then, at a given point, κ is of the form $\kappa = |\kappa_0|e^{i\chi_0}$ and from Eqs. (51) one finds that under the transformation $o_A \mapsto e^{-i\chi_0/4}o_A$ (which preserves the condition $\varepsilon = 0$ and corresponds to a rotation through an angle $-\chi_0/2$ about D), $\kappa \mapsto |\kappa_0|$ at that point. Therefore, Eq. (54) with $\rho = 0$ and κ complex corresponds to a volume-preserving shear with principal axes that form an angle $-(\arg\kappa)/2$ with respect to ∂_1 and ∂_2.

Thus, D is tangent to a shear-free congruence of geodesics if and only if $\alpha = \kappa = 0$ which, according to Eqs. (48), is equivalent to the condition

$$o^A o^C \nabla_{AB} o_C = 0. \tag{56}$$

Similarly, the vector field (41) is tangent to a shear-free congruence of geodesics if and only if

$$\alpha^A \alpha^C \nabla_{AB} \alpha_C = 0, \tag{57}$$

even if $\alpha^A \hat{\alpha}_A$ is not constant. Indeed, assuming that α_A is different from zero, we can define $o_A \equiv (\alpha^R \hat{\alpha}_R)^{-1/2}\alpha_A$, which satisfies Eq. (47), then $\alpha^A \alpha^C \nabla_{AB} \alpha_C = (\alpha^R \hat{\alpha}_R)^{3/2} o^A o^C \nabla_{AB} o_C = 0$, where we have made use of Eq. (56).

5. Massless fields

In this section we consider the usual massless free field equations in flat space-time. These equations are written here in terms of SU(2) spinors, in a form that is manifestly covariant under spatial rotations only.

5.1. *Weyl neutrino field*

The Weyl neutrino equation for the two-component neutrino field, ψ, is given by

$$\frac{1}{c}\frac{\partial}{\partial t}\psi = \boldsymbol{\sigma}\cdot\nabla\psi, \tag{58}$$

where the σ_a are the Pauli matrices [7,8] or, equivalently,

$$\sqrt{2}\,\nabla^B{}_A\,\psi_B = -\frac{1}{c}\frac{\partial}{\partial t}\psi_A. \tag{59}$$

Making use of Eqs. (14) and (20) one finds that the complex conjugate of Eq. (59) is

$$\sqrt{2}\,\nabla^B{}_A\,\hat{\psi}_B = \frac{1}{c}\frac{\partial}{\partial t}\hat{\psi}_A. \tag{60}$$

(It may be noticed that $\hat{\psi}_A$ satisfies the equation for the antineutrino [7,8].)

Equations (59–60) lead to the continuity equation

$$c\sqrt{2}\,\nabla^{AB}\,\psi_A\,\hat{\psi}_B = \psi_A\,\frac{\partial}{\partial t}\,\hat{\psi}^A - \hat{\psi}_B\,\frac{\partial}{\partial t}\,\psi^B = -\frac{\partial}{\partial t}\,(\psi^A\hat{\psi}_A),$$

which is of the form $\operatorname{div}\mathbf{J} + \partial\rho_n/\partial t = 0$, with

$$J_{AB} \equiv -c\sqrt{2}\,\psi_{(A}\hat{\psi}_{B)}, \qquad \rho_n \equiv \psi^A\hat{\psi}_A. \tag{61}$$

(Using Eq. (26) we obtain $J_aJ^a = c^2{\rho_n}^2$, *i.e.*, $|\mathbf{J}| = \rho_n c$; hence, the four-vector $(\rho_n c, \mathbf{J})$ is null.)

Looking for plane wave solutions of Eq. (59) of the form $\psi_A = \alpha_A e^{i(k_a x^a - \omega t)}$, where the x^a are cartesian coordinates, and k_a and α_A are constant, we get

$$\sqrt{2}\,k^B{}_A\,\alpha_B = \frac{\omega}{c}\alpha_A, \tag{62}$$

where k_{AB} are the spinor components of k_a. Since k_a is real, from Eq. (30) it follows that

$$k_{AB} = -\sqrt{2}\left(\frac{\omega}{c}\right)\frac{\alpha_{(A}\hat{\alpha}_{B)}}{\alpha^R\hat{\alpha}_R}. \tag{63}$$

(The minus sign appearing in Eq. (63) corresponds to the fact that, for a neutrino with positive energy, the spin and the momentum are antiparallel.)

5.2. Electromagnetic field

As shown in Ref. [1], the source-free Maxwell equations in vacuum can be written as

$$\sqrt{2}\,\nabla^{C}{}_{(A}F_{B)C} = \frac{1}{c}\frac{\partial}{\partial t}F_{AB}, \qquad \nabla^{AB}F_{AB} = 0, \tag{64}$$

where F_{AB} are the spinor components of the complex vector field $\mathbf{F} \equiv \mathbf{E} + i\mathbf{B}$ (*cf.* Eqs. (59–60)). The Maxwell stress tensor, T_{ab}, is given by $4\pi T_{ab} = E_aE_b + B_aB_b - \frac{1}{2}(E_cE^c + B_cB^c)g_{ab} = F_{(a}\overline{F}_{b)} - \frac{1}{2}F_c\overline{F}^c g_{ab}$; therefore, making use of Eqs. (6), (7) and (38) one finds that the spinor equivalent of T_{ab} is given by $16\pi T_{ABCD} = 2(F_{AB}\overline{F}_{CD} + \overline{F}_{AB}F_{CD}) - F_{RS}F^{RS}(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{BC}\varepsilon_{AD}) = F_{AC}\overline{F}_{BD} + F_{AD}\overline{F}_{BC} + F_{BD}\overline{F}_{AC} + F_{BC}\overline{F}_{AD}$, *i.e.*,

$$4\pi T_{AB}{}^{CD} = F_{(A}{}^{(C}\,\overline{F}_{B)}{}^{D)}. \tag{65}$$

Similarly, using Eq. (16) of Ref. [1], one finds that the spinor equivalent of the Poynting vector, $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B} = (ic/8\pi)\mathbf{F}\times\overline{\mathbf{F}}$, is given by

$$S_{AB} = \frac{c}{4\pi\sqrt{2}}F_{R(A}\,\overline{F}_{B)}{}^{R}. \tag{66}$$

The symmetry of F_{AB} implies that

$$F_{AB} = \alpha_{(A}\,\beta_{B)} \tag{67}$$

(*cf.* Eq. (21)). Thus, $F_aF^a = -F_{AB}F^{AB} = \frac{1}{2}(\alpha^A\beta_A)^2$. On the other hand, $F_aF^a = E_aE^a - B_aB^a + 2iE_aB^a = \mathbf{E}^2 - \mathbf{B}^2 + 2i\mathbf{E}\cdot\mathbf{B}$; therefore, $\beta_A = \lambda\alpha_A$ if and only if $\mathbf{E}^2 = \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B} = 0$. By absorbing the factor $\lambda^{1/2}$ into α_A, we obtain

$$F_{AB} = \alpha_A\alpha_B \quad \Leftrightarrow \quad \mathbf{E}^2 = \mathbf{B}^2 \quad \text{and} \quad \mathbf{E}\cdot\mathbf{B} = 0, \tag{68}$$

where the principal spinor α_A is defined up to sign. The electromagnetic fields with $\mathbf{E}^2 = \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B} = 0$ are called degenerate, algebraically special, null or pure radiation fields.

The principal spinors α_A and β_A of F_{AB} define the real vector fields

$$v_{AB} \equiv \alpha_{(A}\,\hat{\alpha}_{B)}, \qquad w_{AB} \equiv \beta_{(A}\,\hat{\beta}_{B)}. \tag{69}$$

It may be noticed that α_A and β_A are defined by Eq. (67) up to the transformation $\alpha_A \mapsto \lambda\alpha_A$, $\beta_A \mapsto \lambda^{-1}\beta_A$, which induces the transformation $v_a \mapsto |\lambda|^2 v_a$, $w_a \mapsto |\lambda|^{-2}w_a$, on the vector fields (69). This means that a nondegenerate electromagnetic field defines, at each point of the space, two real vectors whose direction and sense are uniquely defined. In general, the direction of v_a or w_a does not coincide with that of the electric or the magnetic field. Substituting Eq. (67) into Eq. (66), making use of Eqs. (24) and (38), one finds that

$$S_{AB} = \frac{c}{8\pi\sqrt{2}}\big(\alpha^R\hat{\alpha}_R w_{AB} + \beta^R\hat{\beta}_R v_{AB}\big), \tag{70}$$

which, by virtue of Eq. (26), amounts to

$$\mathbf{S} = \frac{c}{8\pi}\big(|\mathbf{v}|\mathbf{w} + |\mathbf{w}|\mathbf{v}\big). \tag{71}$$

Thus, $\mathbf{S}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ and it makes equal angles with $\mathbf{v}$ and $\mathbf{w}$. On the other hand, from Eqs. (24), (65) and (67) it follows that the spinor equivalent of the trace-free part of the Maxwell stress tensor, $\tilde{T}_{ab} = T_{ab} - \frac{1}{3}T_c{}^c g_{ab}$, is given by

$$4\pi\tilde{T}_{ABCD} = F_{(AB}\overline{F}_{CD)} = -\alpha_{(A}\,\beta_B\,\hat{\alpha}_C\,\hat{\beta}_{D)} \tag{72}$$

therefore (*cf.* Eqs. (39–40))

$$4\pi\tilde{T}_{ab} = -v_{(a}w_{b)} + \tfrac{1}{3}v^c w_c g_{ab}$$

and

$$4\pi T_{ab} = -v_{(a}w_{b)} + \tfrac{1}{3}(4\pi T_c{}^c + v^c w_c)g_{ab}.$$

Using Eqs. (26), (38), (65), (67) and (69) we obtain

$$\begin{aligned} 4\pi T_c{}^c + v^c w_c &= \tfrac{1}{2}F_{AB}\overline{F}^{AB} - v^{AB}w_{AB} = -\tfrac{3}{4}\alpha^A\beta_A\,\hat{\alpha}^B\hat{\beta}_B \\ &= -\tfrac{3}{4}\big(|\mathbf{v}|\,|\mathbf{w}| - v^c w_c\big), \end{aligned}$$

therefore,

$$4\pi T_{ab} = -v_{(a}w_{b)} - \tfrac{1}{4}\big(|\mathbf{v}|\,|\mathbf{w}| - v^c w_c\big)g_{ab}. \tag{73}$$

In the case of a degenerate electromagnetic field [Eq. (68)], Eqs. (71) and (73) reduce to

$$\mathbf{S} = \frac{c}{4\pi}|\mathbf{v}|\mathbf{v}, \qquad 4\pi T_{ab} = -v_a v_b. \tag{74}$$

If the electromagnetic field satisfies the source-free Maxwell equations (64), the vector field $\mathbf{v}$ (and hence $\mathbf{S}$) is tangent to a shear-free congruence of geodesics. (This result is a special case of the Mariot-Robinson theorem [9,10] which applies in curved space-time.) Indeed, if $F_{AB} = \alpha_A\alpha_B$ represents an algebraically special electromagnetic field that satisfies the source-free Maxwell equations, the only nonvanishing component of F_{AB} with respect to the triad defined by

$$o_A \equiv \left(\alpha^R \hat{\alpha}_R\right)^{-1/2} \alpha_A \tag{75}$$

is F_{22}; then, making use of the Maxwell equations in the form given by Eqs. (66) of Ref. [1], one obtains

$$\kappa = \alpha = 0, \tag{76}$$

and

$$(\delta - 2\beta)F_{-1} = 0,$$
$$(D + 2\varepsilon + \overline{\rho})F_{-1} = -\frac{1}{\sqrt{2}\,c}\frac{\partial}{\partial t}F_{-1}, \tag{77}$$

where $F_{-1} \equiv F_{22}$ (*cf.* Eqs. (41) of Ref. [1]). Equations (76) imply that the congruence with tangent vector $v_{AB} = \alpha_{(A}\hat{\alpha}_{B)}$ is shear-free and geodetic.

Conversely, given a shear-free congruence of geodesics, there exists an algebraically special solution of the source-free Maxwell equations $F_{AB} = \alpha_A \alpha_B$ such that $\alpha_{(A}\hat{\alpha}_{B)}$ is tangent to the congruence. Indeed, by choosing the triad D, δ, $\overline{\delta}$, in such a way that D is tangent to the congruence, $\kappa = \alpha = 0$. Hence, the source-free Maxwell equations for an electromagnetic field with $F_{11} = F_{12} = 0$ reduce to Eqs. (77). Since $[D, \delta] = (2\varepsilon - \rho)\delta$ (Ref. [1], Eq. (33)), the integrability condition of Eqs. (77) is

$$D(2\beta F_{-1}) - \delta\left((-2\varepsilon - \overline{\rho})F_{-1} - \frac{1}{\sqrt{2}\,c}\frac{\partial}{\partial t}F_{-1}\right) = (2\varepsilon - \rho)2\beta F_{-1}.$$

Using again Eqs. (77) and Eqs. (58b–c) of Ref. [1] one finds that this condition is satisfied identically. The solution of Eqs. (77) is not unique; in fact, it contains an arbitrary analytic function of two complex variables. For example, the straight lines through a given point form a shear-free congruence of geodesics. In fact, if we introduce the triad

$$D = \frac{1}{\sqrt{2}}\frac{\partial}{\partial r}, \qquad \delta = \frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial \theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial \phi}\right), \qquad \overline{\delta} = \frac{1}{\sqrt{2}\,r}\left(\frac{\partial}{\partial \theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial \phi}\right), \tag{78}$$

where r, θ, ϕ are spherical coordinates, then D is tangent to the congruence and it is easy to see that the spin-coefficients are [1]

$$\kappa = \alpha = \varepsilon = 0, \qquad \beta = -\frac{1}{2\sqrt{2}\,r}\cot\theta, \qquad \rho = \frac{1}{\sqrt{2}\,r}. \tag{79}$$

Substituting Eqs. (78–79) into Eqs. (77) one finds that

$$F_{-1} = \frac{1}{r\sin\theta} f\left(r - ct, \cot\tfrac{\theta}{2} e^{-i\phi}\right),$$

where f is an arbitrary analytic function. However, in this case, for any choice of the function f, F_{-1} will diverge in some direction.

5.3. The massless free field equations for arbitrary spin

The massless free field equations for spin s are given by

$$\sqrt{2}\,\nabla^R{}_{(A}\phi_{B\ldots L)R} = \pm\frac{1}{c}\frac{\partial}{\partial t}\phi_{AB\ldots L}, \tag{80}$$

and

$$\nabla^{AB}\phi_{AB\ldots L} = 0, \tag{81}$$

where $\phi_{AB\ldots L}$ is a $2s$–index totally symmetric spinor and the sign in the right-hand side of Eq. (80) depends on the helicity of the field (*cf.* Eqs. (59–60) and (64)).

Let

$$\hat{\phi}_{AB\ldots L} \equiv \overline{\phi^{AB\ldots L}} \tag{82}$$

(or, equivalently, $\hat{\phi}^{AB\ldots L} = (-1)^{2s}\overline{\phi_{AB\ldots L}}$, *cf.* Eqs. (13–14) and (20)), then, using the fact that $\overline{\nabla^A{}_B} = \nabla_A{}^B$, it can be readily seen that Eqs. (80–81) are equivalent to

$$\sqrt{2}\,\nabla^R{}_{(A}\hat{\phi}_{B\ldots L)R} = \mp\frac{1}{c}\frac{\partial}{\partial t}\hat{\phi}_{AB\ldots L}, \qquad \nabla^{AB}\hat{\phi}_{AB\ldots L} = 0. \tag{83}$$

(Hence, $\phi_{AB\ldots L}$ and $\hat{\phi}_{AB\ldots L}$ have opposite helicities, *cf.* Eqs. (59–60).) From Eqs. (80) and (83) one obtains the continuity equation

$$\frac{\partial}{\partial t}\left(\phi^{AB\ldots L}\hat{\phi}_{AB\ldots L}\right) = \pm\sqrt{2}\,c\nabla^{AR}\left(\phi_{(A}{}^{B\ldots L}\hat{\phi}_{R)B\ldots L}\right) \tag{84}$$

(*cf.* Eqs. (61) and (66)). Note that $\phi^{AB\ldots L}\hat{\phi}_{AB\ldots L} \geq 0$ and that $\phi_{(A}{}^{B\ldots L}\hat{\phi}_{R)B\ldots L}$ corresponds to a real vector field.

A plane wave solution of Eqs. (80–81) is of the form $\phi_{AB\ldots L} = \chi_{AB\ldots L}e^{i(k_a x^a - \omega t)}$, where $\chi_{AB\ldots L}$ is constant and k_a is a real constant vector. Taking into account Eq. (30), from Eqs. (80–81) it follows that

$$\chi_{AB\ldots L} = \alpha_A\,\alpha_B\cdots\alpha_L,$$

for some α_A, and

$$k_{AB} = \pm\sqrt{2}\left(\frac{\omega}{c}\right)\frac{\alpha_{(A}\hat{\alpha}_{B)}}{\alpha^R\hat{\alpha}_R}$$

(*cf.* Eq. (63)).

6. Concluding remarks

Among the differences between the spinor formalism employed in general relativity and the spinor formalism of three-dimensional spaces is the fact that, in the latter case, any vector can be expressed in terms of two one-index spinors [Eq. (25)]. As we have shown, when the metric has signature $(++-)$, the algebraic classification of the spinor equivalents of vectors amounts to classifying the vectors according to whether $v^a v_a \geq 0$ or $v^a v_a \leq 0$. In the case of a trace-free symmetric tensor, the factorization of its spinor equivalent [Eq. (34)] leads to a canonical form [Eq. (40)] that is not based on the eigenvectors of the tensor.

The example given at the end of Sect. 5.2 shows that, even though Eqs. (77) are integrable for a given shear-free congruence of geodesics, their solution may not be well behaved globally.

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