The Starobinsky constant and the algebraically special perturbations of Carter's A solution

G. SILVA-ORTIGOZA
Facultad de Ciencias Físico-Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1152, Puebla, Pue., México
and
Departamento de Física
Centro de Investigación y de Estudios Avanzados del IPN
Apartado postal 14–740, 07000 México, D.F., México

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ABSTRACT. Assuming that the background space-time is the Carter A solution (CA), an expression for the Starobinsky constant for perturbations of arbitrary spin s by means of a determinant of order $2s$ is obtained. This constant and the corresponding algebraically special perturbations for some interesting cases are determined.

RESUMEN. Asumiendo que el espacio-tiempo de fondo es la solución A de Carter (CA), se obtiene una expresión para la constante de Starobinsky para perturbaciones de espín arbitrario s en términos de un determinante de orden $2s$. Esta constante y las perturbaciones algebraicamente especiales son determinadas para algunos casos de interés.

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1. INTRODUCTION

A method of studying the perturbations of the Kerr solution by spin 1/2, 1, 3/2 and 2 massless free fields, consists in analyzing the maximal spin-weight components of the perturbation, each of which satisfies a second-order differential equation (known as Teukolsky's equation), that admits separable solutions [1–5]. One of the many remarkable properties of the separated functions is the differential relation between the one-variable functions belonging to opposite spin-weight $+s$ and $-s$. This differential relation is given by the so-called Teukosky-Starobinsky identities which contain a constant called Starobinsky's constant. Although this constant can be evaluated by means of a straightforward computation from the Teukolsky equation, an alternative method to obtain this constant has been proposed by Chandrasekhar when the curved space-time is the Kerr solution [6].

On the other hand, just like for the Kerr metric, in all the type D vacuum space-times with a possibly non zero cosmological constant, it has been shown that the maximal spin-weight components of the perturbations satisfy decoupled (second-order) equations that can be solved by separation of variables too. Torres del Castillo has found that in all the type D vacuum space-times with cosmological constant, relation similar to the Teukolsky-Starobinsky identities hold. Furthermore, he showed that analogous relations apply to the case of the spin–3/2 perturbations, if the cosmological constant is set equal to zero [7] and, when the curved space-time is a solution of the Einstein-Maxwell equations, he has shown that the equations for the spin–3/2 perturbations have the same structure as those found when the curved space-time is a solution of the Einstein equations with cosmological constant. Therefore, in that case, the Teukolsky-Starobinsky identities have exactly the same form as those before mentioned [8]. The existence of these identities, turns out be related to that of a two-index Killing spinor [9,10].

In the present paper, following Ref. [11], we show that when the background metric is the Carter A solution, the Starobinsky constant can be evaluated looking for algebraically special perturbations for arbitrary spin s by means of a determinant of order $2s$. In Sect. 2 we give the decoupled equations for the perturbations in the Newman-Penrose notation, we assume that the Teukolsky equations are valid for any value of s and we determine explicitly the expression for the Starobinsky constant and the corresponding special perturbations when $s = 1/2$, 1, 3/2, 2 and 5/2. The Carter A solution contains the Kerr-Newman metric as a particular case and, of course, the Kerr solution.

2. Expression for the Starobinsky Constant

In all the type D solutions of the Einstein-Maxwell equations with an aligned electromagnetic field, one can choose a null tetrad of vectors $(D, \Delta, \delta, \bar{\delta})$ such that D and Δ are double principal null directions of the conformal curvature. If we denote by X_s, and X_{-s} the components with maximal spin-weight of a spin s massless perturbation, then from the corresponding field equations it follows that [1,2,7,8]

$$\Big[(D - (2s-1)\epsilon + \bar{\epsilon} - 2s\rho - \bar{\rho})(\Delta - 2s\gamma + \mu) \\ - (\delta - (2s-1)\beta - \bar{\alpha} - 2s\tau + \bar{\pi})(\bar{\delta} - 2s\alpha + \pi) - (s-1)(2s-1)\Psi_2\Big] X_s = 0, \quad (1)$$

$$\Big[(\Delta + (2s-1)\gamma - \bar{\gamma} + 2s\mu + \bar{\mu})(D + 2s\epsilon - \rho) \\ - (\bar{\delta} + (2s-1)\alpha + \bar{\beta} + 2s\pi - \bar{\tau})(\delta + 2s\beta - \tau) - (s-1)(2s-1)\Psi_2\Big] X_{-s} = 0, \quad (2)$$

when $s = 1/2$, 1; 3/2 and 2; provided that when $s = 1$ or $s = 2$ the background electromagnetic field vanishes.

On the other hand, it is known that in all the type D solutions of the Einstein-Maxwell equations with an aligned electromagnetic field, there exist a coordinate system $\{x, y, u, v\}$ such that ∂_u and ∂_v are Killing vectors. It can be shown that the solutions of Eqs. (1) and (2) are given by [8]

$$X_s = e^{i(ku+lv)} R_{+s}\, S_{+s}, \quad (3)$$

$$X_{-s} = \frac{1}{(-\sqrt{2})^{2s}} \phi^{2s} e^{i(ku+iv)} R_{-s}\, S_{-s}, \quad (4)$$

respectively, where k, l are constants, ϕ is certain complex function depending on the background metric, R_{+s}, S_{+s}, R_{-s}, and S_{-s} satisfy the following equations:

$$\left[QD_{1-s}D_0^+ - (2s-1)iq^{(1)} + \tfrac{1}{6}(s-1)(2s-1)Q^{(2)}\right]Q^sR_{+s} = A_sQ^sR_{+s}, \tag{5}$$

$$\left[\mathcal{L}_{1-s}^+\mathcal{L}_s + (2s-1)p^{(1)} + \tfrac{1}{6}(s-1)(2s-1)P^{(2)}\right]S_{+s} = -A_sS_{+s}, \tag{6}$$

$$\left[QD_{1-s}^+D_0 + (2s-1)iq^{(1)} + \tfrac{1}{6}(s-1)(2s-1)Q^{(2)}\right]R_{-s} = A_{-s}R_{-s}, \tag{7}$$

$$\left[\mathcal{L}_{1-s}\mathcal{L}_s^+ - (2s-1)p^{(1)} + \tfrac{1}{6}(s-1)(2s-1)P^{(2)}\right]S_{-s} = -A_{-s}S_{-s}. \tag{8}$$

Here A_s and A_{-s} are separation constants and

$$D_n \equiv \partial_y + i\frac{q}{Q} + n\frac{Q^{(1)}}{Q}, \tag{9}$$

$$D_n^+ \equiv \partial_y - i\frac{q}{Q} + n\frac{Q^{(1)}}{Q}, \tag{10}$$

$$\mathcal{L}_n \equiv \sqrt{P}\left(\partial_x + \frac{p}{P} + n\frac{P^{(1)}}{2P}\right), \tag{11}$$

$$\mathcal{L}_n^+ \equiv \sqrt{P}\left(\partial_x - \frac{p}{P} + n\frac{P^{(1)}}{2P}\right), \tag{12}$$

$q(y)$ and $p(x)$ are polynomials of degree not greater than 2, which contain the separation constants k and l, $Q(y)$ and $P(x)$ are polynomials of degree not greater than 4, which contain parameters present in the metric and $f^{(n)}$ denotes the n–th derivative of f. These polynomials are reported in Ref. [8] for all the type D solutions of the Einstein-Maxwell equations, in our case (*i.e.*, for the CA metric), these polynomials and the complex function ϕ are given by

$$q(y) = l - ky^2, \tag{13}$$

$$Q(y) = b + e^2 - 2my + \epsilon_0 y^2, \tag{14}$$

$$p(x) = l + kx^2, \tag{15}$$

$$P(x) = b - g^2 + 2nx - \epsilon_0 x^2, \tag{16}$$

$$\phi(x,y) = (y + ix)^{-1}, \tag{17}$$

where b, e, g, n, m and ϵ_0 are constants. The Kerr-Newman metric is obtained if one takes $b = a^2$, $g = 0 = n$, $\epsilon_0 = 1$. In terms of the Boyer-Lindquist coordinates, $y = r$, $x =$

$-a\cos\theta$, $u = -t + a\varphi$ and $v = \varphi/a$. Making use of Eqs. (5), (7), (9) and (10), we obtain that in this case the so-called Teukolsky-Starobinsky identities are given by

$$Q^s D_0^{+2s} Q^s R_{+s} = E_s R_{-s}, \tag{18}$$

$$Q^s D_0^{2s} R_{-s} = B_s Q^s R_{+s}, \tag{19}$$

where E_s, and B_s are complex constants. From Eqs. (18) and (19) one gets

$$Q^s D_0^{+2s} Q^s D_0^{2s} R_{-s} = B_s E_s R_{-s} = |C_s|^2 R_{-s}. \tag{20}$$

From this last equation we see that using Eqs. (7), (9) and (10), we can evaluate in a straightforward way $|C_s|^2 \equiv B_s E_s$. However, the aim of the present paper is to obtain an expression for $|C_s|^2$ as a condition for getting algebraically special perturbations, which is equivalent to assume that only one of the functions R_{+s} or R_{-s} is different to zero. Making use of Eqs. (18) and (19), we have that a solution for R_{+s} different from zero associated with an identically vanishing R_{-s} is possible only if $B_s = 0$ and R_{+s} satisfies

$$D_0^{+2s} Q^s R_{+s} = 0. \tag{21}$$

Physically, the algebraically special perturbations describe waves propagating only in the positive or negative y–direction.

Following Chandrasekhar, we make use of the independent variable, y_* defined by [2]

$$\frac{dy_*}{dy} = -\frac{q}{kQ}. \tag{22}$$

According to this equation, one gets

$$D_0^+ = e^{-iky_*}\partial_y e^{+iky_*}. \tag{23}$$

Therefore, the general solution of Eq. (21) is given by (with $P_{+s} = Q^s R_{+s}$)

$$P_{+s} = e^{-iky_*}\sum_{j=0}^{2s-1} B_j\, y^j; \tag{24}$$

here $B_{2s-1}, \ldots, B_1$ and B_0 are constants of integration. These constants can be evaluated substituting (24) into Eq. (5); thus, in this way, one gets the following restriction:

$$Q\sum_{j=0}^{2s-1} jB_j\left[(j-1)y^{j-2} + 2i\frac{q}{Q}y^{j-1} + (1-s)\frac{Q^{(1)}}{Q}y^{j-1}\right]$$
$$+\left[(s-1)(2s-1)\frac{Q^{(2)}}{6} - iq^{(1)}(2s-1) - A_s\right]\sum_{j=0}^{2s-1} B_i y^j = 0. \tag{25}$$

Taking $\lambda = (1-s)(2s-1)\frac{Q^{(2)}}{6} + A_s$, and using Eqs. (13) and (14), this last equation can be transformed into

$$\left[b + e^2 - 2my + \epsilon_0 y^2\right] \sum_{j=0}^{2s-1} j(j-1)B_j y^{j-2} - \left[\lambda - 2iky(2s-1)\right] \sum_{j=0}^{2s-1} B_j y^j$$
$$+ \left[2i(l - kr^2) + 2(1-s)(\epsilon_0 y - m)\right] \sum_{j=0}^{2s-1} jB_j y^{j-1} = 0. \quad (26)$$

Since this last equation is a polynomial in the variable y of degree $(2s-1)$ equated to zero, and this equality must hold for all values of y, the coefficient of each and every power of y in this equation must vanish separately. Therefore, we are led to consider one system of homogeneous linear equations for the constants B_j $(j = 2s-1, \ldots, 0)$, which has a nontrivial solution (*i.e.*, not all constants equal to zero) provided that the determinant of this system of equations vanishes. Therefore, we have the following restriction:

$$\begin{vmatrix} \lambda & -2ik & 0 & 0 & 0 & \cdots \\ 2(2s-1)[(s-1)m - i\ell] & \lambda + \epsilon_0(2s-2) & -4ik & 0 & 0 & \cdots \\ -(2s-1)(2s-2)(b+e^2) & 2(2s-2)[(s-2)m - i\ell] & \lambda + 2\epsilon_0(2s-3) & -6ik & 0 & \cdots \\ 0 & -(2s-2)(2s-3)(b+e^2) & 2(2s-3)[(s-3)m - i\ell] & \lambda + 3\epsilon_0(2s-4) & -8ik & \cdots \\ 0 & 0 & -(2s-3)(2s-4)(b+e^2) & 2(2s-4)[(s-4)m - i\ell] & \lambda + 4\epsilon_0(2s-5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0. \quad (27)$$

On the other hand, when the background metric is such that it corresponds to Kerr-Newman or the Kerr solution, and $s = 3/2$, or $s = 1, 2$ respectively, the left-hand side of Eq. (27) gives the right expression for the Starobinsky constant, so appealing to these particular results, we conjecture that the Starobinsky constant for arbitrary spin and when the curved space-time is the Carter A solution is given by means of this determinant, *i.e.*,

$$|C_s|^2 = \text{Determinant of Eq. (27)}. \quad (28)$$

Therefore, for determining the expression of $|C_s|^2$, we have to evaluate a determinant of order $2s$, whose evaluation for $s = 5/2$, requires a little effort. Now, we consider special cases of Eq. (28).
$s = 1/2$:

$$|C_{1/2}|^2 = \lambda. \quad (29)$$

$s = 1$:

$$|C_1|^2 = \begin{vmatrix} \lambda & -2ik \\ -2i\ell & \lambda \end{vmatrix} = \lambda^2 + 4k\ell. \tag{30}$$

$s = 3/2$:

$$\begin{aligned} |C_{3/2}|^2 &= \begin{vmatrix} \lambda & -2ik & 0 \\ 2m - 4i\ell & \lambda + \epsilon_0 & -4ik \\ -2(b + e^2) & -m - 2i\ell & \lambda \end{vmatrix} \\ &= \lambda^2(\lambda + \epsilon_0) + 16\, k^2[b + e^2 + \lambda\ell/k]. \end{aligned} \tag{31}$$

$s = 2$:

$$\begin{aligned} |C_2|^2 &= \begin{vmatrix} \lambda & -2ik & 0 & 0 \\ 6(m - i\ell) & \lambda + 2\epsilon_0 & -4ik & 0 \\ -6b & -4i\ell & \lambda + 2\epsilon_0 & -6ik \\ 0 & -2b & -2(m + i\ell) & \lambda \end{vmatrix} \\ &= \lambda^2(\lambda + 2\epsilon_0)^2 + 144\, k^2(m^2 + \ell^2) + 40\, k\lambda^2\ell \\ &\quad + 48\, k\epsilon_0\ell\lambda + 96\, k^2\lambda b. \end{aligned} \tag{32}$$

$s = 5/2$:

$$\begin{aligned} |C_{5/2}|^2 &= \begin{vmatrix} \lambda & -2ik & 0 & 0 & 0 \\ 8\left(\frac{3m}{2} - i\ell\right) & \lambda + 3\epsilon_0 & -4ik & 0 & 0 \\ -12(b + e^2) & 6\left(\frac{m}{2} - i\ell\right) & \lambda + 4\epsilon_0 & -6ik & 0 \\ 0 & -6(b + e^2) & -4\left(\frac{m}{2} + i\ell\right) & \lambda + 3\epsilon_0 & -8ik \\ 0 & 0 & -2(b + e^2) & -2\left(\frac{3m}{2} + i\ell\right) & \lambda \end{vmatrix} \\ &= \lambda(\lambda + 3\epsilon_0)[\lambda(\lambda + 4\epsilon_0)(\lambda + 3\epsilon_0) + 80\, k\ell\lambda + 128\, k\ell\epsilon_0 \\ &\quad + (b + e^2)[336\, k^2\lambda^2 + 576\, \epsilon_0\lambda k^2 + 3072\, \ell k^3] + 1152\, m^2k^2\lambda \\ &\quad + 1024\, k^2\ell^2(\epsilon_0 + \lambda) + 2304\, k^2m^2\epsilon_0. \end{aligned} \tag{33}$$

From Eq. (24), we have that the algebraically special perturbations when the spin is 1/2, 1, 3/2, 2 and 5/2 are given by

$$\begin{aligned} P_{1/2}(y) &= B_{1/2}e^{-iky_*}, \\ P_1(y) &= B_1(2iky + \lambda)e^{-iky_*}, \\ P_{3/2}(y) &= B_{3/2}\left\{-2iky^2 - \lambda y + \frac{i}{4k}[\lambda(\lambda + \epsilon_0) + 4mik + 8k\ell]\right\}e^{-iky_*}, \end{aligned}$$

$$P_2(y) = B_2 \left[-2iky^3 - \lambda y^2 + \left(\frac{iB}{4k} \right) y + C \right] e^{-iky_*},$$

$$B = \lambda(\lambda + 2\epsilon_0) + 12\, imk + 12\, k\ell,$$

$$C = 2b + \frac{2}{3} \frac{\lambda \ell}{k} + \left(\frac{\lambda + 2\epsilon_0}{24\, k^2} \right) B,$$

$$P_{5/2}(y) = B_{5/2} \Bigg\{ -2iky^4 - \lambda y^3 - \left(6m - \frac{iD}{4k} \right) y^2$$

$$+ \frac{i}{6k} [3m\lambda + 6(\lambda + 4\epsilon_0)m - iE] y$$

$$- \frac{i}{8k} [6\lambda(b + e^2) + 12m^2 + \frac{\ell D}{k}$$

$$+ \frac{(\lambda + 3\epsilon_0)}{6k} E + iF] \Bigg\} e^{-iky_*},$$

where

$$D = 16\, k\ell + \lambda(\lambda + 3\epsilon_0),$$

$$E = 24\, k(b + e^2) + 6\lambda\ell + \frac{(\lambda + 4\epsilon_0)}{4k} D,$$

$$F = 24\, m\ell - \frac{m}{2k} D + \frac{m\lambda}{2k} (\lambda + 3\epsilon_0) + \frac{m}{k} (\lambda + 3\epsilon_0)(\lambda + 4\epsilon_0).$$

In these expressions, $B_{1/2}$, B_1, $B_{3/2}$, B_2 and $B_{5/2}$ are arbitrary constants, and k, except for $s = 1/2$ where is arbitrary, has to be such that the corresponding Starobinsky constant must vanish. The solution given by Eq. (24) represents waves propagating in the positive y-direction, while its complex conjugate describes waves propagating in the negative y-direction.

3. CONCLUSIONS

We observe that the determinant by means of which $|C_s|^2$ is obtained, presents one important property, that is given by the following fact: if one determines the nine elements of left top quadrant, then we are in the possibility of generating all elements of the determinant for an arbitrary value of s. On the other hand the results obtained here for $|C_s|^2$, when $s = 1/2, 1, 3/2$ and 2 reduce to the ones obtained when the curved space-time is the Kerr solution. In the case of the algebraically special perturbations, when $s = 3/2$ or $s = 2$, these expressions are in agreement to those reported when the background metric corresponds to Kerr-Newman or Kerr solution respectively [6,12].

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