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Solution of nonscalar equations in cylindrical coordinates. II

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ABSTRACT. Functions analogous to the spin-weighted spherical harmonics, adapted to the parabolic and elliptic coordinates are defined. Some examples of the usefulness of these functions in the solution of partial differential equations for nonscalar fields are given.

RESUMEN. Se definen funciones análogas a los armónicos esféricos con peso de espín, adaptadas a las coordenadas parabólicas y elípticas. Se dan algunos ejemplos de la utilidad de estas funciones en la solución de ecuaciones diferenciales parciales para campos no escalares.

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1. INTRODUCTION

The method of separation of variables is one of the most useful techniques employed in the solution of partial differential equations; however, the partial differential equations governing vector, tensor, or spinor fields written in noncartesian coordinates usually correspond to systems of partial differential equations that cannot be solved by separation of variables in a straightforward way.

In the case of spherical and circular cylindrical coordinates, the use of spin-weighted quantities and of the corresponding raising and lowering operators allows one to reduce nonscalar partial differential equations to sets of ordinary differential equations, by expressing the fields in terms of spin-weighted harmonics (see, *e.g.*, Refs. [1–4]).

The aim of this paper is to extend the main results of Ref. [4], which deals with circular cylindrical coordinates only, to the parabolic cylindrical and elliptic cylindrical coordinates. In Sect. 2, following Ref. [5], the spin-weight and the raising and lowering operators are defined for any system of orthogonal cylindrical coordinates; the usual vector operators are expressed in terms of spin-weighted quantities and the spin-weighted harmonics are defined. In Sect. 3, the eigenfunctions of the curl operator, the divergenceless vector fields, the solution of the vector Helmholtz equation and of the Dirac equation in parabolic cylindrical and elliptic cylindrical coordinates are expressed in terms of the corresponding spin-weighted harmonics.

2. Spin-weighted quantities

We shall consider cylindrical coordinates (u, v, z), where

$$u = u(x, y), \qquad v = v(x, y), \tag{1}$$

and (x, y, z) are cartesian coordinates. We shall further assume that (u, v, z) are orthogonal coordinates and that the induced orthonormal basis $\{\hat{e}_u, \hat{e}_v, \hat{e}_z\}$ is right-handed. A quantity η has spin-weight s if under the rotation through an angle θ about $\hat{e}_z$,

$$\hat{e}_u{}' + i\hat{e}_v{}' = e^{i\theta}(\hat{e}_u + i\hat{e}_v), \tag{2}$$

it transforms according to

$$\eta' = e^{is\theta}\eta \tag{3}$$

(*cf.* Ref. [6]). If η has spin-weight s then its complex conjugate $\overline{\eta}$ has spin-weight $-s$. For an arbitrary vector field $\mathbf{F}$, the scalar fields

$$F_0 \equiv \mathbf{F} \cdot \hat{e}_z, \qquad F_{\pm 1} \equiv \mathbf{F} \cdot (\hat{e}_u \pm i\hat{e}_v) \tag{4}$$

have spin-weight 0 and ± 1, and we have

$$\mathbf{F} = \tfrac{1}{2}F_{-1}(\hat{e}_u + i\hat{e}_v) + \tfrac{1}{2}F_{+1}(\hat{e}_u - i\hat{e}_v) + F_0\hat{e}_z. \tag{5}$$

For a quantity η with spin-weight s, we define

$$\begin{aligned} \eth\eta &\equiv -\left(\frac{1}{h_1}\partial_u + \frac{i}{h_2}\partial_v\right)\eta + \frac{s}{h_1 h_2}(h_{2,u} + ih_{1,v})\eta, \\ \overline{\eth}\eta &\equiv -\left(\frac{1}{h_1}\partial_u - \frac{i}{h_2}\partial_v\right)\eta - \frac{s}{h_1 h_2}(h_{2,u} - ih_{1,v})\eta, \end{aligned} \tag{6}$$

where h_1, h_2 are the scale factors corresponding to the coordinates u and v, respectively, and the comma indicates partial differentiation. Then, $\eth\eta$ and $\overline{\eth}\eta$ have spin-weight $s+1$ and $s-1$, respectively (see the Appendix). Using the definitions (6) one finds that if η has spin-weight s, then

$$\begin{aligned} \overline{\eth}\eth\eta = \eth\overline{\eth}\eta = {}& \frac{1}{h_1 h_2}\left\{\partial_u\left(\frac{h_2}{h_1}\partial_u\eta\right) + \partial_v\left(\frac{h_1}{h_2}\partial_v\eta\right)\right\} - \frac{2is}{h_1 h_2}\left(\frac{h_{1,v}}{h_1}\partial_u\eta - \frac{h_{2,u}}{h_2}\partial_v\eta\right) \\ & + s\left\{\left(\frac{1}{h_1}\partial_u + \frac{i}{h_2}\partial_v\right)\frac{h_{2,u} - ih_{1,v}}{h_1 h_2} - \frac{s-1}{(h_1 h_2)^2}(h_{2,u}^2 + h_{1,v}^2)\right\}\eta. \end{aligned} \tag{7}$$

Similarly, one finds that the gradient of a function f with spin-weight 0 is given by

$$\nabla f = -\tfrac{1}{2}(\overline{\eth} f)(\hat{e}_u + i\hat{e}_v) - \tfrac{1}{2}(\eth f)(\hat{e}_u - i\hat{e}_v) + (\partial_z f)\hat{e}_z, \tag{8}$$

and the divergence and the curl of a vector field $\mathbf{F}$ can be expressed as

$$\nabla \cdot \mathbf{F} = -\tfrac{1}{2}\eth F_{-1} - \tfrac{1}{2}\overline{\eth} F_{+1} + \partial_z F_0, \tag{9}$$

$$\begin{aligned}\nabla \times \mathbf{F} = {}& \tfrac{1}{2i}(\overline{\eth} F_0 + \partial_z F_{-1})(\hat{e}_u + i\hat{e}_v) - \tfrac{1}{2i}(\eth F_0 + \partial_z F_{+1})(\hat{e}_u - i\hat{e}_v) \\ & + \tfrac{1}{2i}(\eth F_{-1} - \overline{\eth} F_{+1})\hat{e}_z,\end{aligned} \tag{10}$$

in terms of the spin-weighted components F_s defined by Eq. (4). (Note that Eqs. (8–10) hold for all the orthogonal cylindrical coordinate systems.)

From Eqs. (8–9) and the commutativity of $\eth$ and $\overline{\eth}$ it follows that the laplacian of a function of spin-weight 0 is given by

$$\nabla^2 f = \overline{\eth}\eth f + \partial_z^2 f, \tag{11}$$

which also follows from Eq. (7) with $s = 0$. Using the identity $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$ and Eqs. (8–10), one finds that

$$\begin{aligned}\nabla^2\mathbf{F} = {}& \tfrac{1}{2}(\overline{\eth}\eth F_{-1} + \partial_z^2 F_{-1})(\hat{e}_u + i\hat{e}_v) + \tfrac{1}{2}(\overline{\eth}\eth F_{+1} + \partial_z^2 F_{+1})(\hat{e}_u - i\hat{e}_v) \\ & + (\overline{\eth}\eth F_0 + \partial_z^2 F_0)\hat{e}_z.\end{aligned} \tag{12}$$

Let ${}_sF_\alpha$ be a function of u and v with spin-weight s such that

$$\overline{\eth}\eth({}_sF_\alpha) = -\alpha^2\, {}_sF_\alpha, \tag{13}$$

where α is a (real or complex) constant. Since $\eth$ and $\overline{\eth}$ commute, we can normalize the functions ${}_sF_\alpha$ in such a way that, for $\alpha \neq 0$,

$$\begin{aligned}\eth\, {}_sF_\alpha &= \alpha\, {}_{s+1}F_\alpha, \\ \overline{\eth}\, {}_sF_\alpha &= -\alpha\, {}_{s-1}F_\alpha.\end{aligned} \tag{14}$$

(The solution of Eq. (13) is not unique; as we shall show below, the solutions of Eq. (13) can be characterized by an additional label λ, which takes values in a discrete set. Furthermore, for given values of s, α, and λ, with real α, there is only one linearly independent bounded solution of Eq. (13).)

The simplest case of Eq. (13) corresponds to $s = 0$ [see Eq. (7)], in which case Eq. (13) reduces to the two-dimensional Helmholtz equation [see Eqs. (7) and (11)],

$$\frac{1}{h_1 h_2}\left\{\partial_u\left(\frac{h_2}{h_1}\partial_u({}_0F_\alpha)\right) + \partial_v\left(\frac{h_1}{h_2}\partial_v({}_0F_\alpha)\right)\right\} + \alpha^2\, {}_0F_\alpha = 0, \tag{15}$$

which admits separable solutions in cartesian, polar, parabolic and elliptic coordinates (see, *e.g.*, Ref. [7]). Since Eq. (13) has been solved in polar coordinates in Ref. [4], in what follows we shall restrict ourselves to parabolic and elliptic coordinates, which are defined by

$$x = uv, \qquad y = \tfrac{1}{2}(v^2 - u^2) \tag{16}$$

and

$$x = a\cosh u \cos v, \qquad y = a \sinh u \sin v, \tag{17}$$

where a is a constant scale factor, respectively.

The coordinate transformations (16–17) satisfy the Cauchy-Riemann conditions

$$\partial_u x = \partial_v y, \qquad \partial_v x = -\partial_u y. \tag{18}$$

Therefore, the scale factors h_1 and h_2 coincide

$$h_1 = h_2 = \sqrt{(\partial_u x)^2 + (\partial_v x)^2} \equiv h, \tag{19}$$

and Eqs. (6–7) reduce to

$$\begin{aligned} \eth\eta &= -h^{s-1}(\partial_u + i\partial_v)(h^{-s}\eta), \\ \overline{\eth}\eta &= -h^{-s-1}(\partial_u - i\partial_v)(h^{s}\eta), \end{aligned} \tag{20}$$

and

$$\begin{aligned} \overline{\eth}\eth\eta = \eth\overline{\eth}\eta = \frac{1}{h^2}\left\{\partial_u^{\,2}\eta + \partial_v^{\,2}\eta\right\} - \frac{2is}{h^3}(h_{,v}\partial_u\eta - h_{,u}\partial_v\eta) \\ + s\left\{\frac{1}{h^3}(h_{,uu} + h_{,vv}) - \frac{s+1}{h^4}(h_{,u}^{\,2} + h_{,v}^{\,2})\right\}\eta, \end{aligned} \tag{21}$$

respectively. We shall consider now the solutions of Eq. (13) in parabolic and elliptic coordinates separately.

2.1. *Spin-weighted parabolic harmonics*

The scale factor h for the parabolic coordinates defined by Eqs. (16) is given by

$$h = \sqrt{u^2 + v^2} \tag{22}$$

[Eq. (19)], therefore, using Eqs. (21–22) one finds that Eq. (13) amounts to

$$\left[\frac{1}{u^2+v^2}(\partial_u^{\,2} + \partial_v^{\,2}) - \frac{2is}{(u^2+v^2)^2}(v\partial_u - u\partial_v) - \frac{s^2}{(u^2+v^2)^2} + \alpha^2\right] {}_sF_\alpha(u,v) = 0. \tag{23}$$

This last equation admits separable solutions only if $s = 0$, in which case it has the separable solutions

$$ {}_0F_{\alpha\lambda}(u,v) \equiv U_{\alpha\lambda}(u)V_{\alpha\lambda}(v), \tag{24}$$

where

$$\frac{d^2U_{\alpha\lambda}}{du^2} + (\alpha^2u^2 - \lambda^2)U_{\alpha\lambda} = 0, \qquad \frac{d^2V_{\alpha\lambda}}{dv^2} + (\alpha^2v^2 + \lambda^2)V_{\alpha\lambda} = 0, \tag{25}$$

and λ is a separation constant. Hence, if $\alpha \neq 0$, $U_{\alpha\lambda}$ and $V_{\alpha\lambda}$ can be expressed in terms of the parabolic cylinder functions (Weber functions) (see, *e.g.*, Refs. [7–9]).

By virtue of Eqs. (14), we can obtain the functions ${}_sF_{\alpha\lambda}$, for integral values of s and $\alpha \neq 0$, in terms of ${}_0F_{\alpha\lambda}$. In fact, using Eqs. (14) and (20) one finds that

$$ {}_sF_{\alpha\lambda} = \begin{cases} \dfrac{1}{\alpha^s}\eth^s\, {}_0F_{\alpha\lambda} = \left(-\dfrac{h}{\alpha}\right)^s \left[\dfrac{1}{h^2}(\partial_u + i\partial_v)\right]^s {}_0F_{\alpha\lambda}, & s \geq 0, \\ \left(-\dfrac{1}{\alpha}\right)^{-s} \bar{\eth}^{-s}\, {}_0F_{\alpha\lambda} = \left(\dfrac{h}{\alpha}\right)^{-s} \left[\dfrac{1}{h^2}(\partial_u - i\partial_v)\right]^{-s} {}_0F_{\alpha\lambda}, & s \leq 0. \end{cases} \tag{26}$$

Since the functions ${}_{\pm1/2}F_{\alpha\lambda}$ appear in the solution of the Dirac equation (see Sect. 3.4), one can obtain these functions from the solutions to the Dirac equation given in Ref. [10]; in this manner we get

$$ {}_{\pm1/2}F_{\alpha\lambda} = (1 \mp i)h^{-1/2}\left(\sqrt{h+u} \mp i\sqrt{h-u}\right)\left(\widetilde{U}(u)V(v) \mp U(u)\widetilde{V}(v)\right), \tag{27}$$

where

$$\begin{aligned} \left(\frac{d}{du} + i\alpha u\right) U &= \lambda \widetilde{U}, \\ \left(\frac{d}{du} - i\alpha u\right) \widetilde{U} &= \lambda U, \\ \left(\frac{d}{dv} + i\alpha v\right) V &= i\lambda \widetilde{V}, \\ \left(\frac{d}{dv} - i\alpha v\right) \widetilde{V} &= i\lambda V, \end{aligned} \tag{28}$$

and λ is a separation constant. Combining Eqs. (28) one obtains the parabolic cylinder equations

$$\begin{aligned} \frac{d^2U}{du^2} + (\alpha^2u^2 + i\alpha - \lambda^2)U &= 0, \\ \frac{d^2\widetilde{U}}{du^2} + (\alpha^2u^2 - i\alpha - \lambda^2)\widetilde{U} &= 0, \\ \frac{d^2V}{dv^2} + (\alpha^2v^2 + i\alpha + \lambda^2)V &= 0, \\ \frac{d^2\widetilde{V}}{dv^2} + (\alpha^2v^2 - i\alpha + \lambda^2)\widetilde{V} &= 0, \end{aligned} \tag{29}$$

(*cf.* Eqs. (25)). Using Eqs. (14) and (27) one can find ${}_sF_{\alpha\lambda}$ for half-integral values of s and $\alpha \neq 0$.

Finally, using the fact that

$$(\partial_u^2 + \partial_v^2) \ln h = 0, \tag{30}$$

it can be verified that the most general solution of Eq. (13) with $\alpha = 0$ is given by

$${}_sF_0 = h^s f(u + iv) + h^{-s} g(u - iv), \tag{31}$$

where f and g are arbitrary (differentiable) functions. As in the case of the circular cylindrical coordinates, in some applications, the boundary conditions exclude the spin-weighted harmonics with $\alpha = 0$ (note that the functions (31) either diverge at the origin or at infinity, or do not vanish at infinity (unless, of course, they are identically zero)).

2.2. *Spin-weighted elliptic harmonics*

In the case of the elliptic coordinates defined by Eqs. (17), the scale factor (19) is

$$h = a\sqrt{\sinh^2 u + \sin^2 v} = a\sqrt{\cosh^2 u - \cos^2 v}, \tag{32}$$

and, using Eq. (21) one finds that Eq. (13) takes the explicit form

$$\begin{aligned}\Bigg[\frac{1}{\sinh^2 u + \sin^2 v}(\partial_u^2 + \partial_v^2) &- \frac{2is}{(\sinh^2 u + \sin^2 v)^2}(\sin v \cos v\, \partial_u - \sinh u \cosh u\, \partial_v) \\ &- \frac{s^2(\cosh^2 u - \sin^2 v)}{(\sinh^2 u + \sin^2 v)^2} + a^2\alpha^2\Bigg] {}_sF_\alpha(u, v) = 0.\end{aligned} \tag{33}$$

This partial differential equation admits separable solutions only if $s = 0$. Substituting

$${}_0F_{\alpha\lambda}(u, v) \equiv U_{\alpha\lambda}(u) V_{\alpha\lambda}(v) \tag{34}$$

into Eq. (33) with $s = 0$ one finds that

$$\frac{d^2U_{\alpha\lambda}}{du^2} + (a^2\alpha^2 \sinh^2 u + \lambda^2)U_{\alpha\lambda} = 0, \qquad \frac{d^2V_{\alpha\lambda}}{dv^2} + (a^2\alpha^2 \sin^2 v - \lambda^2)V_{\alpha\lambda} = 0, \tag{35}$$

where λ is a separation constant. The solutions of Eqs. (35) are linear combinations of Mathieu functions (see, *e.g.*, Refs. [7,8]).

As in the preceding case, the functions ${}_sF_{\alpha\lambda}$, for integral values of s and $\alpha \neq 0$, are given by Eq. (26) with h and ${}_0F_{\alpha\lambda}$ given by Eqs. (32) and (34–35), respectively. Using the results of Ref. [10] one finds that

$$\begin{aligned}{}_{\pm 1/2}F_{\alpha\lambda} = {} & \mp i a h^{-3/2} \sqrt{\cosh u + \cos v}\, \left(\sqrt{h + a \sin v} \mp i\sqrt{h - a \sin v}\right) \\ & \times \left(\widetilde{V}(v) U(u) \mp V(v) \widetilde{U}(u)\right),\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\left(\frac{d}{du} + ia\alpha \sinh u\right) U &= i\lambda \widetilde{U}, \\
\left(\frac{d}{du} - ia\alpha \sinh u\right) \widetilde{U} &= i\lambda U, \\
\left(\frac{d}{dv} - ia\alpha \sin v\right) V &= -\lambda \widetilde{V}, \\
\left(\frac{d}{dv} + ia\alpha \sin v\right) \widetilde{V} &= -\lambda V,
\end{aligned} \tag{37}$$

and λ is a separation constant. By combining the first-order differential equations (37) one gets

$$\begin{aligned}
\frac{d^2 U}{du^2} + (a^2\alpha^2 \sinh^2 u + ia\alpha \cosh u + \lambda^2) U &= 0, \\
\frac{d^2 \widetilde{U}}{du^2} + (a^2\alpha^2 \sinh^2 u - ia\alpha \cosh u + \lambda^2) \widetilde{U} &= 0, \\
\frac{d^2 V}{dv^2} + (a^2\alpha^2 \sin^2 v - ia\alpha \cos v - \lambda^2) V &= 0, \\
\frac{d^2 \widetilde{V}}{dv^2} + (a^2\alpha^2 \sin^2 v + ia\alpha \cos v - \lambda^2) \widetilde{V} &= 0,
\end{aligned} \tag{38}$$

which are Whittaker-Hill equations (see Ref. [10] and the references cited therein). (Note that Ref. [10] contains several misprints.) Then, the functions ${}_sF_{\alpha\lambda}$, with half-integral values of s and $\alpha \neq 0$, can be obtained from Eqs. (14) and (36).

Since the scale factor (32) also satisfies Eq. (30), the most general solution of Eq. (13) with $\alpha = 0$ is also given by Eq. (31).

3. Applications

In this section we give some examples of the usefulness of the spin-weighted functions ${}_sF_{\alpha\lambda}(u, v)$ in the solution of nonscalar equations in parabolic cylindrical and elliptic cylindrical coordinates.

3.1. Solution of the vector Helmholtz equation

According to Eqs. (5) and (12), the vector Helmholtz equation, $\nabla^2 \mathbf{F} + k^2 \mathbf{F} = 0$, in circular, parabolic, or elliptic cylindrical coordinates, is equivalent to the three uncoupled equations

$$\overline{\partial}\, \partial\, F_s + \partial_z^{\,2} F_s + k^2 F_s = 0, \qquad s = 0, \pm 1, \tag{39}$$

which admit solutions of the form

$$F_s = {}_sF_{\alpha\lambda}(u,v)\, g_s(z), \tag{40}$$

where the $g_s(z)$ are functions of z only that, owing to Eqs. (13) and (39–40), satisfy the differential equations

$$\frac{d^2 g_s}{dz^2} + (k^2 - \alpha^2) g_s = 0, \qquad s = 0, \pm 1. \tag{41}$$

Following the steps given in Ref. [4], one can show that any divergenceless solution of the vector Helmholtz equation can be written in the form

$$\mathbf{F} = \hat{e}_z \times \nabla\psi_1 + \nabla \times (\hat{e}_z \times \nabla\psi_2), \tag{42}$$

where ψ_1 and ψ_2 are solutions of the scalar Helmholtz equation.

3.2. Divergenceless vector fields

Let $\mathbf{F}$ be a vector field with vanishing divergence, then its spin-weighted components satisfy

$$-\tfrac{1}{2}\eth\, F_{-1} - \tfrac{1}{2}\overline{\eth}\, F_{+1} + \partial_z F_0 = 0 \tag{43}$$

[Eq. (9)]. Assuming that any function with spin-weight s can be expanded in terms of the ${}_sF_{\alpha\lambda}(u,v)$, with $\alpha \neq 0$, we can write (*cf.* Refs. [1,3])

$$F_s = \int d\alpha \sum_\lambda g_s(\alpha,\lambda,z)\, {}_sF_{\alpha\lambda}(u,v), \qquad s = 0, \pm 1. \tag{44}$$

Substitution of Eqs. (44) into Eq. (43), making use of Eqs. (14), yields

$$\tfrac{1}{2}\Big(g_{+1}(\alpha,\lambda,z) - g_{-1}(\alpha,\lambda,z)\Big) = -\tfrac{1}{\alpha}\partial_z g_0(\alpha,\lambda,z), \tag{45}$$

hence, using Eqs. (13–14) and (45), from Eqs. (44) one finds that

$$\begin{aligned} F_{+1} &= -i\eth\,\psi_1 + \partial_z \eth\,\psi_2, \\ F_0 &= \overline{\eth}\eth\,\psi_2, \\ F_{-1} &= i\overline{\eth}\,\psi_1 + \partial_z \overline{\eth}\,\psi_2, \end{aligned} \tag{46}$$

where

$$\psi_1 \equiv \int d\alpha \sum_\lambda \frac{i}{2\alpha}\Big(g_{+1}(\alpha,\lambda,z) + g_{-1}(\alpha,\lambda,z)\Big)\, {}_0F_{\alpha\lambda}(u,v),$$

$$\psi_2 \equiv -\int d\alpha \sum_\lambda \frac{1}{\alpha^2} g_0(\alpha,\lambda,z)\, {}_0F_{\alpha\lambda}(u,v).$$

Owing to Eqs. (8) and (10), Eqs. (46) are equivalent to

$$\mathbf{F} = \hat{e}_z \times \nabla\psi_1 + \nabla \times (\hat{e}_z \times \nabla\psi_2). \tag{47}$$

Thus, any divergenceless vector field can be expressed in the form (47), where ψ_1 and ψ_2 are two scalar functions. (Note that Eq. (42) is a special case of Eq. (47).)

3.3. Eigenfunctions of the curl operator

By using Eqs. (5) and (10) one finds that the eigenvalue equation $\nabla \times \mathbf{F} = \mu\mathbf{F}$ in circular, parabolic, or elliptic cylindrical coordinates amounts to (*cf.* Ref. [3])

$$\begin{aligned} i(\eth\, F_0 + \partial_z F_{+1}) &= \mu F_{+1}, \\ i(\overline{\eth}\, F_{+1} - \eth\, F_{-1}) &= 2\mu F_0, \\ -i(\overline{\eth}\, F_0 + \partial_z F_{-1}) &= \mu F_{-1}. \end{aligned} \tag{48}$$

Looking for separable solutions of Eqs. (48) of the form

$$F_s = {}_sF_{\alpha\lambda}(u,v)\, g_s(z), \qquad s = 0, \pm 1, \tag{49}$$

one gets

$$\begin{aligned} -i\alpha G &= \mu g_0, \\ i\frac{dH}{dz} + i\alpha g_0 &= \mu G, \\ i\frac{dG}{dz} &= \mu H, \end{aligned} \tag{50}$$

where

$$G \equiv \tfrac{1}{2}(g_{+1} + g_{-1}), \qquad H \equiv \tfrac{1}{2}(g_{+1} - g_{-1}). \tag{51}$$

From Eqs. (50) it follows that

$$\frac{d^2G}{dz^2} + (\mu^2 - \alpha^2)G = 0, \tag{52}$$

and making use of Eqs. (13–14) and (49–51) one obtains

$$\begin{aligned} F_{+1} &= (G + H)\frac{1}{\alpha}\eth\, {}_0F_{\alpha\lambda} = \left(G + \frac{i}{\mu}\partial_z G\right)\frac{1}{\alpha}\eth\, {}_0F_{\alpha\lambda} = -i\mu\eth\,\psi + \partial_z\eth\,\psi, \\ F_0 &= -\frac{i\alpha G}{\mu}{}_0F_{\alpha\lambda} = \overline{\eth}\,\eth\left(\frac{iG}{\alpha\mu}{}_0F_{\alpha\lambda}\right) = \overline{\eth}\,\eth\,\psi, \\ F_{-1} &= (G - H)\left(-\frac{1}{\alpha}\right)\overline{\eth}\,{}_0F_{\alpha\lambda} = -\left(G - \frac{i}{\mu}\partial_z G\right)\frac{1}{\alpha}\overline{\eth}\,{}_0F_{\alpha\lambda} = i\mu\overline{\eth}\,\psi + \partial_z\overline{\eth}\,\psi, \end{aligned} \tag{53}$$

(*cf.* Eqs. (46)) where

$$\psi \equiv \frac{iG(z)}{\alpha\mu}\, {}_0F_{\alpha\lambda}(u,v),$$

which, owing to Eqs. (11), (13) and (52), satisfies the scalar Helmholtz equation, $\nabla^2\psi + \mu^2\psi = 0$. Equations (53) are equivalent to

$$\mathbf{F} = \mu\hat{e}_z \times \nabla\psi + \nabla \times (\hat{e}_z \times \nabla\psi). \tag{54}$$

(An alternative derivation of Eq. (54) is given in Ref. [11].)

3.4. Solution of the Dirac equation

Using Eq. (63) of Ref. [5] and Eqs. (A1), (A3) and (A4), one finds that the Dirac equation written in terms of spin-weighted quantities is given by

$$\begin{aligned}
\frac{1}{c}\frac{\partial u^1}{\partial t} &= -\partial_z v^1 + \bar{\eth} v^2 - \frac{imc}{\hbar}u^1,\\
\frac{1}{c}\frac{\partial u^2}{\partial t} &= \eth v^1 + \partial_z v^2 - \frac{imc}{\hbar}u^2,\\
\frac{1}{c}\frac{\partial v^1}{\partial t} &= -\partial_z u^1 + \bar{\eth} u^2 + \frac{imc}{\hbar}v^1,\\
\frac{1}{c}\frac{\partial v^2}{\partial t} &= \eth u^1 + \partial_z u^2 + \frac{imc}{\hbar}v^2,
\end{aligned} \tag{55}$$

where u^A, v^A are the components of the Dirac spinor with respect to the spin basis induced by the coordinates (u, v, z); u^1 and v^1 have spin-weight $-\frac{1}{2}$, while u^2 and v^2 have spin-weight $\frac{1}{2}$. (Alternative derivations of Eqs. (55) are given in Refs. [4,10,12–14].) Equations (55) admit solutions of the form

$$\begin{aligned}
u^1 &= {}_{-1/2}F_{\alpha\lambda}(u,v)g(z)e^{-iEt/\hbar},\\
u^2 &= {}_{1/2}F_{\alpha\lambda}(u,v)G(z)e^{-iEt/\hbar},\\
v^1 &= {}_{-1/2}F_{\alpha\lambda}(u,v)f(z)e^{-iEt/\hbar},\\
v^2 &= {}_{1/2}F_{\alpha\lambda}(u,v)F(z)e^{-iEt/\hbar}.
\end{aligned} \tag{56}$$

Substituting Eqs. (56) into Eqs. (55) one obtains [4]

$$\begin{aligned}
\frac{dA}{dz} + \alpha A &= \frac{E + mc^2}{\hbar c}C,\\
-\frac{dC}{dz} + \alpha C &= \frac{E - mc^2}{\hbar c}A,
\end{aligned} \tag{57}$$

and

$$\frac{dB}{dz} - \alpha B = \frac{E + mc^2}{\hbar c} D,$$
$$-\frac{dD}{dz} - \alpha D = \frac{E - mc^2}{\hbar c} B, \tag{58}$$

where

$$A \equiv \tfrac{1}{2}(g + G), \qquad B \equiv \tfrac{1}{2}(G - g), \qquad C \equiv \tfrac{1}{2i}(F - f), \qquad D \equiv \tfrac{1}{2i}(f + F). \tag{59}$$

Therefore, Eqs. (55) admit solutions of the form

$$\begin{bmatrix} u^1 \\ u^2 \\ v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} A(z)X_{\alpha\lambda} \\ iC(z)X_{-\alpha\lambda} \end{bmatrix} e^{-iEt/\hbar} + \begin{bmatrix} B(z)X_{-\alpha\lambda} \\ iD(z)X_{\alpha\lambda} \end{bmatrix} e^{-iEt/\hbar}, \tag{60}$$

where

$$X_{\alpha\lambda} \equiv \begin{bmatrix} {}_{-1/2}F_{\alpha\lambda} \\ {}_{1/2}F_{\alpha\lambda} \end{bmatrix}, \qquad X_{-\alpha\lambda} \equiv \begin{bmatrix} -({}_{-1/2}F_{\alpha\lambda}) \\ {}_{1/2}F_{\alpha\lambda} \end{bmatrix}, \qquad (\alpha \neq 0). \tag{61}$$

(The bounded solutions with $\alpha = 0$ correspond to plane waves traveling along the z-axis [4,14].)

Using the fact that

$$\widetilde{Q} X_{\pm\alpha\lambda} = \pm\alpha X_{\pm\alpha\lambda}, \tag{62}$$

where [4]

$$\widetilde{Q} \equiv \begin{bmatrix} 0 & -\overline{\eth} \\ \eth & 0 \end{bmatrix}, \tag{63}$$

it is easy to see that each term in the right-hand side of Eq. (60) is an eigenfunction of the operator

$$\widetilde{K} \equiv \hbar \begin{bmatrix} -\widetilde{Q} & 0 \\ 0 & \widetilde{Q} \end{bmatrix}, \tag{64}$$

with eigenvalue $-\hbar\alpha$ and $\hbar\alpha$, respectively (*cf.* also Refs. [2,4]).

4. CONCLUDING REMARKS

By contrast with the spin-weighted harmonics in spherical and circular cylindrical coordinates, in the cases of parabolic cylindrical and elliptic cylindrical coordinates, the functions ${}_sF_{\alpha\lambda}(u,v)$ with $s \neq 0$, are not separable; however, one can find the solutions of Eqs. (23) and (33) for $s = \pm 1/2, \pm 1, \ldots$, by means of Eqs. (14).

Since the equations for fields of any spin, written in terms of spin-weighted quantities and the operators $\eth$ and $\overline{\eth}$, have the same form in circular cylindrical coordinates as in parabolic cylindrical and elliptic cylindrical coordinates, the solutions to such equations, given in terms of the spin-weighted harmonics and the operators $\eth$ and $\overline{\eth}$, have the same form in any of these coordinate systems. Thus, for instance, the solution of the Helmholtz equation for spin-2 fields in parabolic cylindrical or elliptic cylindrical coordinates is given by Eqs. (42) of Ref. [15], which were obtained in circular cylindrical coordinates.

APPENDIX

If (u, v, z) are orthogonal cylindrical coordinates, then

$$\partial_1 \equiv \frac{1}{h_1}\partial_u, \qquad \partial_2 \equiv \frac{1}{h_2}\partial_v, \qquad \partial_3 \equiv \partial_z, \tag{A1}$$

form an orthonormal triad. A straightforward computation shows that

$$[D, \delta] = 0, \qquad [\delta, \overline{\delta}] = \frac{1}{\sqrt{2}\, h_1 h_2}\Big\{(h_{2,u} - ih_{1,v})\delta - (h_{2,u} + ih_{1,v})\overline{\delta}\Big\}, \tag{A2}$$

where

$$D \equiv \tfrac{1}{\sqrt{2}}\partial_3, \qquad \delta \equiv \tfrac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \qquad \overline{\delta} \equiv \tfrac{1}{\sqrt{2}}(\partial_1 - i\partial_2) \tag{A3}$$

(see Ref. [5], Eq. (32b)). Comparing Eqs. (A2) with Eqs. (33) of Ref. [5] one finds that the only nonvanishing spin-coefficient for the triad (A3) is given by

$$\beta = -\frac{1}{2\sqrt{2}\, h_1 h_2}(h_{2,u} + ih_{1,v}). \tag{A4}$$

Therefore, the spin-weight raising and lowering operators $(\delta + 2s\beta)$ and $(\overline{\delta} - 2s\overline{\beta})$ are

$$\begin{aligned} \delta + 2s\beta &= \frac{1}{\sqrt{2}}\left(\frac{1}{h_1}\partial_u + \frac{i}{h_2}\partial_v\right) - \frac{s}{\sqrt{2}\, h_1 h_2}(h_{2,u} + ih_{1,v}), \\ \overline{\delta} - 2s\overline{\beta} &= \frac{1}{\sqrt{2}}\left(\frac{1}{h_1}\partial_u - \frac{i}{h_2}\partial_v\right) + \frac{s}{\sqrt{2}\, h_1 h_2}(h_{2,u} - ih_{1,v}). \end{aligned} \tag{A5}$$

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