Thermodynamic optimization of endoreversible engines

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ABSTRACT. Within the field of the so-called endoreversible thermodynamics (ET) it has been established that most of the optimum performance conditions for finite-time thermal engines models are highly sensitive to the heat transfer law used. For example, this is the case for maximum power output conditions. In the present work, by means of the De Vos' method we develop a systematic way to obtain several typical ET–results and we find that there exists a general property which is maintained for endoreversible engines under changes in the heat-transfer law.

RESUMEN. En el campo de la llamada termodinámica endorreversible (TE) se ha establecido que la mayoría de las condiciones para la operación óptima de modelos de máquinas térmicas a tiempo finito dependen fuertemente de la ley de transferencia de calor usada. Este es el caso del régimen de máxima potencia, por ejemplo. En este trabajo, mediante el metodo de De Vos, desarrollamos una manera sistemática de obtener varios resultados típicos de la TE, y encontramos que existe una propiedad general que se mantiene bajo cambios de la ley de transferencia térmica usada en máquinas endorreversibles.

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1. INTRODUCTION

Endoreversible thermodynamics (ET) can be considered as an extension of classical equilibrium thermodynamics in the domain of endoreversible processes [1, 2]. ET has been capable to describe physical systems in which predominant irreversible processes occur at the coupling between the system and its surroundings. In the ET approach, the internal processes within the system may be treated as reversible transformations and the global entropy production is ascribed only to energy exchanges occurring at the links connecting the system with its surroundings. This separation in internal reversible processes and frontier-irreversible processes is known as the endoreversibility hypothesis, and can be considered as reasonable for cases in which the internal relaxation times of the system are negligibly short compared to the duration of the global process to be considered. A typical endoreversible system is the so-called Curzon & Ahlborn (CA) engine [3] (see Fig. 1). This heat engine is a Carnot-type cycle in which there is not thermal equilibrium between the working fluid and the reservoirs, at the isothermal branches of the cycle. CA showed that such an engine is a non zero power output engine (in contrast with the reversible Carnot cycle that in practice does not deliver power output) which working at the maximum power regime has an efficiency given by

$$\eta_{\mathrm{CA}} = 1 - \sqrt{\frac{T_2}{T_1}}, \tag{1}$$

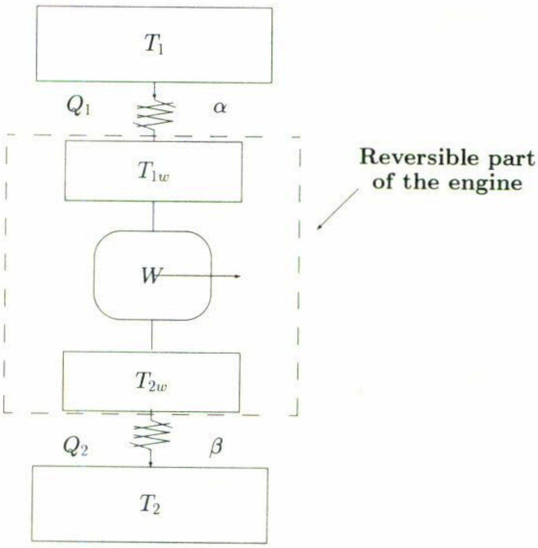

FIGURE 1. De Vos' scheme for the Curzon and Ahlborn endoreversible engine.

where T_2 and T_1 are the absolute temperatures of the cold and hot reservoirs, respectively. Equation (1) compares well with observed efficiencies for some power sources as can be seen in Table I of Ref. [3]. Eq. (1) has been obtained by several authors [4–6] by using alternative approaches to the CA–cycle. Since the CA paper, an extensive work has been made in the field of the ET, also called finite-time thermodynamics [7]. Among the several approaches proposed for endoreversible engines, the De Vos' treatment [1, 5, 8] is remarkable because of its simplicity and generality. In this paper, we obtain a general outline for some ET results following the De Vos' approach, including the recovery of those results in Refs. [9] and [10]. They consist in some properties of the CA–cycle operating under an ecological optimization criterion, the first one, and an improvement of finite-time efficiencies calculations, the second one. In this work, we also suggest that there exists a general property for CA–cycles which is independent of several heat transfer laws.

2. TYPICAL ENDOREVERSIBLE ENGINE CHARACTERISTICS

Figure 1 shows the De Vos' scheme for a typical endoreversible engine (the CA engine) constituted by the following parts: two heat reservoirs (T_1 and T_2); two irreversible components (thermal conductances α and β) and a reversible Carnot engine between the intermediate heat reservoir at T_{1w} and the intermediate heat reservoir at T_{2w}. By the condition of endoreversibility [1], we have

$$\frac{Q_1}{T_{1w}} = \frac{Q_2}{T_{2w}}, \tag{2}$$

where Q_1 and Q_2 are the heat flows (heat transfer per unit time). Because of first law of thermodynamics, it follows that

$$Q_1 = W + Q_2, \tag{3}$$

where W denotes the power output (work per unit time). It has been usual to propose (for the thermal conductors) constitutive laws of the type

$$Q_1 = \alpha(T_1^k - T_{1w}^k) \tag{4}$$

and

$$Q_2 = \beta(T_{2w}^k - T_2^k), \tag{5}$$

with $k = 1$ for the Newton's law of cooling or $k = 4$ for Stefan-Boltzmann heat exchange [5, 12]. By substitution of Eqs. (4) and (5) in Eq. (2), we obtain

$$\frac{\alpha(T_1^k - T_{1w}^k)}{T_{1w}} = \frac{\beta(T_{2w}^k - T_2^k)}{T_{2w}}. \tag{6}$$

For the reversible part of Fig. 1 it is clear that

$$\eta = 1 - \frac{T_{2w}}{T_{1w}}, \tag{7}$$

where η is obviously the endoreversible Carnot efficiency. By solving Eqs. (6) and (7) for T_{1w} and T_{2w}, we obtain

$$T_{1w}^k = \frac{\alpha}{\alpha + \beta(1-\eta)^{k-1}} T_1^k + \frac{\beta}{\alpha + \beta(1-\eta)^{k-1}} \, \frac{T_2^k}{(1-\eta)} \tag{8}$$

and

$$T_{2w}^k = \frac{\alpha(1-\eta)^k}{\alpha + \beta(1-\eta)^{k-1}} T_1^k + \frac{\beta(1-\eta)^{k-1}}{\alpha + \beta(1-\eta)^{k-1}} T_2^k. \tag{9}$$

By means of Eqs. (8) and (4) we get the function $Q_1 = Q_1(\eta)$,

$$Q_1 = \gamma \frac{(1-\eta)^k T_1^k - T_2^k}{\frac{\alpha}{\alpha+\beta}(1-\eta) + \frac{\beta}{\alpha+\beta}(1-\eta)^k}, \tag{10}$$

with $\gamma = \frac{\alpha\beta}{\alpha+\beta}$. Multiplying Eq. (10) by η, immediately becomes

$$W = \gamma\eta \frac{(1-\eta)^k T_1^k - T_2^k}{\frac{\alpha}{\alpha+\beta}(1-\eta) + \frac{\beta}{\alpha+\beta}(1-\eta)^k}, \tag{11}$$

which give us the cycle's power output in terms of η. Following the same procedure, we can obtain the mean entropy production of the CA engine. The entropy production σ for the entire engine (working fluid plus reservoirs) is

$$\sigma = \frac{Q_2}{T_2} - \frac{Q_1}{T_1} \tag{12}$$

which in terms of η becomes [by substitution of Eqs. (4) and (5) in (12)]

$$\sigma = \frac{\gamma}{T_1 T_2} \frac{[(1-\eta)T_1 - T_2][(1-\eta)^k T_1^k - T_2^k]}{\frac{\alpha}{\alpha+\beta}(1-\eta) + \frac{\beta}{\alpha+\beta}(1-\eta)^k}. \tag{13}$$

Equations (10), (11) and (13) give us the main characteristics of an endoreversible CA cycle, as functions of the efficiency η, for heat flows governed by Eqs. (4) and (5). Figures 2a, 2b and 2c show the characteristics curves of heat input, power output and entropy production, respectively. As it is expected these characteristics are compatible with the full reversible limit case, that is $W(\eta_c) = 0$ and $\sigma(\eta_c) = 0$ for $\eta_c = 1 - \frac{T_2}{T_1}$.

3. Thermodynamic optimization

3.1. Newton's law of cooling

Curzon and Ahlborn obtained Eq. (1) maximizing the cycle's power output as function of the variables $X = T_1 - T_{1w}$ and $Y = T_{2w} - T_2$. This procedure implies a cumbersome algebraic handling of two variable functions [3]. By means of expressions as Eqs. (11) and (13) the maximization problem is reduced to one independent variable, namely, η (which is the De Vos' procedure). For example, to maximize the power output by using Eq. (11), immediately it follows that, $\frac{dW(\eta)}{d\eta} = 0$ implies

$$\frac{T_1^k}{T_2^k}\left(\alpha(1-\eta)^{k+1} + \beta(1-\eta)^{2k}\right) - \left(\alpha + \beta(1-\eta)^{k-1}\right) - (k-1)\eta\left(\alpha\frac{T_1^k}{T_2^k}(1-\eta)^k + \beta(1-\eta)^{k-1}\right) = 0, \tag{14}$$

which for the Newton's law of cooling case ($k = 1$) reduces to

$$(1-\eta)^2 T_1 - T_2 = 0, \tag{15}$$

whose solution is Eq. (1).

In Ref. [9] a criterion of merit to optimize a CA cycle was proposed. This criterion consists in the maximizing of the function E, which represents the best compromise between high power output and low entropy production. The function E is given by

$$E = W - T_2\sigma, \tag{16}$$

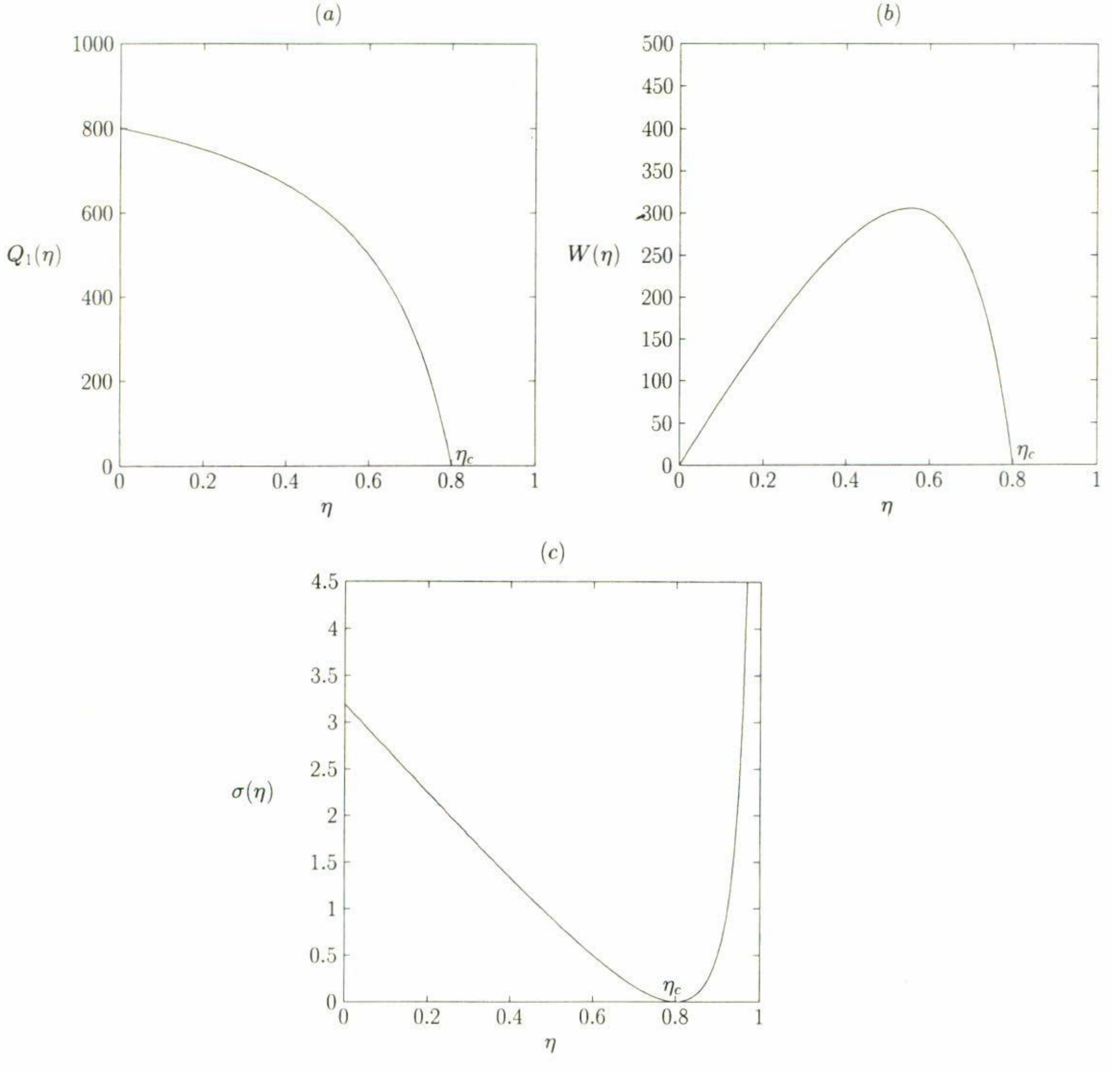

FIGURE 2. Thermodynamic characteristics for the CA–engine in terms of η, with $k = 1$ and arbitrary values for α, β, T_1 and T_2: (a) heat input; (b) mean power output and (c) mean entropy production.

where W is power output, T_2 the temperature of the cold reservoir and σ the entropy production (per unit time). By substituting Eqs. (11) and (13) in Eq. (16), we get

$$E(\eta) = \frac{\gamma}{T_1}\left[\frac{(1-\eta)(2\eta-1)T_1^{k+1} + (1-2\eta)T_1T_2^k + (1-\eta)^k T_1^k T_2 - T_2^{k+1}}{\frac{\alpha}{\alpha+\beta}(1-\eta) + \frac{\beta}{\alpha+\beta}(1-\eta)^k}\right]. \qquad (17)$$

For fixed k, $\frac{T_2}{T_1}$ and $\frac{\alpha}{\beta}$, $E(\eta)$ is depicted in Fig. 3. This is a convex curve with an unique maximum point. To obtain η_{ME}, where $E(\eta)$ has its maximum, we calculate $\frac{dE}{d\eta} = 0$, which

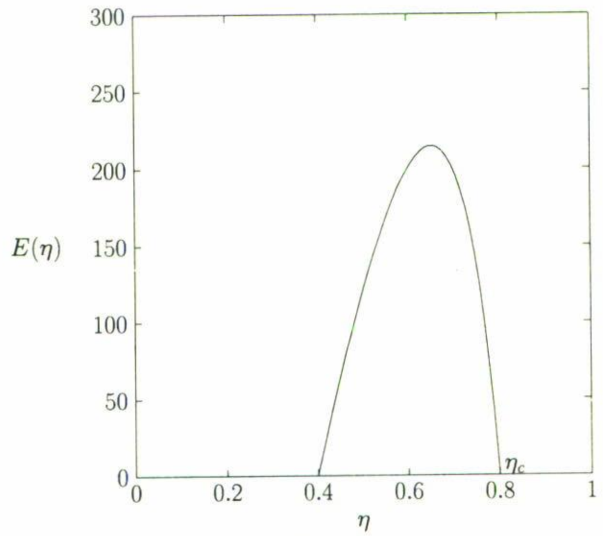

FIGURE 3. The ecological function in terms of η for $k = 1$ and arbitrary values of α, β, T_1 and T_2.

implies (for $k = 1$)

$$2T_1^2(1-\eta)^2 - T_1T_2 - T_2^2 = 0, \tag{18}$$

whose solution is

$$\eta_{\rm ME} = 1 - \sqrt{\frac{T_2(T_2+T_1)}{2T_1^2}}, \tag{19}$$

in agreement with the solution reported in Ref. [9], where it was obtained by means of a conventional CA–approach. Equation (19) has the following property [9]:

$$\eta_{\rm ME} \approx \eta_{\rm ME}^{\dagger} \equiv \frac{\eta_{\rm C} + \eta_{\rm CA}}{2}, \tag{20}$$

with $\eta_{\rm C}$ the Carnot efficiency and $\eta_{\rm CA}$ the CA efficiency given by Eq. (1), as can be seen in Fig. 4.

The point of maximum $E(\eta)$ has the interesting property of giving us down to about 80% of the maximum power output, but with an entropy production down to about 30% of the entropy that would be produced by the maximum power regime. This behavior can be seen in Figs. 5a and 5b. For this reason the function $E(\eta)$ is called ecological function [9].

3.2. Dulong–Petit's law of cooling

A more realistic description of the heat exchange between the working fluid and its surroundings must include a T^4–term due to radiative contributions. The so-called Dulong-Petit (DP) law of cooling given by

$$Q_1 = \alpha(T_1 - T_{1w})^k \tag{21}$$

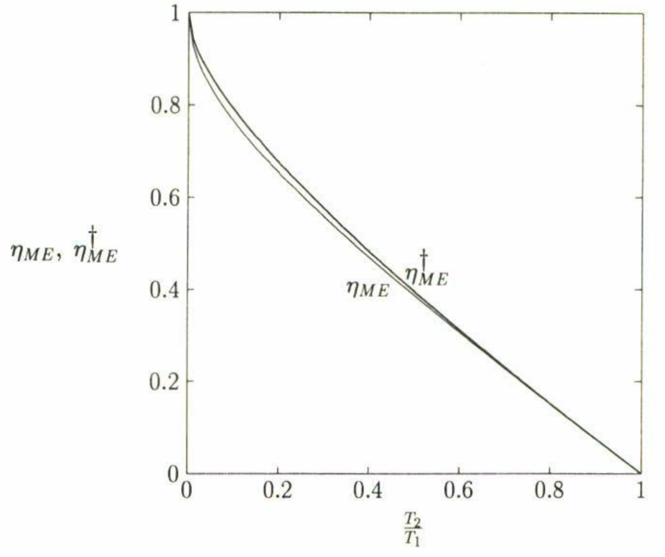

FIGURE 4. Comparison between η_{ME} and $\eta^{\dagger}_{\mathrm{ME}}$.

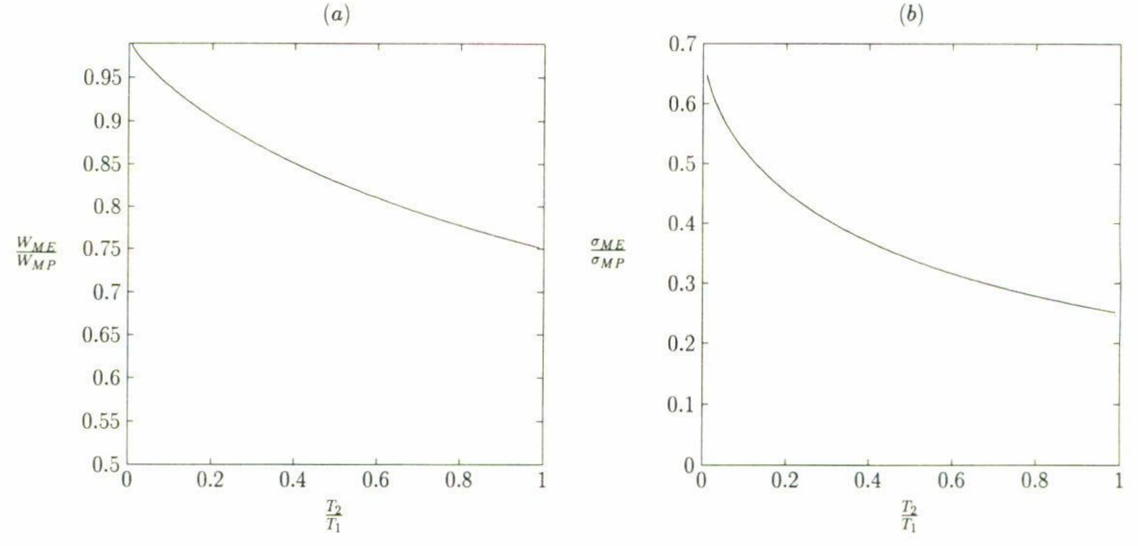

FIGURE 5. Comparisons: (a) between power output at maximum W and maximum E regimes and (b) between entropy production at the same both regimes.

and

$$Q_2 = \beta(T_{2w} - T_2)^k, \tag{22}$$

with $k = 5/4$ takes into account combined conductive-convective and radiative cooling [10,11]. This law may be seen as a curve fitting of the Newton and Stefan-Boltzmann contributions to the heat exchange between the working fluid and the heat reservoirs (for at least certain temperature interval) [11]. In Ref. [10], by using this law of cooling, the

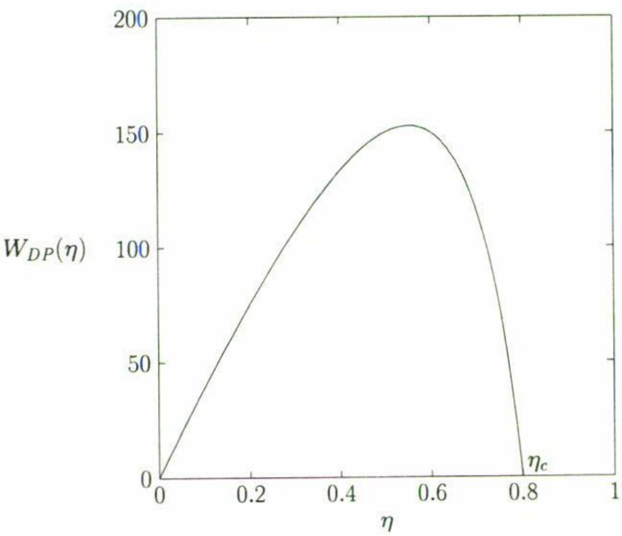

FIGURE 6. Power output in terms of η by using a DP–law of cooling.

authors improved efficiency calculations made with Newton's law of cooling only [3, 12] (see Table I of Ref. [10]). Following the same procedure as in the previous cases [Eqs. (10), (11) and (13)] for the DP–case, we obtain

$$W_{\mathrm{DP}} = \alpha\beta\frac{\eta}{1-\eta}\left[\frac{(1-\eta)T_1 - T_2}{\alpha^{1/k} + \beta^{1/k}(1-\eta)^{\frac{k-1}{k}}}\right]^k, \tag{23}$$

which is a convex curve with an unique maximum point, as is depicted in Fig. 6.

To obtain $\eta_{\mathrm{MP_{DP}}}$ which maximizes Eq. (23) we solve the equation

$$\frac{dW_{\mathrm{DP}}(\eta)}{d\eta} = 0 \tag{24}$$

by means of numerical methods. Some of the numerical results of Eq. (24) can be seen in Table I (third column), and they agree with those reported in Ref. [10] for the same power plants. As it was stated in that reference, by using the Dulong-Petit's law of cooling the efficiency calculations are improved in comparison with the CA–efficiency results which use the Newton's law of heat transfer.

For the case of the DP–law of cooling, we also can express an ecological function as

$$E_{\mathrm{DP}}(\eta) = W_{\mathrm{DP}}(\eta) - T_2\sigma_{\mathrm{DP}}(\eta). \tag{25}$$

To find this function we need to construct the entropy production σ_{DP} for this law of cooling,

$$\sigma_{\mathrm{DP}} = \frac{Q_2}{T_2} - \frac{Q_1}{T_1}, \tag{26}$$

TABLE I. Comparison between $\eta_{\mathrm{MP_{DP}}}$, $\eta_{\mathrm{ME_{DP}}}$ and η_{obs} for several power plants.

Power plant	T_2 (K)	T_1 (K)	$\eta_{\mathrm{MP_{DP}}}$	$\eta_{\mathrm{ME_{DP}}}$	η_{obs}	k
West Thurrock (U.K.) 1962 conventional coal fired steam plant	298	838	0.366	0.49	0.36	5/4
Lardarello (Italy) geothermal steam plant	353	523	0.16	0.239	0.16	5/4
1936–1940 Central steam-power stations in the U.K.	298	698	0.281	0.416	0.28	1.55
1956 steam–power plant in the U.S.	298	923	0.392	0.52	0.40	5/4
1949 combined–cycle (steam and mercury) plant in the U.S.	298	783	0.347	0.47	0.34	5/4
Doel 4 (Belgium)	283	566	0 258	0.373	0.35	5/4

which implies [by substitution of Eqs. (21) and (22) in (26)]

$$\sigma_{\mathrm{DP}} = \frac{\alpha\beta}{T_1T_2}\frac{(1-\eta)T_1 - T_2}{1-\eta}\left[\frac{(1-\eta)T_1 - T_2}{\alpha^{1/k} + \beta^{1/k}(1-\eta)^{\frac{k-1}{k}}}\right]^k. \tag{27}$$

By substituting Eqs. (27) and (23) in (25) we obtain

$$E_{\mathrm{DP}}(\eta) = \frac{\alpha\beta\eta}{1-\eta}\left[\frac{(1-\eta)T_1 - T_2}{\alpha^{1/k} + \beta^{1/k}(1-\eta)^{\frac{k-1}{k}}}\right]^k - \frac{\alpha\beta}{T_1}\frac{(1-\eta)T_1 - T_2}{1-\eta}\left[\frac{(1-\eta)T_1 - T_2}{\alpha^{1/k} + \beta^{1/k}(1-\eta)^{\frac{k-1}{k}}}\right]^k, \tag{28}$$

which also is a convex curve in η with one maximum point (see Fig. 7).

If we take data for the Doel nuclear power plant reported in Ref. [7], which are $T_1 = 566$ K and $T_2 = 283$ K, and we suppose that it was designed with an ecological criterion (maximizing E_{DP}) we obtain by means of Eq. (28) $\eta_{\mathrm{ME_{DP}}} = 0.37$ in good agreement with the observed efficiency, $\eta_{\mathrm{obs}} = 0.35$. (see last row in Table I)

In Table II we show the so-called ecological efficiency $\eta_{\mathrm{E_{DP}}}$ calculated by means two procedures: Solving numerically $\frac{dE_{\mathrm{DP}}}{d\eta} = 0$, to find the point ($\eta_{\mathrm{ME_{DP}}}$) where E_{DP} has a maximum, and by means of expression

$$\eta_{\mathrm{ME_{DP}}} \approx \eta^{\dagger}_{\mathrm{ME_{DP}}} \equiv \frac{\eta_C + \eta_{\mathrm{MP_{DP}}}}{2}, \tag{29}$$

which is the analogous of Eq. (20). We see that the property showed in Ref. [9] (expressed by Eq. (20)) for the Newton's law of cooling case, is retained for the DP-case. The k exponent in Eqs. (21) and (22) may have other values in the interval [1.1, 1.6] [11]. For the case of the Steam Power plant (1936–1940 Central steam-power stations in the UK) the better value for k is 1.55 [10]. In this case, we also observe, that the property expressed

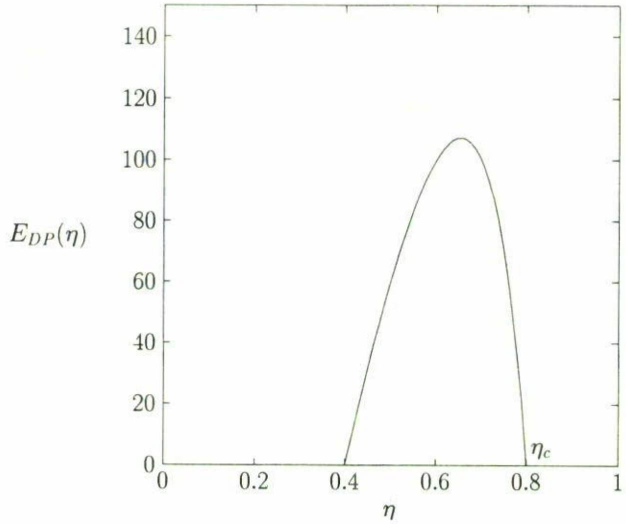

FIGURE 7. Ecological function in terms of η for the DP–law of cooling case with $k = 5/4$.

TABLE II. Semisum property [Eq. (29)] for $k = 5/4$ and for $k = 1.55$: 1936–1940 Central steam-power stations in the UK.

Power plant	η_C	$\eta_{MP_{DP}}$	$\eta_{ME_{DP}}$	$\eta^{\dagger}_{ME_{DP}}$	k
West Thurrock (U.K.) 1962 conventional coal fired steam plant	0.64	0.366	0.49	0.50	5/4
Lardarello (Italy) geothermal steam plant	0.32	0.16	0.239	0.24	5/4
1936–1940 Central steam-power stations in the U.K.	0.57	0.281	0.416	0.42	1.55
1956 steam–power plant in the U.S.	0.67	0.392	0.52	0.53	5/4
1949 combined–cycle (steam and mercury) plant in the U.S.	0.61	0.347	0.47	0.47	5/4
Doel 4 (Belgium)	0.50	0.258	0.373	0.37	5/4

by Eq. (29) (but with $k = 1.55$) is approximately maintained. The results summarized in Table II suggest that the property

$$\eta_{ME} \approx \frac{\eta_C + \eta_{MP}}{2}, \tag{30}$$

with η_{MP} the efficiency at the maximum power regime, is practically independent of the heat transfer law employed, in contrast with η_{MP} which changes with different heat laws [13,14]. In our opinion, the previous property of η_{ME} is a consequence derived from the fact, that the geometrical fashion of the curves representing $W(\eta)$ (see Figs. 2b and 6) is very close to a parabolic behavior. For the case of a true parabola, as $W(\eta) = W_0\eta(\eta_C - \eta)$, the property expressed by Eq. (30) is an equality (see Appendix).

4. CONCLUDING REMARKS

In this paper we showed that the formalism proposed by De Vos consisting in to express relevant quantities in terms of η, is suitable for embracing several optimization criteria for endoreversible engines. In Section 3, we extended the De Vos' results for the case of the ecological optimization criterion by using different heat transfer laws. Our results include a nice reproduction of those reported in Ref. [10](which were calculated by means of a CA–formalism) and numerical calculations which suggest that the semisum property (Eq. (30)) is maintained for at least the three heat transfer laws previously mentioned. As far as we know this property is the first candidate to be independent of the heat transfer law used in endoreversible engines.

APPENDIX

If we take a typical power output *vs.* efficiency curve (for example, see Fig. 6) and propose that it has a parabolic behavior given by

$$W(\eta) = W_0\eta(\eta_C - \eta), \tag{A1}$$

immediately we obtain

$$\eta_{MP} = \tfrac{1}{2}\eta_C. \tag{A2}$$

On the other hand, the ecological function is given by

$$E(\eta) = W(\eta) - T_2\sigma(\eta), \tag{A3}$$

which has a maximum at η_{ME}. Then

$$\left(\frac{dE(\eta)}{d\eta}\right)_{\eta_{ME}} = \left(\frac{dW(\eta)}{d\eta}\right)_{\eta_{ME}} - T_2\left(\frac{d\sigma(\eta)}{d\eta}\right)_{\eta_{ME}} = 0 \tag{A4}$$

and

$$\left(\frac{dW(\eta)}{d\eta}\right)_{\eta_{ME}} = T_2\left(\frac{d\sigma(\eta)}{d\eta}\right)_{\eta_{ME}}. \tag{A5}$$

In Refs. [9] and [10] was showed that the functions $W(\eta)$ and $\sigma(\eta)$ are linked by

$$W(\eta) = g(\eta)\sigma(\eta), \tag{A6}$$

with

$$g = \frac{T_1T_2\eta}{T_1 - T_2 - \eta T_1}. \tag{A7}$$

Equation (A7) holds for both Newton and Dulong-Petit's laws of cooling [9, 10].

By substitution of Eqs. (A1), (A6) and (A7) in (A5), it follows that

$$\eta_{\mathrm{ME}} = \tfrac{3}{4}\eta_{\mathrm{C}}, \tag{A8}$$

then

$$\eta_{\mathrm{ME}} = \tfrac{1}{2}\left[\tfrac{3}{2}\eta_{\mathrm{C}}\right] = \tfrac{1}{2}\left[\eta_{\mathrm{C}} + \tfrac{1}{2}\eta_{\mathrm{C}}\right], \tag{A9}$$

finally, substituting Eq. (A2) in (A9), we obtain

$$\eta_{\mathrm{ME}} = \frac{\eta_{\mathrm{C}} + \eta_{\mathrm{MP}}}{2}, \tag{A10}$$

that is, Eq. (30) as a true equality.

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