Quantization of 3D Chern-Simons theories

C. RAMÍREZ
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
Apartado postal 1364, 72000 Puebla, Pue., México
Recibido el 8 de abril de 1994; aceptado el 25 de agosto de 1994

ABSTRACT. Quantization of Chern-Simons theory is considered. Canonical quantization is analyzed and it is shown that it cannot be consistently done unless renormalization of integrated connections is taken into account. Otherwise quantization at the level of Wilson loop algebra can be done consistently.

RESUMEN. Se estudia la cuantización de las teorías de Chern-Simons en 3 dimensiones. Se analiza la cuantización canónica y se muestra que no puede ser hecha consistentemente a menos que se tome en cuenta la renormalización de las conexiones integradas. De otra manera, la cuantización del álgebra de lazos de Wilson puede ser realizada consistentemente.

PACS: 03.50.Kk; 03.70.+k; 11.15.Tk

1. INTRODUCTION

Chern-Simons (CS) theories in 3 dimensions have been object of study due to their multiple applications: 3D gravity [1,2], knot theory [2,3] and 2D (rational) conformal field theories [3,4].

Let A_i, $i = 0, 1, 2$ be a Lie algebra valued connection of a group G on the 3 dimensional manifold $\mathcal{M}$, the corresponding CS theory is described by the action

$$I = \frac{k}{4\pi} \int_{\mathcal{M}} d^3x\, \epsilon^{ijk}\, \mathrm{tr}\left(A_i \partial_j A_k + \tfrac{2}{3} A_i A_j A_k\right), \tag{1}$$

where k is the coupling constant with dimensions of action and tr is the bilinear form of the Lie algebra of the group G. CS theories are topological field theories, invariant under spacetime diffeomorphisms and independent on the metric of the corresponding manifold.

Under gauge transformations, CS action is gauge invariant up to an additive topological constant, that is

$$A'_i = U^{-1}\, \partial_i\, U + U^{-1}\, A_i U \tag{2}$$

induces

$$I' = I + \frac{k}{4\pi}\, \mathrm{tr} \int_{\mathcal{M}} d^3x\, \epsilon^{ijk}\, \partial_i\, (U^{-1}\, A_j\, \partial_k U) \tag{3}$$

$$- \frac{k}{12\pi}\, \mathrm{tr} \int_{\mathcal{M}} d^3x\, \epsilon^{ijk}\, (U^{-1}\, \partial_i U\, U^{-1}\, \partial_j U\, U^{-1} \partial_k U). \tag{4}$$

The second term on the r.h.s., a total derivative, is a surface term whereas the third, locally also a total derivative, is proportional to the winding number of the transforming group element $U \in G$

$$W(U) = \frac{1}{24\pi^2} \operatorname{tr} \int_{\mathcal{M}} d^3x\, \epsilon^{ijk} (U^{-1}\, \partial_i U\, U^{-1}\, \partial_j U\, U^{-1} \partial_k U), \tag{5}$$

which is given by the homotopy group $\pi_3(G)$, for compact groups an integer number.

Pure C-S theories describe "trivial" motions given by flat connections as can be seen from their equations of motion:

$$F_{ij} = \partial_i A_j - \partial_j A_i - [A_i, A_j] = 0, \tag{6}$$

which are gauge covariant, consistently with the gauge invariance of the action.

In all the mentioned three cases of applications of CS theories, Wilson loops play an important role in defining the corresponding degrees of freedom. The reason is the topological character of the theory, which can be studied only by means of nonlocal objects.

Thus, for the group $\mathrm{ISO}(2,1)$, if $\pi_1(M) \neq 0$, CS theory contains nontrivial degrees of freedom described by Wilson loops [2], integrated along noncontractible curves and which represent motions of point particles. In fact, the resulting theory is equivalent to 3D Einstein gravity [1].

For quantum CS theories with compact groups, with nonvanishing values of the winding number (5), the expectation values of Wilson lines along knotted closed curves give the corresponding Jones polynomials.

Finally, the quantum Hilbert space of a 2D spacelike section punctured by the intersection of Wilson loops, describes the space of conformal blocks of the WZW model with gauge group G [3].

Therefore, the calculation of the commutator algebra of Wilson loops is an important issue. In Ref. [5], this commutator algebra has been considered for the group $\mathrm{ISO}(2,1)$. As usual, a foliation of $M = R \times \Sigma$ has been chosen so that time runs along R and Σ is a spacelike two-manifold. In this case the topology of M is summarized by the one of Σ, and it is enough to study spacelike Wilson loops.

The study of this algebra has been further pursued in Ref. [6] for 3D de Sitter gravity, that is, CS theory of the groups $\mathrm{SO}(3,1)$ and $\mathrm{SO}(2,2)$, where the resulting observable algebra has been quantized obtaining, after suitable reparametrizations, two copies of $\mathrm{SU}(2)_q$. Further work has been done, in the genus one sector of the theory, to calculate the Wilson loop algebra for the following cases: 3D gravity [7], conformal gravity [8] and the Sitter supergravity [9]. Generalizations to include the remaining handles of Σ in the case of the Sitter gravity, have been pursued in Ref. [10].

In all these works the quantized observable algebra turned out to have the structure of a quantum algebra. In Ref. [17] this structure has been considerate for the classical Wilson loop algebra as well as for the quantized one, with an operator product depending on one parameter. In both cases, the structure of the theory turned out to be the same,

differing only the deformation parameter, in the last case having a nontrivial dependence on the Planck constant.

In all works mentioned before, quantization has been performed at the level of the Wilson loop algebra. That is, the Wilson loop Poisson bracket algebra has been converted into a commutator algebra following canonical quantization, so that ambiguous operator products are fixed by means of some ordering prescription as already mentioned.

In this work, starting from the algebra of integrated connections, we consider quantization of Chern-Simons theories. In order to have this algebra in a closed form, a gauge fixing is in order [17]. This form of the algebra is the one taken here into account. In Sect. 2 we consider the classical theory. The Poisson brackets algebra of integrated connections has the structure of a braid-like algebra. Even at the classical level the computation of this algebra is ambiguous and in order to have a consistent result, satisfying Jacobi identities, a "regularization" prescription is necessary [17]. In Sect. 3 the algebra of traces of integrated connections, Wilson loops, is considered, in particular for the group $\mathrm{SL}(2, R)$. In Sect. 4 quantization is tackled. Due to the ambiguities in the definition of integrated conexions, the computations implied are in general not well defined. Thus we have to appeal to consistency criteria in order to obtain a regularization prescription for the computation of products of two integrated connections, defined along the same curve. It turns out that it is not possible to maintain the Jacobi identities and at the same time to have an invariant determinant. One way to step out of these problems is to avoid the ambiguities of the definition of integrated connections considering Wilson loops instead. This way was the one followed in earlier work where quantization was done at this level. In Sect. 5 some conclusions are drawn.

2. Classical theory

In this section we consider the classical Chern-Simons theory described by the action (1). We show how the algebra of integrated connections has to be computed in order to have the Jacobi identities verified.

Integrated connections are defined as follows: if $\gamma\colon R \to \Sigma$ is a curve on Σ, the integrated connection $\Psi(\gamma)\colon \gamma \to G$, is given by the path ordered exponential

$$\Psi(\gamma) = P \exp \int_\gamma A\, dx, \tag{7}$$

is a solution of the differential equation [5]

$$\frac{d\Psi}{ds} = A_s \Psi. \tag{8}$$

Here the curve γ is parameterized by $s \in [0, 1]$ and A_s is the connection tangent to γ at s. Integrated connections depend on the homotopy class of the chosen curve, and closed curves have to be considered in order to produce topological invariants.

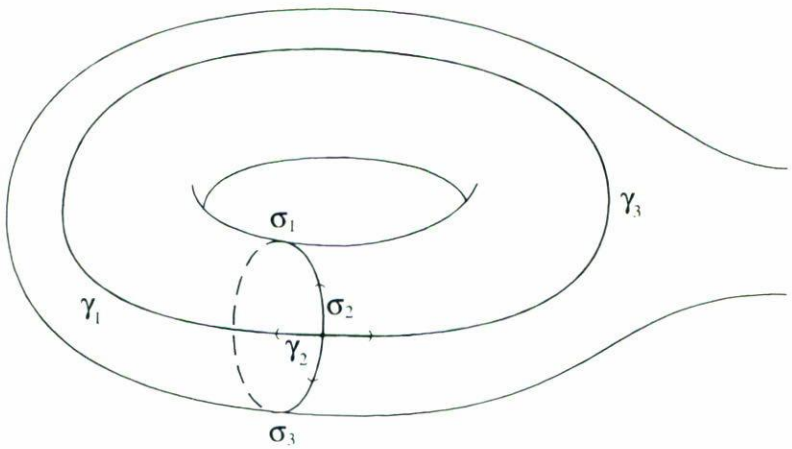

FIGURE 1. Decomposition of two intersecting cycles γ and σ into $\gamma = \gamma_3\gamma_2\gamma_1$ and $\sigma = \sigma_3\sigma_2\sigma_1$. The points B are the base points and C is the crossing point.

The canonical Poisson brackets of the theory can be derived [1,11] from the action (1):

$$\{A_{a\alpha}(t,\vec{x}), A^b{}_\beta(t,\vec{y})\}_{\text{PB}} = \frac{2\pi}{k}\epsilon_{\alpha\beta}\,\delta^b_a\,\delta^2(\vec{x}-\vec{y}), \tag{9}$$

where $\alpha, \beta = 1, 2$ and a is an index of the adjoint representation of G.

Therefore, Poisson brackets of integrated connections differ from zero only if the corresponding curves intersect. Indeed, as shown in Refs. [5,6], we can consider two intersecting closed curves γ and σ, taken for consistency as the two independent cycles of a one genus sector of a genus g two-dimensional surface. Taking arbitrary base points for γ and σ, both curves can be decomposed into three pieces, the central one being in the neighborhood of the crossing point (Fig. 1). Thus

$$\Psi(\gamma) = \Psi(\gamma_3)\Psi(\gamma_2)\Psi(\gamma_1), \qquad \Psi(\sigma) = \Psi(\sigma_3)\Psi(\sigma_2)\Psi(\sigma_1). \tag{10}$$

Taking the canonical Poisson brackets (9) intó account we obtain

$$\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{\text{PB}} = \Psi_1(\gamma_3)\Psi_2(\sigma_3)\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{\text{PB}}\Psi_1(\gamma_1)\Psi_2(\sigma_1), \tag{11}$$

where, as usual, the notation is

$$\Psi_1 = \Psi \otimes 1, \qquad \Psi_2 = 1 \otimes \Psi. \tag{12}$$

Further, the crossing pieces γ_2 and σ_2 are taken to be of infinitesimal length in such a way that

$$\Psi(\gamma_2) = 1 + \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, A_\alpha[\vec{x}(s)]\frac{dx^\alpha(s)}{ds} + \mathcal{O}(\epsilon^2), \tag{13}$$

$$\Psi(\sigma_2) = 1 + \int_{u_0-\epsilon}^{u_0+\epsilon} du\, A_\alpha[\vec{y}(u)] \frac{dy^\alpha(u)}{du} + \mathcal{O}(\epsilon^2). \tag{14}$$

Using this we can calculate

$$\begin{aligned}\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{\text{PB}} &= \frac{2\pi}{k} \int_{s_0-\epsilon}^{s_0+\epsilon} ds \int_{u_0-\epsilon}^{u_0+\epsilon} du\, \epsilon_{\alpha\beta} \frac{dx^\alpha(s)}{ds} \frac{dy^\beta(u)}{du} \delta^2(\vec{x}-\vec{y})(T^a \otimes T_a) \\ &= -\frac{2\pi}{k} s(\gamma,\sigma)\,(T^a \otimes T_a),\end{aligned} \tag{15}$$

where $s(\gamma,\sigma) = \pm 1$ is the signature of the relative orientation of γ and σ and is responsible for the antisymmetry of the Poisson brackets. Therefore, we can take the limit $\epsilon \to 0$ of (15):

$$\begin{aligned}\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{\text{PB}} &= \lim_{\epsilon\to 0} \Psi_1(\gamma_3)\Psi_2(\sigma_3)\{\Psi_1(\gamma_2), \Psi_2(\sigma_2)\}_{\text{PB}}\Psi_1(\gamma_1)\Psi_2(\sigma_1) \\ &= -\frac{2\pi}{k}\, s(\gamma,\sigma)\Psi_1(\gamma_3)\Psi_2(\sigma_3)(T^a \otimes T_a)\Psi_1(\gamma_1)\Psi_2(\sigma_1).\end{aligned} \tag{16}$$

Of course, Eq. (16) depends on the base points of the loops γ and σ. This dependence is a reflection of the gauge covariance of this result. Indeed, under a base point change, integrated connections transform under the adjoint representation

$$\Psi \to G\Psi G^{-1}, \tag{17}$$

where G is the group element corresponding to a curve which joins the two different base points.

The invariance under (17) of the Casimir element

$$C_{12} \equiv T^a \otimes T_a, \tag{18}$$

allows us to rewrite the algebra (16) in a closed form. Indeed, given two crossing inequivalent loops γ and σ, we can always choose them to have a common base point B, different from the crossing point C (Fig. 2), in such a way that in the limit $\epsilon \to 0$ the curve $\gamma_3\sigma_3$ is contractible and the corresponding group elements coincide: $\Psi(\gamma_3) = \Psi(\sigma_3)$. After this gauge fixing, the algebra (16) becomes braid-like:

$$\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{\text{PB}} = r_{12}(\gamma,\sigma)\Psi_1(\gamma)\Psi_2(\sigma), \tag{19}$$

where

$$r_{12}(\gamma,\sigma) = -\frac{2\pi}{k} s(\gamma,\sigma) C_{12}. \tag{20}$$

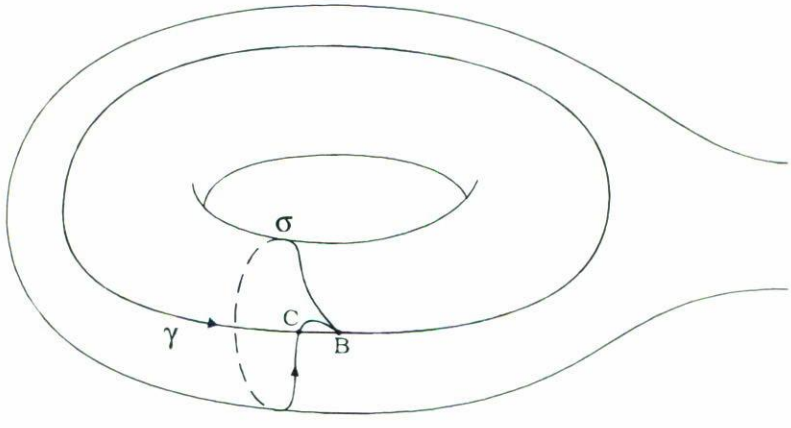

FIGURE 2. Gauge fixing for the cycles γ and σ. B is the common base point and C is the crossing point.

Consistency of such braid equations, in our case the Jacobi identities, would usually imply classical Yang-Baxter equations for the matrix $r_{12}(\gamma,\sigma)$. It is easy to see that the mentioned equations are not satisfied by (20). In fact, the corresponding algebra is seemingly generated by the matrix elements of $\Psi(\gamma)$ and $\Psi(\sigma)$ and such an algebra, even in the classical theory, is not well defined. For instance we expect that the Poisson bracket $\{\Psi_1(\gamma),\Psi_2(\gamma)\}_{\rm PB}$ vanishes. An explicit computation of it cannot be done unless we adopt a sort of regularization which takes care of the infinity of crossing points of γ with itself, for example by

$$\{\Psi_1(\gamma),\Psi_2(\gamma)\}_{\rm PB} = \lim_{\gamma\to\gamma'}\{\Psi_1(\gamma),\Psi_2(\gamma')\}_{\rm PB}, \tag{21}$$

where γ' belongs to the homotopy class of γ. The necessity of such a regularization can be seen if we try to check the Jacobi identities for the algebra generated by $\Psi(\gamma)$ and $\Psi(\sigma)$ in a formal way. Indeed, if we suppose that the matrix elements of $\Psi(\gamma)$ commute among them, we get

$$\begin{aligned}\big\{\{\Psi_1(\gamma),\Psi_2(\sigma)\},\Psi_3(\gamma)\big\} + \big\{\{\Psi_2(\sigma),\Psi_3(\gamma)\},\Psi_1(\gamma)\big\} + \big\{\{\Psi_3(\gamma),\Psi_1(\gamma)\},\Psi_2(\sigma)\big\} &= \\ [r_{12}(\gamma,\sigma),r_{23}(\gamma,\sigma)]\Psi_1(\gamma)\Psi_2(\sigma)\Psi_3(\gamma) \\ = \frac{4\pi}{k^2}f^{abc}T_a\Psi(\gamma)\otimes T_b\Psi(\sigma)\otimes T_c\Psi(\gamma),\end{aligned} \tag{22}$$

which does not identically vanish.

As mentioned, in order to avoid such problems, we can take three different, but equally based elements, say $\Psi(\gamma),\Psi(\sigma)$ and $\Psi(\sigma')$, where σ and σ' belong to the same homotopy

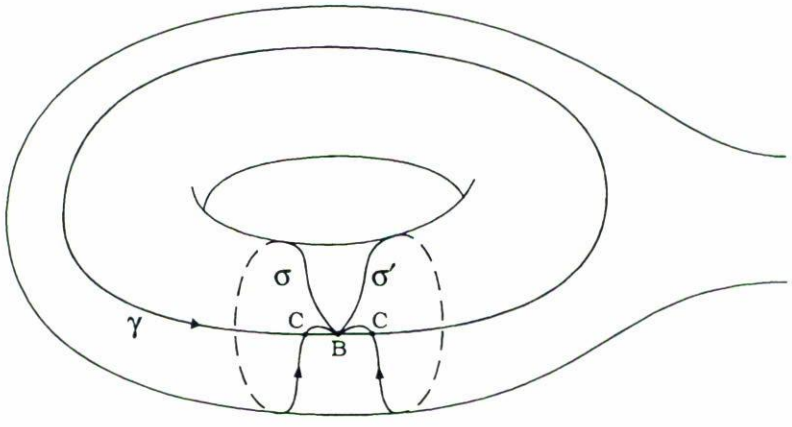

FIGURE 3. Point splitting regularization of two cycles σ into σ and σ'. The points C are crossing points.

class (Fig. 3). However, the gauge (19) cannot be implemented simultaneously for all the loops involved. For each of the brackets we must use (11) and from Fig. 3 we get

$$\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{\text{PB}} = r_{12}(\gamma, \sigma)\Psi_1(\gamma)\Psi_2(\sigma), \tag{23}$$

$$\{\Psi_1(\gamma), \Psi_2(\sigma')\}_{\text{PB}} = \Psi_1(\gamma)\Psi_2(\sigma')r_{12}(\gamma, \sigma'). \tag{24}$$

Taking this into account it is easy to show that the Jacobi identities of (11) are fulfilled since we obtain

$$\left\{\{\Psi_1(\gamma), \Psi_2(\sigma)\}_{\text{PB}}, \Psi_3(\sigma')\right\}_{\text{PB}} = r_{12}\Psi_1(\gamma)\Psi_2(\sigma)\Psi_3(\sigma')r_{13}, \tag{25}$$

$$\left\{\{\Psi_1(\sigma'), \Psi_2(\gamma)\}_{\text{PB}}, \Psi_3(\sigma)\right\}_{\text{PB}} = -r_{12}\Psi_1(\gamma)\Psi_2(\sigma)\Psi_3(\sigma')r_{13}. \tag{26}$$

Another consistency check is related to the determinant. If the elements of the group we started with have determinant one, then the algebra (19) must let the determinant invariant. In fact, it is not difficult to show that this is indeed the case. Let us write for an $n \times n$ matrix

$$\epsilon^{j_1 \dots j_n} \det \Psi = \epsilon^{i_1 \dots i_n} \Psi^{j_1}_{i_1} \dots \Psi^{j_n}_{i_n} \tag{27}$$

or

$$\det \Psi = \frac{1}{n!} \epsilon_{j_1 \dots j_n} \epsilon^{i_1 \dots i_n} \Psi^{j_1}_{i_1} \dots \Psi^{j_n}_{i_n}. \tag{28}$$

Then, if we use the identity

$$\epsilon^{i_1 \dots i_k \dots i_n} \epsilon_{i_1 \dots j_k \dots i_n} = (n-1)!\, \delta^{i_k}_{j_k}, \tag{29}$$

we obtain

$$\{\Psi_1(\gamma), \det \Psi(\sigma)\}_{\rm PB} = -\frac{2\pi}{k} s(\gamma,\sigma)\, \mathrm{tr}(T^a)[T_a \Psi(\gamma)]_1 \det \Psi(\gamma), \tag{30}$$

which vanishes because, as a consequence of the fact that the determinant is one, the Lie algebra generators are traceless.

The algebra of integrated connections obtained in this section is not gauge covariant, it depends on a gauge fixing which cannot be imposed for any situation. If we wish to avoid such problems we must restrict ourselves to invariant quantities as Wilson loops.

3. WILSON LOOP ALGEBRA

Wilson loops are traces of integrated connections. Thus, they do not depend on the base point of the curve along which the connection was integrated and taking the trace of (16) their algebra can be written as

$$\{W(\gamma), W(\sigma)\}_{\rm PB} = -\frac{2\pi}{k} s(\gamma,\sigma)\, \mathrm{tr}\big[T^a \tilde{\Psi}(\gamma)\big]\, \mathrm{tr}\big[T_a \tilde{\Psi}(\sigma)\big], \tag{31}$$

where

$$\tilde{\Psi}(\gamma) = \Psi(\gamma_1)\Psi(\gamma_3), \tag{32}$$

$$\tilde{\Psi}(\sigma) = \Psi(\sigma_1)\Psi(\sigma_3) \tag{33}$$

are the integrated connections corresponding to γ and σ but with base point located at the crossing of the two curves.

The expression (31) can be written more explicitly in terms of Clebsch-Gordan coefficients. Let us consider the tensor product of the irreducible representation $D^{(\lambda)}(\alpha) = \exp \alpha^a T^a$ with itself and its decomposition in irreducible representations. We have

$$D^{(\lambda)}_{m_1 m_1'} D^{(\lambda)}_{m_2 m_2'} = \sum_{\Lambda M M'} \langle \lambda m_1, \lambda m_2 | \Lambda M\rangle\, D^{(\Lambda)}_{MM'}\, \langle \Lambda M' | \lambda m_1', \lambda m_2'\rangle. \tag{34}$$

Expanding both sides of this expression in powers of the group parameters α^a and comparing the quadratic terms we obtain for the Casimir element (18)

$$T^{a(\lambda)}{}_{m_1'}{}^{m_1}\, T_a{}^{(\lambda)}{}_{m_2'}{}^{m_2} = -C(\lambda)\, \delta^{m_1}_{m_1'}\, \delta^{m_2}_{m_2'} + \sum_{\Lambda} C(\Lambda)\, \langle \lambda m_1', \lambda m_2' | \Lambda M\rangle\, \langle \Lambda M | \lambda m_1, \lambda m_2\rangle. \tag{35}$$

Of course, in general this expression is quite complicated to handle. Fortunately, for the fundamental representation, as considered in CS theories, explicit expressions in

terms of the invariant tensors of the group are known. For example for SU(n) the group parametrization can be chosen so that

$$T^{am_1}_{m'_1}\, T_{am'_2}{}^{m_2} \tfrac{1}{2}\left(-\tfrac{1}{n}\delta_{m'_1}{}^{m_1}\,\delta_{m'_2}{}^{m_2} + \delta_{m'_1}{}^{m_2}\,\delta_{m'_2}{}^{m_1}\right) \tag{36}$$

and so on.

As we are rather concerned on the quantization of the theory, in the following we will restrict ourselves to the simplest non-abelian case of $G = \mathrm{SL}(2, R)$ which encloses many of the cases discussed in the literature. In this case we have

$$T^a{}_{m'_1}{}^{m_1}\, T_{am'_2}{}^{m_2} = -\tfrac{1}{4}\delta_{m'_1}{}^{m_1}\,\delta_{m'_2}{}^{m_2} + \tfrac{1}{2}\delta_{m'_1}{}^{m_2}\,\delta_{m'_2}{}^{m_1}, \tag{37}$$

and the resulting algebra is [5,6]

$$\{W(\gamma), W(\sigma)\}_{\mathrm{PB}} = \frac{\pi}{k} s(\gamma,\sigma)\left[-\tfrac{1}{2}W(\gamma)W(\sigma) + W(\gamma\sigma)\right]. \tag{38}$$

The algebra generated by (38) closes due to the following Mandelstam identity, satisfied by 2×2 matrices [5,6,13]:

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B) - (\det A)\,\mathrm{tr}(A^{-1}B). \tag{39}$$

In our case the determinant is one and we have for example $W(\gamma^2\sigma) = -W(\sigma) + W(\gamma)W(\gamma\sigma)$. Using these identities it is possible to show that the only independent generators are [5,6]

$$X_1 = W(\gamma), \quad X_2 = W(\sigma), \quad X_3 = W(\gamma\sigma), \tag{40}$$

which satisfy the algebra [6]

$$\{X_i, X_j\}_{\mathrm{PB}} = \frac{\pi}{k}(\epsilon_{ij}X_iX_j + \epsilon_{ijk}X_k), \tag{41}$$

where $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1$ and ϵ_{ijk} is the 3D Levi-Civita symbol.

In Ref. [17] it has been shown that (41) can be identified with the quantum algebra $\mathrm{SU}(2)_q$. This algebra can be brought to the canonical form of Drinfeld and Jimbo by means of the following nonlinear reparametrization [17,14]:

$$K^{\pm} = X_1 \pm iX_2 e^{\mp\frac{\pi}{k}H}, \tag{42}$$

$$X_3 = \tfrac{i}{2}\left(e^{-\frac{\pi}{k}H} - e^{\frac{\pi}{k}H}\right), \tag{43}$$

resulting

$$\{K^+, K^-\}_{\mathrm{PB}} = \frac{e^{-\frac{2\pi}{k}H} - e^{\frac{2\pi}{k}H}}{e^{\frac{2\pi}{k}} - e^{-\frac{2\pi}{k}}}, \tag{44}$$

$$\{H, K^{\pm}\}_{\mathrm{PB}} = \pm K^{\pm}; \tag{45}$$

that is, the deformation parameter is real and is given by

$$q = e^{-\frac{\pi}{k}}, \tag{46}$$

where we must have in mind that the right meaningful combination is the one which appears in the algebra, that is q^H, k^{-1} having dimensions of action.

Therefore, as mentioned in Ref. [17], the "quantum" symmetry of CS theory is not related to quantization, it seems rather to be related to the "interactions" associated to the nonabelian gauge group, in a way similar to the case of the Heisenberg ferromagnet model when anisotropies are introduced.

4. QUANTIZATION

Quantization of CS theories cannot be done following the ordinary schemes of field theory: operator ordering criteria are missing and Wilson loops do not have a simple physical interpretation. As a consequence quantization will imply a certain degree of arbitrariness.

A natural starting point would be canonical quantization, promoting the connections $A_{a\alpha}$ to the status of operators and substituting the basic Poisson brackets relations (9) by commutators.

If we ignore renormalization of integrated connections, possible if we are interested in the quantization of the algebra of integrated connections (16), we can compute the corresponding commutator in the same way as for the classical case, only taking care of the noncommutativity of the factors. In this case the result is the same:

$$[\Psi_1(\gamma), \Psi_2(\sigma)] = -\frac{2\pi}{k} s(\gamma,\sigma)\Psi_1(\gamma_3)\Psi_2(\sigma_3)(T^a \otimes T_a)\Psi_1(\gamma_1)\Psi_2(\sigma_1), \tag{47}$$

or after fixing the gauge as in Fig. 1:

$$[\Psi_1(\gamma), \Psi_2(\sigma)] = r_{12}(\gamma,\sigma)\Psi_1(\gamma)\Psi_2(\sigma). \tag{48}$$

The same is valid for the Jacobi identities when calculated as in Eqs. (25) and (26) but based on the algebra (47).

So far it seems to be all right. However, one key argument in the classical theory was the fact that the determinant could be set equal to one. If the determinant cannot be set equal to one, the Mandelstam relations (39) become very complicated and we cannot restrict ourselves to the algebra generated by $\Psi(\gamma), \Psi(\sigma)$ and $\Psi(\gamma\sigma)$ in order to get a consistent and closed Wilson loop algebra.

Indeed, in the preceding section we have shown that the Jacobi identities are satisfied for $\Psi(\gamma), \Psi(\sigma)$ and $\Psi(\sigma')$. However, if we wish to check them, instead of $\Psi(\sigma')$, for composite expressions like $\Psi(\gamma\sigma)$ we need to consider these expressions as well in a regularized form (Fig. 4):

$$\Psi(\gamma\sigma) \to \Psi(\gamma'\sigma') \tag{49}$$

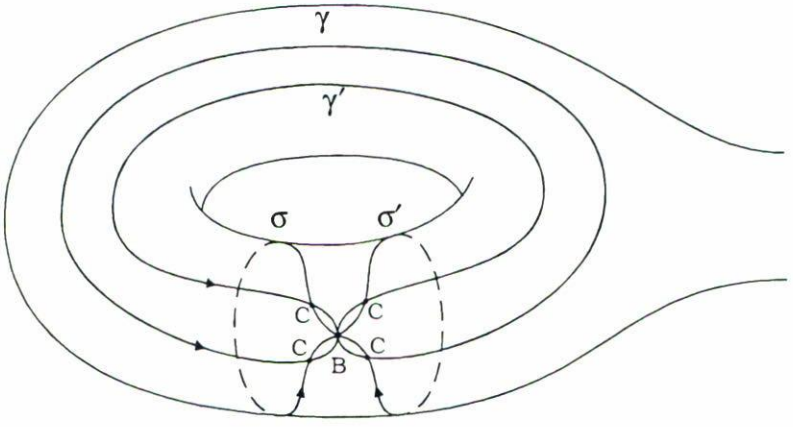

FIGURE 4. Point splitting regularization of two cycles γ and two cycles σ into γ and γ', and σ and σ' respectively. The points C are crossing points. The paths going from crossing points and back are contractible.

If we must take into account relations more complicated than $\Psi(\gamma\sigma)$, then it seems not possible to have a regularization consistent with the gauge (23) and (24). This fact can be checked if we try to add one more curve γ'' or σ'' in Fig. 4 in such a way that it has the same base point B as the others and respects the gauge fixing leading to (23) or (24). The reason is that, in order to fulfil the gauge fixing, the closed path going from the base point to the intersecting point and viceversa must be contractible, *i.e.* it must no contain inserted operators. In particular, if it contains a portion of an integrated connection, it will not be contractible.

Once this point clarified, the main problem in the quantized theory is that the determinant is not invariant and cannot be set equal to one. Indeed, an explicit computation in the case of $G = \mathrm{SL}(2, R)$ gives

$$[\Psi_1(\gamma), \det \Psi(\sigma)] = -\frac{6\pi^2}{k^2}\Psi_1(\gamma) \det \Psi(\sigma), \tag{50}$$

which obviously does not vanish. We could try to modify the definition of the determinant, for instance we could take the "regularized" versions

$$\det_q \Psi(\sigma) = \tfrac{1}{2}\epsilon^{i_1 i_2}\, \epsilon^{j_1 j_2}\, \Psi(\sigma)^{j_1}_{i_1}\, \Psi(\sigma')^{j_2}_{i_2} \tag{51}$$

or, as in the case of the quantum group $\mathrm{SL}(2, R)_q$

$$\det_q \Psi(\sigma) = \Psi(\sigma)_1{}^1 \Psi(\sigma)_2{}^2 - q\Psi(\sigma)_1{}^2 \Psi(\sigma)_2{}^1 \tag{52}$$

and so on. It turns out that none of these versions is invariant.

Another possibility is to observe the fact that in the computation of (47), after the limit $\epsilon \to 0$, the order of the operators on the r.h.s. is ambiguous. That can be taken into account by means of an ordering prescription as done in Ref. [17] for the Wilson loop algebra. If we take the gauge as in Fig. 1 and adopt the ordering prescription

$$\Psi_1(\gamma)\Psi_2(\sigma) \to \frac{1}{1+a}\big[\Psi_1(\gamma)\Psi_2(\sigma) + a\Psi_2(\sigma)\Psi_1(\gamma)\big] \tag{53}$$

we get, instead of (48), the commutators

$$\big[\Psi_1(\gamma), \Psi_2(\sigma)\big] = r_{12}(\gamma,\sigma)\big[\Psi_1(\gamma)\Psi_2(\sigma) + a\Psi_2(\sigma)\Psi_1(\gamma)\big], \tag{54}$$

which can be rewritten as

$$\big[\Psi_1(\gamma), \Psi_2(\sigma)\big] = \tilde{r}_{12}(\gamma,\sigma)\Psi_1(\gamma)\Psi_2(\sigma), \tag{55}$$

where

$$\tilde{r}_{12}(\gamma,\sigma) = -\frac{2\pi}{k}\left[1 - \frac{2a\pi}{k(1+a)}\, r_{12}(\gamma,\sigma)\right]^{-1} r_{12}(\gamma,\sigma). \tag{56}$$

Actually, taking (53) instead of (48) is not very helpful for our purposes. In fact, similarly to the classical case, the explicit form of the r–matrix is not relevant for the fulfilment of the Jacobi identities. Further, the only thing we can adjust in order to have an invariant determinant is the gauge fixing. On the other side, if the gauge covariant expression (47) lets the determinant invariant, then any gauge fixed expression will do the same thing. It turns out that, independently of the taken definition for the determinant, there is no way to have simultaneously both det $\Psi(\gamma)$ and det $\Psi(\sigma)$ invariant.

For instance, following Fig. 4 we obtain the algebra

$$\big[\Psi_1(\gamma), \Psi_2(\sigma)\big] = r_{12}(\gamma,\sigma)\Psi_1(\gamma)\Psi_2(\sigma), \tag{57}$$

$$\big[\Psi_1(\gamma), \Psi_2(\sigma'^{-1})\big] = -\Psi_1(\gamma)\Psi_2(\sigma'^{-1})r_{12}(\gamma,\sigma), \tag{58}$$

$$\big[\Psi_1(\gamma'^{-1}), \Psi_2(\sigma)\big] = -\Psi_1(\gamma'^{-1})\Psi_2(\sigma)r_{12}(\gamma,\sigma), \tag{59}$$

$$\big[\Psi_1(\gamma'^{-1}), \Psi_2(\sigma'^{-1})\big] = r_{12}(\gamma,\sigma)\Psi_d 1(\gamma'^{-1})\Psi_2(\sigma'^{-1}), \tag{60}$$

which satisfies the Jacobi identities. In this case we get

$$\big[\Psi_(\gamma), \det\Psi(\sigma)\big] = 0. \tag{61}$$

However $[\Psi_1(\sigma), \det\Psi(\gamma)]$ does not vanish.

Taking the mentioned "point-splitting" sort of regularization require in fact careful conventions because the limit $\sigma' \to \sigma$ may not give the same result as the limit $\sigma \to \sigma'$. This is a reflection of the lack of gauge covariance of the chosen algebra.

A solution to this problem would be to redefine integrated connections dividing them by the square root of their determinant, so that their determinant would be one. However, in this case the algebra could not be put in the closed form of the type (23) or (24).

An observation is here in order. We cannot neglect the fact that the equations (57–60) can be written in the form of a braid algebra:

$$\Psi_1(\gamma)\Psi_2(\sigma) = R_{12}(\gamma,\sigma)\Psi_2(\sigma)\Psi_1(\gamma), \tag{62}$$

$$\Psi_1(\gamma)\Psi_2(\sigma'^{-1}) = \Psi_2(\sigma'^{-1})\Psi_1(\gamma)\tilde{R}_{12}(\gamma,\sigma), \tag{63}$$

and so on. The corresponding associativity conditions, which usually lead to the Yang-Baxter equations for the R matrix, are of a more general nature than the Jacobi identities of (57–60), they are more restrictive. It is easy to see that (62) and (63) satisfy in an equally trivial way these associativity conditions, regardless of the form of the matrices R or $\tilde{R}$. The same observation is valid for the equally equivalent q–Jacobi identities corresponding to the q–commutation relations associated to (62) and (63).

Therefore, we can conclude that it is not possible in our heuristic way to perform the canonical quantization of the theory, the main problem being the ambiguities of the definition of the integrated connections in the quantum theory. Actually, we had supposed that we can ignore a possible renormalization of integrated connections and in fact, as far as multilinearity is respected, there is no problem. However, if this condition is not respected, for instance if we wish to set the determinant equal to one, we can expect to have problems as these.

It seems that the only way to avoid such problems is to proceed in the same way as done in earlier work: quantizing the theory at the level of the Wilson loop algebra.

Of course, we could have started taking from the beginning some sort of regularization, for instance considering CS theory in the frame of lattice field theory. Unfortunately, such approaches are much more involved and would be the subject of a separated work.

The quantization of the theory at the level of the Wilson loop algebra was given in Refs. [5], [6], [10] and [17]. the most general form was taken in the last work, where, as mentioned in the introduction, the ordering prescription (53), in this case for Wilson loops, was considered.

The Wilson loop algebra obtained in this way is given by

$$[X_1, X_2] = \frac{i\hbar u}{1 + ia\frac{\hbar u}{1+a}}(X_1X_2 + X_3), \tag{64}$$

which is shown to be equivalent to the same $\mathrm{SU}(2)_q$ algebra as in the classical case

$$[K^+, K^-] = i\hbar\,\frac{\tilde{q}^{2H} - \tilde{q}^{-2H}}{\tilde{q}^2 - \tilde{q}^{-2}}, \tag{65}$$

but with the deformation parameter

$$\tilde{q} = \left(\frac{1 - i\frac{\pi\hbar/k}{1+a}}{1 + ia\frac{\pi\hbar/k}{1+a}}\right)^{\frac{1}{i\hbar}} \tag{66}$$

Of course, our results are based on a naive quantization of the traces algebra (19), even if we have introduced the factor ordering prescription (53). Nevertheless, the resulting deformation parameters are consistent with the classical limit ($\hbar \to 0$) as can be seen in (46) together with (66).

5. CONCLUSIONS

Quantization of 3-dimensional Chern-Simons theory was considered from a heuristic point of view. First, the general structure of the algebra of integrated connections in the classical theory was shown. It turned out that even at the classical level the computation of this algebra is ambiguous. Further, the fact that the determinant is invariant allows to make a consistent formulation. In the quantum theory the invariance of the determinant cannot be insured. In fact, under the regularization prescription (16), the determinant is invariant. However, in the quantum theory this regularization is not realized rigorously, that is, the corresponding limit $\epsilon \to 0$ is not performed explicitly. Instead of that, by means of general considerations and guided by consistency checks, we try to guess how the result should look like. Therefore, in this approach it turns out that the only way to consistently quantize the theory is to consider only invariant and to quantize directly the Wilson loop algebra, thus ignoring the problems related to the invariance of the determinant. Another way is the one proposed in Ref. [10] where the condition on the determinant being one is imposed as a constraint. Nevertheless, in this work the consistency of this constraint with the Wilson loop algebra was not shown.

ACKNOWLEDGEMENTS

This work was partially supported by CONACyT, proyect 3544-E9311.

REFERENCES

1. E. Witten, *Nucl. Phys.* **B311** (1988) 46.
2. E. Witten, *Nucl. Phys.* **B323** (1989) 113.
3. E. Witten, *Comm. Math. Phys.* **121** (1989) 351; S. Carlip, *Nucl. Phys.* **B324** (1989) 106.
4. H. Verlinde, *Nucl. Phys.* **B337** (1990) 652.
5. J.E. Nelson and T. Regge, *Nucl. Phys.* **B328** (1989) 190.
6. J.E. Nelson, T. Regge and F. Zertuche, *Nucl. Phys.* **B339** (1990) 516.
7. F. Zertuche and L.F. Urrutia, *Clas. Quantum Grav.* **9** (1992) 641.
8. F. Zertuche and L.F. Urrutia, *Phys. Letts.* **254 B** (1991) 424.
9. L.F. Urrutia, H. Waelbroeck and F. Zertuche, *Mod. Phys. Lett.* **A7** (1992) 2715.
10. J.E. Nelson and T. Regge, *Comm. Math. Phys.* **141** (1991) 211; J.E. Nelson and T. Regge, *Phys. Letts.* **B272** (1991) 213; J.E. Nelson and T. Regge, *Comm. Math. Phys.* **155** (1993) 561.
11. S. Hojman and L.F. Urrutia, *J. Math. Phys.* **22** (1981) 1896.
12. K. Gawedzki, *Comm. Math. Phys.* **139** (1991) 201.
13. D. Beresntein and L.F. Urrutia, "The equivalence between the Mandelstam and the Cayley-Hamilton identities", preprint ICN-UNAM-1993, to be published in *J. Math. Phys.*

14. T.L. Curtright and C.K. Zachos, *Phys. Letts.* **B243** (1990) 237.
15. L.D. Fadeev, N. Yu. Reshetikin and L.A. Takhtajan, *Algebra i Anal.* **1** (1989) 178 (in russian).
16. A. Alekseev, L. Fadeev and M. Semenov-Tian-Shansky, *Comm. Math. Phys.* **149** (1992) 335.
17. C. Ramírez and L.F. Urrutia, Puebla preprint CF/FCFM-94-1.