Reduced matrix elements for the leading spin zero states in the SU(3) scheme*

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ABSTRACT. We present a method for evaluating reduced matrix elements of two particle transfer, one and two-body operators which are necessary for many applications within the SU(3) scheme. The procedure is applicable for nuclear states with an even number of particles which are coupled at spin zero and belong to the leading SU(3) irreps. Explicit expressions of the highest-weight states are constructed and a *Mathematica* code is used for evaluating the matrix elements.

RESUMEN. Se presenta un método para evaluar los elementos de matriz de los operadores de transferencia de dos partículas, de uno y de dos cuerpos, que son necesarios en varias aplicaciones dentro del esquema SU(3). El procedimiento es aplicable a estados nucleares con un número par de partículas, que estén acoplados a espín cero y pertenezcan a la *irrep* líder de SU(3). Se construyen expresiones explícitas para los estados de máximo peso, y se usa un programa en *Mathematica* para evaluar los elementos de matriz.

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1. INTRODUCTION

The standard problem in analyzing the structure of heavy deformed nuclei within the nonrelativistic spherical shell model is how to reduce the dimensionality of the Hilbert space. In the past decade various truncation schemes [1] have been proposed which exploit symmetries of the interactions that dominate the low-energy structure. There is experimental evidence that strongly supports the view that the nuclear effective interaction appropriate to low-energy excitations must have strong correlation with the pairing

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and quadrupole-quadrupole ($Q \cdot Q$) interactions. Uncovering an SU(3) symmetry in the structure of the higher major shells is therefore crucial to the truncation issue.

The nuclear shell structure is not much different from the three-dimensional harmonic oscillator (HO) up to the *ds*-shell, thus SU(3) was proposed by Elliott [2] as a reasonable *ds*-shell symmetry, which has proven to be useful for truncating the full space down to a tractable size. The quadrupole-quadrupole interaction $Q \cdot Q$ is dominant when many valence nucleons are present, and can be expressed in terms of C_2 which is the second order Casimir invariant of SU(3) and L^2 which is the square of the orbital angular momentum, as $Q \cdot Q = 4C_2 - 3L^2$. Considering the quadrupole-quadrupole interaction with a negative sign, the states which lie lowest in energy belong to the irreducible representation (irrep) of SU(3) with the largest eigenvalues of C_2, which is called the leading irrep. These basis states are those which have the largest intrinsic quadrupole deformation and represent a severe truncation of the shell model space. The SU(3) coupling scheme is a good approximation when the asymptotic Nilsson quantum numbers $[\eta n_z \Lambda]\Omega$ for the single particle states are approximately good and when the Nilsson spin-orbit doublets with $\Omega = \Lambda \pm \frac{1}{2}$ are nearly degenerate. Full space *ds*-shell model calculations have confirmed that the leading irrep do indeed comprise 60 – 80% of the yrast (lowest state of a given spin) eigenstates [2].

For higher shells the magnitude of the spin-orbit splitting is so large that the deformed Nilsson $\Omega = \Lambda \pm \frac{1}{2}$ levels are widely separated and SU(3) is not a good symmetry. In addition, the spin-orbit interaction pushes the state of maximum j down into the next lower shell. But the normal parity levels that remain have the same total angular momentum content as the levels of an oscillator shell of one less quantum and the orbits with $j = l - \frac{1}{2}$ and $j' = (l-2) + \frac{1}{2}$ are nearly degenerated [3]. The pseudo SU(3) scheme exploits this degeneracy. As an example consider the $\eta = 4$ harmonic oscillator shell. The $g_{9/2}$ orbital is pushed down by the spin-orbit interaction into the $\eta = 3$ oscillator shell. This accounts for the fact that 50 rather than 40 defines a shell closure. The remaining normal parity levels are relabeled by the mapping

$$(g_{7/2}, d_{5/2}, d_{3/2}, s_{1/2}) \longrightarrow (\tilde{f}_{7/2}, \tilde{f}_{5/2}, \tilde{p}_{3/2}, \tilde{p}_{1/2}) \tag{1}$$

where $\mathbf{l} + \mathbf{s} = \mathbf{j} = \tilde{\mathbf{l}} + \tilde{\mathbf{s}}$ and the pairs of orbits $(g_{7/2}, d_{5/2})$ and $(d_{3/2}, s_{1/2})$ are close in energy. This mapping defines the pseudo $(\tilde{f}\tilde{p}, \tilde{\eta} = 3)$ shell. The $h_{11/2}$ intruder orbit of the next ($\eta = 5$) harmonic oscillator shell that is pushed down into this region by the spin-orbit term is called the *unique* or abnormal parity level [4]. The mathematical formalism necessary for performing these transformations was developed recently [5] and the mapping is related with a supersymmetry transformation [6].

It has been shown [7] that diagonalizing a general one plus two-body interactions in a space with frozen unique parity states is equivalent to diagonalizing a phenomenological Hamiltonian comprised of products of generators of SU(3) coupled to angular momentum zero, which in the SU(3) $\longrightarrow$ O(3) basis contains only five independent operators, two of them independent of K and L, one which produces the $L(L+1)$ splitting of L states but remains independent of K, and two with both K and L dependence. Slow variations in the five parameters associated with the above mentioned operators are able to give a complete and very accurate description of the ground and gamma band rotational structure of rare earth and actinide nuclei, with the concomitant interband and intraband $E2$ strengths, as well as the 1^+ states with strong $M1$ transitions, and the $E2$ and $M3$ transitions of these nuclei, providing a rigourous test for the pseudo SU(3) model [7,8].

In this work we will give the mathematical formalism needed in addition to that given in [7,8] in order to evaluate the matrix elements of a general two-body interaction. The necessary technology for performing calculations in the SU(3) scheme has been widely developed in the past years. Clebsch-Gordan [9], Wigner and Racah [10] and 9-$(\lambda\mu)$ [11] coefficients are available. Expansion of one and two-body operators in terms of their SU(3) components are given explicitly in the Appendix of the work of Draayer *et al.* [7] and Castaños *et al.* [8]. The Wigner-Eckart theorem allows one to evaluate any matrix elements of tensorial operators in terms of SU(2) and SU(3) Clebsch-Gordan coefficients and a reduced matrix element. Reduced matrix elements are explicitly given in [8] for some one-body operators. In the general case, these can be evaluated in terms of triple-barred reduced matrix elements of the creation operators using the coefficients of fractional parentage introduced by Hecht [12]. A special computer code is also available, where the highest weight states are obtained solving a system of linear equations [13].

The main contribution of this work is to give a simple method for evaluating reduced matrix elements for one and two-body operators in the SU(3) scheme. The method is valid for states with even number of particles, coupled pairwise to spin zero, and belonging to the leading SU(3) irrep $(\lambda\mu)$. Although this may seem quite restrictive, these states are the most important ones for even-even nuclei. Our method utilizes the SU(3) cylindrical basis for explicitly constructing the highest-weight state for a given number of particles, in a way making simple the evaluation of the matrix elements. This procedure has been intensively used in the recent years [14].

The structure of this paper is the following: In Sec. 2 the necessary notation for describing the states and the SU(3) tensorial expansion of the operators is given, together with a brief review of the Wigner-Eckart theorem. The highest weight states are constructed explicitly in Sec. 3, Sec. 4 contains the expressions for the matrix elements of the two-particle transfer, and one and two-body operators between the highest-weight states. In Sec. 5 these matrix elements are used to obtain the reduced matrix elements, including some numerical examples. Final conclusions are drawn in Sec. 6. Appendix A resumes the U(3) group properties and the cylindrical basis, in Appendix B we include the formulas describing the cylindrical to spherical transformation brackets and in Appendix C some SU(3) multiplicities are discussed.

2. STATES AND OPERATORS IN THE SU(3) SCHEME

The many-particle states of N nucleons in a shell of dimension Ω are characterized by a totally antisymmetric irrep of a unitary group of dimension Ω, that is [15],

$$\begin{aligned} &U(\Omega) \leftarrow \text{group symbol},\\ &[1^N] \leftarrow \text{irrep label}. \end{aligned} \tag{2}$$

The basis states are specified in terms of the group chains

$$\begin{array}{ccccccccc} \mathrm{U}(\Omega_N^\sigma) & \rightarrow & \mathrm{U}(\Omega_N^\sigma/2) \times U(2) & \rightarrow & \mathrm{SU}(3) \times \mathrm{SU}(2) & \rightarrow & \mathrm{O}(3) \times \mathrm{SU}(2) & \rightarrow & \mathrm{SU}(2) \\ {[1^{n_N^\sigma}]} & & [f_\sigma] \quad [\tilde{f}_\sigma] & & \rho_\sigma(\lambda_\sigma \mu_\sigma) \quad S_\sigma & & K_\sigma L_\sigma & & J_N^\sigma . \end{array} \tag{3}$$

Under each group the quantum numbers that characterize its irreps are given. The indices ρ and β are the multiplicity labels of the indicated reductions. Note that the decomposition of $\mathrm{U}(\Omega_N)$ into $\mathrm{U}(\Omega_N/2) \times \mathrm{U}(2)$ is a factorization of the normal parity space into orbital and spin degrees of freedom. This is an LS-coupling scheme, where O(3) is the orbital angular momentum group and the final SU(2) refers to the total angular momentum.

In the description of even-even nuclei it is usual to make the following additional assumptions [8]:

i) The most important normal parity configurations are those with highest spatial symmetry, $[f] = [2^{n_N/2}]$. This implies that $S_\pi = S_\nu = 0$; that is, only real (light nuclei) or pseudo (heavy nuclei) spin zero configurations are taken into account.

ii) Leading SU(3) irreps in the proton and neutron spaces will dominate. For these representations $\rho_\pi = \rho_\nu = 1$.

One-body and two-body operators acting in a single harmonic oscillator shell η can be expanded in terms of their SU(3) tensorial components using the SU(3) Clebsch-Gordan coefficients and usual SU(2) algebra [7,8].

We obtain for the one-body terms

$$\begin{aligned} \left\{a_a^\dagger \tilde{a}_b\right\}^{JM} = & \sum_{L,S} U \begin{Bmatrix} l_a & \frac{1}{2} & j_a \\ l_b & \frac{1}{2} & j_b \\ L & S & J \end{Bmatrix} \\ & \times \sum_{(\lambda,\mu)K} \left\langle (\eta,0)l_a; (0,\eta)l_b \| (\lambda,\mu)KL \right\rangle_{\rho=1} \left\{ a^\dagger_{(\eta,0),\frac{1}{2}} \, \tilde{a}_{(0,\eta),\frac{1}{2}} \right\}^{(\lambda,\mu)KL,S;JM} , \end{aligned} \tag{4}$$

where $\{\ \}^\alpha$ means standard angular momentum coupling ($\alpha = JM$ or LM_L), spin coupling ($\alpha = SM_S$) or SU(3), spin and angular momentum coupling ($\alpha = (\lambda,\mu)KLS, JM$), the sub-index in the creation and annihilation operators represent a whole set of single particle labels ($a \equiv \eta_a, l_a, j_a$) and the others symbols are explained below.

There is a similar expression for the two-particle transfer operator

$$\begin{aligned} \left\{a_a^\dagger a_b^\dagger\right\}^{JM} = & \sum_{L,S} U \begin{Bmatrix} l_a & \frac{1}{2} & j_a \\ l_b & \frac{1}{2} & j_b \\ L & S & J \end{Bmatrix} \\ & \times \sum_{(\lambda,\mu)K} \left\langle (\eta,0)l_a; (\eta,0)l_b \| (\lambda,\mu)KL \right\rangle_{\rho=1} \left\{ a^\dagger_{(\eta,0),\frac{1}{2}} \, a^\dagger_{(\eta,0),\frac{1}{2}} \right\}^{(\lambda,\mu)KL,S;JM} , \end{aligned} \tag{5}$$

The two-body operator has a more involved expression [7]:

$$\{a_a^\dagger a_b^\dagger\}^J \cdot \{a_c a_d\}^J =$$

$$\sum_{L_{ab}L_{cd}L_0 S_{ab}S_{cd}} U\left\{\begin{matrix} l_a & \frac{1}{2} & j_a \\ l_b & \frac{1}{2} & j_b \\ L_{ab} & S_{ab} & J \end{matrix}\right\} U\left\{\begin{matrix} l_c & \frac{1}{2} & j_c \\ l_d & \frac{1}{2} & j_d \\ L_{cd} & S_{cd} & J \end{matrix}\right\} U\left\{\begin{matrix} L_{ab} & S_{ab} & J \\ L_{cd} & S_{cd} & J \\ L_0 & L_0 & 0 \end{matrix}\right\}$$

$$\sum_{(\lambda_{ab},\mu_{ab})K_{ab};(\lambda_{cd},\mu_{cd})K_{cd}} \langle(\eta,0)l_a;(\eta,0)l_b\|(\lambda_{ab},\mu_{ab})K_{ab}L_{ab}\rangle_{\rho=1}$$

$$\langle(0,\eta)l_c;(0,\eta)l_d\|(\lambda_{cd},\mu_{cd})K_{cd}L_{cd}\rangle_{\rho=1}$$

$$\sum_{\rho_0(\lambda_0,\mu_0)K_0} \langle(\lambda_{ab},\mu_{ab})K_{ab}L_{ab};(\lambda_{cd},\mu_{cd})K_{cd}L_{cd}\|(\lambda_0,\mu_0)K_0L_0\rangle_{\rho_0}$$

$$\left\{\left\{a^\dagger_{(\eta,0),\frac{1}{2}}\,a^\dagger_{(\eta,0),\frac{1}{2}}\right\}^{(\lambda_{ab},\mu_{ab}),S_{ab}}\left\{\tilde{a}_{(0,\eta),\frac{1}{2}}\,\tilde{a}_{(0,\eta),\frac{1}{2}}\right\}^{(\lambda_{cd},\mu_{cd}),S_{cd}}\right\}^{\rho_0(\lambda_0,\mu_0)K_0L_0,\,L_0;00} \tag{6}$$

We have introduced the $\tilde{a}$ annihilation operator which has the appropriate transformation properties under the SU(3) $\rightarrow$ O(3) schemes, *i.e.*

$$\tilde{a}_{(0,\eta)lm;\frac{1}{2}m_s} \equiv (-1)^{\eta+l+m+\frac{1}{2}+m_s} a_{(\eta,0)l-m;\frac{1}{2}-m_s}. \tag{7}$$

The $U\{-\}$ are SU(2) unitary (Jahn-Hope) 9–j coefficients; in Eq. (6) the first two of them allow the to recoupling of the pairs of creation or annihilation operators from the jj to the L-S scheme, respectively, and the third one gives the total coupling to zero angular momentum, in order to assure the interaction is a scalar under rotations (spherical symmetry). This last assumption has important effects: it implies $L_0 = S_0$ and restricts the available orbital angular momentum. Additionally, for even-even nuclei the normal parity states have zero total spin, and this requires that only $L_0 = 0$ tensor contribute. The $\langle -;-\|-\rangle$ are isoscalar SU(3) coupling coefficients, giving each pair of operators definite SU(3) tensorial properties, and coupling them to a total SU(3) irrep (λ_0,μ_0).

In order to evaluate the matrix elements of the above operators between states of the leading SU(3) irreps, for any weights K,L,M, we will use the Wigner-Eckart theorem. Representing any SU(3) tensor operator by $T^{\rho_0(\lambda_0\mu_0);S_0}_{K_0L_0M_0;M_{S_0}}$, the Wigner-Eckart theorem for the SU(3) algebra states that

$$\langle(\lambda,\mu),\,KLM;\,S=0\,|\,T^{\rho_0(\lambda_0\mu_0);S_0}_{K_0L_0M_0;M_{S_0}}\,|\,(\lambda',\mu'),\,K'L'M';\,S=0\rangle$$

$$= \delta_{S_0,0}\sum_\rho \langle(\lambda',\mu'),\,K'L';\,(\lambda_0,\mu_0),\,K_0L_0\|(\lambda,\mu),\,KL\rangle_\rho (L'M',\,L_0M_0|LM)$$

$$\times\langle(\lambda,\mu);\,S=0|||T^{\rho_0(\lambda_0\mu_0);S_0}|||(\lambda',\mu');S=0\rangle_\rho, \tag{8}$$

where ρ and ρ_0 are multiplicity labels.

In the rest of this work we will describe a method to evaluate the reduced matrix elements that enter in this expression. The next step is the explicit construction of the highest-weight state in each irrep, in a form that allows us to evaluate the full matrix elements. Then by means of the Wigner-Eckart theorem we will obtain from these results the reduced matrix elements.

3. The highest-weight states

The generators of the U(r) group can be expressed in the cylindrical basis as

$$C_{n_1 n_0 n_{-1}}^{n'_1 n'_0 n'_{-1}} = \sum_s a^\dagger_{n_1 n_0 n_{-1},s}\, a^{n'_1 n'_0 n'_{-1},s} = \sum_s C_{n_1 n_0 n_{-1},s}^{n'_1 n'_0 n'_{-1},s}, \tag{9}$$

assuming we are dealing with configuration space states in a single shell η. Using the second quantization formalism the generators of U(3) given in (51) of Appendix A can be expressed in terms of those of U(r) as

$$\mathcal{C}_q^{q'} = \sum_{\mu,\mu'} \langle \mu | c_q^{q'} | \mu' \rangle C_\mu^{\mu'} = \sum_{n_1 n_0 n_{-1}, n'_1 n'_0 n'_{-1}} \langle n_1 n_0 n_{-1} | c_q^{q'} | n'_1 n'_0 n'_{-1} \rangle C_{n_1 n_0 n_{-1}}^{n'_1 n'_0 n'_{-1}}. \tag{10}$$

The weight generators have the simple form

$$\mathcal{C}_q^q = \sum_{n_1 n_0 n_{-1}} n_q\, C_{n_1 n_0 n_{-1}}^{n_1 n_0 n_{-1}}, \qquad q = 1, 0, -1, \tag{11}$$

exhibiting the weights as additive quantities for multiparticle systems. As an example, the 10 one-particle states of the $\eta = 5$ shell are enumerated and listed in the first two columns of Table I. They are ordered in the usual way, giving the maximum weight to the state which has the highest n_1 weight; given they were equal, the highest n_0 weight and so on. The first column enumerates the states and the second one gives their cylindrical weights.

We can now construct the state of maximum weight of a given irrep $[2^{\frac{n}{2}}]$ (n even) of the group U(r). In as much as the states are coupled to spin $S = 0$, we introduce here the following operator, antisymmetric under permutation of the spin s [15]:

$$\Delta_{\mu\mu'}^{12} \equiv a^\dagger_{\mu 1} a^\dagger_{\mu' 2} - a^\dagger_{\mu 2} a^\dagger_{\mu' 1}, \tag{12}$$

where the μ are spatial labels, and the numbers 1 and 2 account for the two possible spin projections $m_s = \frac{1}{2}$ and $-\frac{1}{2}$ respectively.

The two-particle system includes all the irreps available by the direct product $(\eta, 0, 0) \otimes (\eta, 0, 0)$. Continuing with the $\eta = 5$ example we have

$$(5,0,0) \otimes (5,0,0) = (10,0,0) \oplus (9,1,0) \oplus (8,2,0) \oplus (7,3,0) \oplus (6,4,0) \oplus (5,5,0). \tag{13}$$

TABLE I. Weights of the 10 one-particle and 10 one-hole states in the $\eta = 5$ shell. The first and third columns ennumerate the weights, columns 2 and 4 give the cylindrical weights n_1, n_0, n_{-1} for particles and holes, respectively.

i	$\mu(i)$			h	$\mu(h)$		
	n_1	n_0	n_{-1}		n_1	n_0	n_{-1}
1	5	0	0	1	0	0	5
2	4	1	0	2	0	1	4
3	4	0	1	3	1	0	4
4	3	2	0	4	0	2	3
5	3	1	1	5	1	1	3
6	3	0	2	6	2	0	3
7	2	3	0	7	0	3	2
8	2	2	1	8	1	2	2
9	2	1	2	9	2	1	2
10	2	0	3	10	3	0	2

or, in the more familiar (λ, μ) notation,

$$(5,0) \otimes (5,0) = (10,0) \oplus (8,1) \oplus (6,2) \oplus (4,3) \oplus (2,4) \oplus (0,5). \tag{14}$$

We have stated above that there are physical reasons, related with the operator $Q \cdot Q$ and its expression in terms of the Casimir C_2, for selecting from this set of irreps only the irrep with highest $(\lambda + \mu)$ value. In this case, for two particle in the $\eta = 5$ shell, this means the $(10,0)$ irrep. We will show now that the state of highest weight of this irrep, *i.e.*, the state with weight $(10,0,0)$ is

$$\Delta^{12}_{\mu(1)\,\mu(1)}|\ \rangle = \Delta^{12}_{(5,0,0)(5,0,0)}|\ \rangle, \tag{15}$$

where $\mu(i)$ are the cylindrical labels n_1, n_0, n_{-1} of the i–th one-particle state, as listed in Table I. It is easy to show that this state is of highest weight of U(21), because the raising operators $c_{\mu}^{\mu'}$ cannot further raise the label $(5,0,0)$ (it is the state of highest weight for one-particle). This state must also be the highest-weight state of SU(3), in the leading irrep for two particles in the fifth oscillator shell. In this particular case this is also easy to show because the raising operators c_1^0, c_1^{-1}, c_0^{-1} all give zero when acting on it, the first two because it is not possible to raise the n_1 label above 10, given we have twice $\eta = 5$ phonons, and the third because it is not possible to lower the n_{-1} label below 0.

Systems with more particle pairs present slightly more difficulties. Given the additivity of the weights, the state of highest weight must have a pair of particles in the state 1, another in the state 2, and so on, up to complete their total number. And because the particles are fermions, one can put only two in each state, one with spin up and the other with spin down. But one can have combinations of pairs giving the same weight, for example $\Delta^{12}_{(5,0,0)(5,0,0)}\Delta^{12}_{(4,1,0)(4,1,0)}$ and $\Delta^{12}_{(5,0,0)(4,1,0)}\Delta^{12}_{(5,0,0)(4,1,0)}$. There is an important

TABLE II. The maximum weights, characterizing the leading irreps in the $\eta = 5$ shell for $n = 2$ to $n = 16$ even particle states. Columns 2, 3 and 4 give their cylindrical weights n_1, n_0, n_{-1} respectively, while columns 5, 6 give their associated labels λ_m, μ_m.

n	n_1	n_0	n_{-1}	λ_m	μ_m
2	10	0	0	10	0
4	18	2	0	16	2
6	26	2	2	24	0
8	32	6	2	26	4
10	38	8	4	30	4
12	44	8	8	36	0
14	48	14	8	34	6
16	52	18	10	34	8

result which will help us solve this ambiguity:

$$\Delta^{12}_{\mu\mu'}\,\Delta^{12}_{\mu\mu'}|\,\rangle = -\Delta^{12}_{\mu\mu}\,\Delta^{12}_{\mu'\mu'}|\,\rangle. \tag{16}$$

The proof of this statement can be obtained by expressing the Δ's in their creation and annihilation operator components, rearranging them, and recognizing that two identical creation operators acting over the vacuum give zero.

The above result gives us the form of the four-particle highest-weight state (hws) of U(r). Generalizing this result we obtain, for a system of n (even) particles

$$\big|[2^{\frac{n}{2}}],\,\text{hws}\big\rangle = \frac{1}{2^{\frac{n}{2}}}\prod_{i=1}^{\frac{n}{2}}\Delta^{12}_{\mu(i)\,\mu(i)}|\,\rangle. \tag{17}$$

Acting with the raising operators of the U(r) group onto this state will give zero by construction. There exist only one state of highest weight, therefore we have constructed its wave function explicitly. The determination of the SU(3) weights of this state is easy. We need only to sum twice the weights of the single-particle states, up to the numbers of pairs in the state, *i.e.*,

$$n_q(n) = \sum_{i=1}^{\frac{n}{2}} 2n_q(i), \qquad q = 1, 0, -1,$$

$$\lambda_m(n) = n_1(n) - n_0(n), \qquad \mu_m(n) = n_0(n) - n_{-1}(n). \tag{18}$$

For the example in the $\eta = 5$ shell, in Table II we show the highest weights characterizing the leading irreps for one to eight pairs of particles.

The leading irrep (λ_m, μ_m) contains once and only once the state with the highest angular momentum $L_{\max} = \lambda_m + \mu_m$, and magnetic projection $M_{\max} = L_{\max}$. Again, using the uniqueness of this highest-weight state and the fact that it has spin $S = 0$, we have

$$\big|[2^{\frac{n}{2}}],\,\mathrm{hws}\big\rangle = \big|[2^{\frac{n}{2}}],\,(\lambda_m,\mu_m)\,K=1\,L_{\max}M_{\max},\,S=0\big\rangle. \tag{19}$$

If the shell is more than half filled, *i.e.*, $\Omega/2 = r < n \leq \Omega$ the above described formalism *does not* describe the highest-weight state. Instead, it is necessary to replace particles by holes. We start with a completely filled ground state $|h\rangle$, and the highest-weight state is

$$\begin{aligned}\big|[2^{n_h/2}],\,\mathrm{hws}\big\rangle &= \big|[2^{n_h/2}],\,(\lambda_m,\mu_m)\,K=1\,L_{\max}M^h_{\max},\,S=0\big\rangle \\ &= \frac{1}{2^{n_h/2}}\prod_{h=1}^{n_h/2}\big\{\Delta^{12}_{\mu(h)\,\mu(h)}\big\}^{\dagger}|h\rangle,\end{aligned} \tag{20}$$

where we defined $n_h \equiv \Omega - n$ and $M^h_{\max} = -L_{\max}$. The hole states are inversely ordered than the particles one and, for the μ–th one particle-state weight, with $\Omega/2 < \mu \leq \Omega$

$$\big(n_1(h), n_0(h), n_{-1}(h)\big)_{\text{holes}} = \big(n_1(i), n_0(i), n_{-1}(i)\big)_{\text{particles}}; \qquad i = r - h. \tag{21}$$

These one-hole state weights are also exhibited in the last two columns of Table I.

An additional useful relationship is

$$\big(\lambda_m(n_h), \mu_m(n_h)\big) = \big(\mu_m(n), \lambda_m(n)\big). \tag{22}$$

These are the key results in our construction. Using the Wigner-Eckart theorem we will be able to evaluate any matrix elements between states in these irreps (λ_m, μ_m), whatever the weights K, L, M they could have.

4. The matrix elements for the highest-weight states

In order to evaluate the matrix elements for the highest-weight states of the operators introduced in Eqs. (4), (5) and (6) we reexpand them in the cylindrical basis. The wanted expressions are:

$$\begin{aligned}&\Big\{a^{\dagger}_{(\eta 0)\frac{1}{2}}\,\tilde{a}_{(0\eta)\frac{1}{2}}\Big\}^{(\lambda,\mu),S=0}_{KLM} \\ &= \frac{1}{\sqrt{2}}\sum_{l_1 l_2}\big\langle(\eta 0)1l_1,(0\eta)1l_2\|(\lambda,\mu)KL\big\rangle_{\rho=1}\sum_{m_1 m_2}(-1)^{-m_2}(l_1m_1,l_2{-}m_2|LM) \\ &\quad \sum_{n^1_1 n^1_0 n^1_{-1},\,n^2_1 n^2_0 n^2_{-1}}\big\langle\eta l_1 m_1|n^1_1 n^1_0 n^1_{-1}\big\rangle\big\langle\eta l_2 m_2|n^2_1 n^2_0 n^2_{-1}\big\rangle\sum_{\sigma}a^{\dagger}_{n^1_1 n^1_0 n^1_{-1};\frac{1}{2}\sigma}a_{n^2_1 n^2_0 n^2_{-1};\frac{1}{2}\sigma}\end{aligned} \tag{23}$$

and

$$\left\{ \left\{ a^{\dagger}_{(\eta,0),\frac{1}{2}} \, a^{\dagger}_{(\eta,0),\frac{1}{2}} \right\}^{(\lambda,\mu),S} \left\{ \tilde{a}_{(0,\eta),\frac{1}{2}} \, \tilde{a}_{(0,\eta),\frac{1}{2}} \right\}^{(\lambda',\mu'),S} \right\}^{\rho_0(\lambda_0,\mu_0);S_0}_{K_0L_0=S_0M_{L_0}=-M_{S_0}}$$

$$= \sum_{KK'LL'MM'} \left\langle (\lambda,\mu)KL, (\lambda',\mu')K'L' \| (\lambda_0,\mu_0)K_0L_0 \right\rangle_{\rho_0}$$

$$\sum_{l_1l_2l_3l_4} \left\langle (\eta 0)1l_1, (\eta 0)1l_2 \| (\lambda\mu)KL \right\rangle_{\rho=1} \left\langle (0\eta)1l_3, (0\eta)1l_4 \| (\lambda'\mu')K'L' \right\rangle_{\rho=1}$$

$$\sum_{m_1m_2m_3m_4} (l_1m_1, l_2m_2|LM)\,(l_3m_3, l_4m_4|L'M')\,(LM, L'-M'|L_0M_0)\,(-1)^{L'-M'}$$

$$\sum_{n^1_1n^1_0n^1_{-1},n^2_1n^2_0n^2_{-1}} \left\langle \eta l_1m_1|n^1_1n^1_0n^1_{-1}\right\rangle \left\langle \eta l_2m_2|n^2_1n^2_0n^2_{-1}\right\rangle$$

$$\sum_{n^3_1n^3_0n^3_{-1},n^4_1n^4_0n^4_{-1}} \left\langle \eta l_3m_3|n^3_1n^3_0n^3_{-1}\right\rangle \left\langle \eta l_4m_4|n^4_1n^4_0n^4_{-1}\right\rangle$$

$$\left\{ \left\{ a^{\dagger}_{n^1_1n^1_0n^1_{-1};\frac{1}{2}} \, a^{\dagger}_{n^2_1n^2_0n^2_{-1};\frac{1}{2}} \right\}^{S} \left\{ a_{n^3_1n^3_0n^3_{-1};\frac{1}{2}} \, a_{n^4_1n^4_0n^4_{-1};\frac{1}{2}} \right\}^{S} \right\}^{S_0}_{M_{S_0}}, \tag{24}$$

where the $\langle \eta lm|n_1n_0n_{-1}\rangle$ are the transformation brackets between the spherical and cylindrical basis [16] and are given in Appendix B. It must be noted that, using Eq. (17),

$$\left\{ a^{\dagger}_{\mu,\frac{1}{2}} \, a^{\dagger}_{\mu',\frac{1}{2}} \right\}^{S=0} = 2^{-\frac{1}{2}} \Delta^{12}_{\mu\mu'}. \tag{25}$$

Using the above expression, we obtain

$$\left\{ \tilde{a}_{(0\eta)\frac{1}{2}} \, \tilde{a}_{(0\eta)\frac{1}{2}} \right\}^{(\lambda,\mu),S=0}_{KLM}$$

$$= -\frac{1}{\sqrt{2}}(-1)^{L-M} \sum_{l_1l_2} \left\langle (0\eta)1l_1, (0\eta)1l_2 \| (\lambda,\mu)KL \right\rangle_{\rho=1} \sum_{m_1m_2} (l_1m_1, l_2m_2|L-M)$$

$$\sum_{n^1_1n^1_0n^1_{-1},n^2_1n^2_0n^2_{-1}} \left\langle \eta l_1m_1|n^1_1n^1_0n^1_{-1}\right\rangle \left\langle \eta l_2m_2|n^2_1n^2_0n^2_{-1}\right\rangle \left(\Delta^{12}_{(n^1_1n^1_0n^1_{-1})(n^2_1n^2_0n^2_{-1})} \right)^{\dagger}. \tag{26}$$

The second step is to calculate the matrix elements of the spatially decoupled cylindrical operators which appear in the right hand of Eqs. (23), (24) and (26) between the highest-weight states defined in Eq. (17). As we work only with states coupled to $S = 0$, it follows that the above operators will have matrix elements different from zero only if $S_0 = 0$. For the one-body operator we obtain

$$\left\langle \left[2^{\frac{n}{2}}\right], \text{hws} \left| \sum_{\sigma} a^{\dagger}_{\mu_1;\frac{1}{2}\sigma} a_{\mu_2;\frac{1}{2}\sigma} \right| \left[2^{\frac{n}{2}}\right], \text{hws} \right\rangle = 2\delta_{\mu_1\mu_2} \sum_{i=1}^{\frac{n}{2}} \delta_{\mu_2\mu(i)}, \tag{27}$$

where if $\mu_1 \neq \mu_2$ when the one-body operator acts over the hws state generates a state which is orthogonal to it and the matrix element becomes null, and if $\mu_1 = \mu_2$ it counts the two nucleons occupying the state or gives zero for unoccupied states.

For the two particle transfer operator we need to evaluate

$$\begin{aligned} &\left\langle \left[2^{\frac{n}{2}}\right], \text{hws} \middle| \left(\Delta^{12}_{\mu_1\mu_2}\right)^{\dagger} \middle| \left[2^{\frac{n}{2}+1}\right], \text{hws} \right\rangle \\ &\qquad = \tfrac{1}{2}\left\langle \left[2^{\frac{n}{2}}\right], \text{hws} \middle| \left(\Delta^{12}_{\mu_1\mu_2}\right)^{\dagger} \Delta^{12}_{\mu(\frac{n}{2}+1)\mu(\frac{n}{2}+1)} \middle| \left[2^{\frac{n}{2}}\right], \text{hws} \right\rangle = 2\delta_{\mu_1\mu_2}\delta_{\mu_1\mu(\frac{n}{2}+1)}. \end{aligned} \tag{28}$$

In this case we used the definition (12) of $\Delta^{12}_{\mu_1\mu_2}$ and (17) of the highest-weight state, evaluated the commutator and recognized that $\left\langle \left[2^{\frac{n}{2}}\right], \text{hws} \middle| a^{\dagger}_{\mu(\frac{n}{2}+1)} = 0\right.$. The $\mu(\frac{n}{2}+1)$ are the cylindrical labels of the $\frac{n}{2}+1$ one-particle state.

The matrix element of the spatially decoupled two body operator is

$$\begin{aligned} &\left\langle \left[2^{\frac{n}{2}}\right], \text{hws} \left| \left\{ \left\{ a^{\dagger}_{\mu_1;\frac{1}{2}} a^{\dagger}_{\mu_2;\frac{1}{2}} \right\}^{S'} \left\{ \tilde{a}_{\mu_3;\frac{1}{2}} \tilde{a}_{\mu_4;\frac{1}{2}} \right\}^{S'} \right\}^{S_0}_{M_{S_0}} \right| \left[2^{\frac{n}{2}}\right], \text{hws} \right\rangle \\ &\qquad = \delta_{S_0 0}(\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3} \pm \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}) \sum_{i\leq j=1}^{\frac{n}{2}} (\delta_{\mu_1\mu(i)}\delta_{\mu_2\mu(j)} + \delta_{\mu_1\mu(j)}\delta_{\mu_2\mu(i)}). \end{aligned} \tag{29}$$

The + sign in the above expression holds for intermediate spin $S' = 0$, the − sign for $S' = 1$. Evaluating the above matrix elements implied the iterative use of the commutation relationships between a and $a^{\dagger}$. This task is difficult when there are, for example, eight creation operators in the maximum weight state on the right, plus two creation and two annihilation operators in the two-body operator, plus another eight annihilation operators in the state on the left. A *Mathematica* [17] code was developed to do for us this algebraic manipulation. This code is available under request.

Finally the desired matrix elements for the coupled operators given in Eqs. (23), (24) and (26) are evaluated. Given $L_0 = S_0$ due to the spherical symmetry of the interaction and $S_0 = 0$, we obtain important simplifications in this expression. Specifically, as shown in Appendix C, this implies that only even λ_0 and μ_0 are allowed.

For one-body coupled operators the matrix elements are

$$\begin{aligned} &\left\langle \left[2^{\frac{n}{2}}\right](\lambda_m, \mu_m), 1L_{\max}M_{\max}; S = 0 \middle| \left\{ a^{\dagger}_{(\eta,0),\frac{1}{2}} \tilde{a}_{(0,\eta),\frac{1}{2}} \right\}^{(\lambda_0,\mu_0)K_0L_0M_0=0;S_0} \right. \\ &\qquad\qquad \left. \middle| \left[2^{\frac{n}{2}}\right](\lambda_m, \mu_m), 1L_{\max}M_{\max}; S = 0 \right\rangle \end{aligned}$$

$$= -\sqrt{2}\,\delta_{S_00} \sum_{l_1 l_2} \langle (\eta 0) 1 l_1, (0\eta) 1 l_2 \| (\lambda_0, \mu_0) K_0 L_0 \rangle_{\rho=1}$$

$$\sum_m (-1)^m (l_1 m, l_2 - m | L_0 0) \times \sum_{i=1}^{\frac{n}{2}} \langle \eta l_1 m | \mu(i) \rangle \langle \eta l_2 m | \mu(i) \rangle, \quad (30)$$

and for the two-particle transfer operators

$$\Big\langle [2^{\frac{n}{2}}](\lambda_m, \mu_m), 1 L_{\max} M_{\max}; S = 0 \Big| \Big\{ \tilde{a}_{(0,\eta)\frac{1}{2}} \tilde{a}_{(0,\eta),\frac{1}{2}} \Big\}^{(\lambda_0,\mu_0)K_0 L_0 M_0; S_0}$$

$$\Big| [2^{\frac{n}{2}+1}](\lambda'_m, \mu'_m), 1 L'_{\max} M'_{\max}; S' = 0 \Big\rangle$$

$$= -\delta_{S_00}\sqrt{2} \sum_{l_1 l_2, m_1 m_2} \langle (0\eta) 1 l_1, (0\eta) 1 l_2 \| (\lambda_0, \mu_0) K_0 L_0 \rangle_{\rho=1}$$

$$(-1)^{L_0 - M_0} (l_1 m_1, l_2 m_2 | L_0 - M_0) \langle \eta l_1 m_1 | \mu(\tfrac{n}{2}+1) \rangle \langle \eta l_2 m_2 | \mu(\tfrac{n}{2}+1) \rangle. \quad (31)$$

In these expressions the $\mu(\frac{n}{2}+1)$ are again the cylindrical labels of the $\frac{n}{2}+1$ one-particle state, and L_0 and M_0 must respectively satisfy $M_0 = M_{\max} - M'_{\max}$ and $L_0 \geq |M_0|$, otherwise the matrix element is zero.

The matrix elements of the two-body coupled operators become

$$\langle [2^{\frac{n}{2}}](\lambda_m, \mu_m) 1 L_{\max} M_{\max}; S = 0 |$$

$$\Big\{ \Big\{ a^\dagger_{(\eta,0),\frac{1}{2}} a^\dagger_{(\eta,0),\frac{1}{2}} \Big\}^{(\lambda,\mu),S} \Big\{ \tilde{a}_{(0,\eta),\frac{1}{2}} \tilde{a}_{(0,\eta),\frac{1}{2}} \Big\}^{(\lambda',\mu'),S} \Big\}^{\rho_0(\lambda_0,\mu_0)K_0 L_0 M_0 = 0; S_0}$$

$$\Big| [2^{\frac{n}{2}}](\lambda_m, \mu_m) 1 L_{\max} M_{\max}; S = 0 \Big\rangle$$

$$= \delta_{S_00} \sum_{KK'LM} \langle (\lambda, \mu) K L, (\lambda', \mu') K' L \| (\lambda_0, \mu_0) 1 0 \rangle_{\rho_0} \frac{1}{\sqrt{2L+1}}$$

$$\sqrt{2S+1} \sum_{l_1 l_2 l_3 l_4} \langle (\eta 0) 1 l_1, (\eta 0) 1 l_2 \| (\lambda \mu) K L \rangle_{\rho=1} \langle (0\eta) 1 l_3, (0\eta) 1 l_4 \| (\lambda' \mu') K' L \rangle_{\rho=1}$$

$$\sum_{m_1 m_2 m_3 m_4} (l_1 m_1, l_2 m_2 | L M)(l_3 m_3, l_4 m_4 | L M) \sum_{i,j=1}^{\frac{n}{2}} \langle \eta l_1 m_1 | \mu(i) \rangle \langle \eta l_2 m_2 | \mu(j) \rangle$$

$$\big(\langle \eta l_3 m_3 | \mu(j) \rangle \langle \eta l_4 m_4 | \mu(i) \rangle \pm \langle \eta l_3 m_3 | \mu(i) \rangle \langle \eta l_4 m_4 | \mu(j) \rangle \big). \quad (32)$$

5. THE REDUCED MATRIX ELEMENTS

As it was mentioned in Sec. 3, using the Wigner-Eckart theorem, Eq. (8), it is easy to obtain the reduced matrix elements. In matrix notation, one must solve the linear system

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{B}, \tag{33}$$

with

$$\mathbf{A}_{ij} = \left\langle (\lambda'_m, \mu'_m) 1 L'_{\max}, (\lambda_0, \mu_0) K_0 L_0 \| (\lambda_m, \mu_m) 1 L_{\max} \right\rangle_\rho (L'_{\max} M'_{\max}, L_0 M_0 | L_{\max} M_{\max})$$

$$i \to K_0, L_0 \quad j \to \rho \tag{34}$$

$$\mathbf{B}_{ik} = \left\langle (\lambda_m, \mu_m), 1 L_{\max} M_{\max}; S = 0 \middle| T^{\rho_0 (\lambda_0 \mu_0); S_0}_{K_0 L_0 M_0; M_{S_0}} \middle| (\lambda'_m, \mu'_m), 1 L'_{\max} M'_{\max}; S = 0 \right\rangle$$

$$i \to K_0, L_0 \quad k \to \rho_0 \tag{35}$$

$$\mathbf{X}_{jk} = \left\langle (\lambda_m, \mu_m); S = 0 \middle\| \middle| T^{\rho_0 (\lambda_0 \mu_0); S_0} \middle\| \middle| (\lambda'_m, \mu'_m); S = 0 \right\rangle_\rho$$

$$j \to \rho \quad k \to \rho_0. \tag{36}$$

The U(3) multiplicity appears twice in the above expression: in the number $N_{\rm st}$ of times the coupling $(\lambda_m, \mu_m) \otimes (\lambda_0 \mu_0) = (\lambda'_m, \mu'_m)$ can occur, labeled with ρ; and in the number $N_{\rm op}$ associated with the operator irrep (λ_0, μ_0) and labeled by ρ_0. The latter multiplicity $N_{\rm op}$ is always equal to one for the one-body and two-particle transfer operators, but can have values greater than one for the two-particle transfer operators. The $\mathbf{A}$ is a $N_{\rm st} \times N_{\rm st}$ matrix, while $\mathbf{B}$ and $\mathbf{X}$ are $N_{\rm st} \times N_{\rm op}$ matrices. The $\mathbf{A}$ matrix must be an invertible matrix. This property is obtained by construction. System (33) is solved for shell number η, number of particle n and operator $T^{\rho_0 (\lambda_0 \mu_0); S_0}$, each column in $\mathbf{A}$ belongs to a different ρ, the operator labels K_0 and L_0 characterize the rows and are selected taking care of that there are no two linearly dependent ones. Thus $\mathbf{A}$ is non-singular.

As an example let us consider the case of $n = 4$ particles in the $\eta = 4$ shell. The highest-weight state (19) is

$$\left| [2^{\frac{n}{2}}], \text{hws} \right\rangle = \left| [2^2], (12,2)\, K = 1\, L_{\max} = 14\, M_{\max} = 14, S = 0 \right\rangle. \tag{37}$$

For the tensor operator we select

$$T^{\rho_0 (\lambda_0 \mu_0); S_0} = \left\{ \left\{ a^\dagger_{(4,0),\frac{1}{2}}\, a^\dagger_{(4,0),\frac{1}{2}} \right\}^{(4,2), S=0} \left\{ \tilde{a}_{(0,4),\frac{1}{2}}\, \tilde{a}_{(0,4),\frac{1}{2}} \right\}^{(0,8), S=0} \right\}^{\rho_0 = 1 (2,2); S_0 = 0} \tag{38}$$

In this particular example, the direct product $(4,2) \otimes (0,8)$ contains only once the irrep $(2,2)$. Then $N_{\rm op} = 1$ and ρ_0 can only take the value $\rho_0 = 1$. By the other side, the coupling $(12,2) \otimes (2,2) = (12,2)$ can occur $N_{\rm st} = 3$ times, and the values $\rho = 1, 2, 3$ are allowed.

We are looking for the reduced matrix elements and need to solve the linear equations

$$\sum_{\rho}\big\langle(12,2)1\,14,(2,2)K_0L_0\|(12,2)1\,14\big\rangle_{\rho}(14\;14,2\;2|14\;14)$$

$$\big\langle[2^2],(12,2);S=0\big|\!\big|\!\big|\left\{\left\{a^{\dagger}_{(4,0),\frac{1}{2}}\,a^{\dagger}_{(4,0),\frac{1}{2}}\right\}^{(4,2),S=0}\left\{\tilde{a}_{(0,4),\frac{1}{2}}\,\tilde{a}_{(0,4),\frac{1}{2}}\right\}^{(0,8),S=0}\right\}^{1(2,2);0}$$

$$\big|\!\big|\!\big|[2^2],(12,2);S=0\big\rangle_{\rho}$$

$$=\big\langle[2^2],(12,2)1\,14\,14;S=0\big|$$

$$\left\{\left\{a^{\dagger}_{(4,0),\frac{1}{2}}\,a^{\dagger}_{(4,0),\frac{1}{2}}\right\}^{(4,2),S=0}\left\{\tilde{a}_{(0,4),\frac{1}{2}}\,\tilde{a}_{(0,4),\frac{1}{2}}\right\}^{(0,8),S=0}\right\}^{1(2,2);0}_{K_0\,L_0\,M_0=0}$$

$$\big|[2^2],(12,2)1\,14\,14;S=0\big\rangle. \quad (39)$$

Given $N_{\rm st}=3$ we need three independent equations. They are obtained selecting $K_0=1$, $L_0=0,2,3$, which were found numerically to generate linear independent equation. The right side of the above equation was evaluated using Eq. (32), while the SU(3) and SU(2) Clebsch-Gordan coefficientes are available as computer codes [10]. The wanted reduced matrix elements are exhibited in the first row of Table IV.

The values of many reduced matrix elements for one-body operators are explicitly given in Ref. [8] and we will not repeat them here. Table III exhibits some reduced matrix elements of two-particle transfer operators. In order to check the correctness of the reduced matrix elements for the two-body operators obtained in the way described above, we have performed the evaluation of the matrix elements of the two-body operators N^2 and $Q\cdot Q$, which have the well know values n^2 and $4C_2-3L(L+1)$ respectively. We used the expressions

$$\big\langle\big[2^{\frac{n}{2}}\big](\lambda_m,\mu_m)KLM;S=0|N^2|\big[2^{\frac{n}{2}}\big](\lambda_m,\mu_m)KLM;S=0\big\rangle$$

$$=n-\frac{(\eta+1)(\eta+2)}{2}\sum_{(\lambda,\mu)(\lambda',\mu')}\left\{\begin{array}{cccc}(\eta,0)&(0,\eta)&(0,0)&1\\(\eta,0)&(0,\eta)&(0,0)&1\\(\lambda,\mu)&(\lambda',\mu')&(0,0)&1\\1&1&1&\end{array}\right\}$$

$$\sum_{S}\sqrt{2S+1}\big\langle\big[2^{\frac{n}{2}}\big](\lambda_m,\mu_m),S=0\big|\!\big|\!\big|$$

$$\left\{\left\{a^{\dagger}_{(\eta,0),\frac{1}{2}}\,a^{\dagger}_{(\eta,0),\frac{1}{2}}\right\}^{(\lambda,\mu),S}\left\{\tilde{a}_{(0,\eta),\frac{1}{2}}\,\tilde{a}_{(0,\eta),\frac{1}{2}}\right\}^{(\lambda',\mu'),S}\right\}^{1(0,0);0}$$

$$\big|\!\big|\!\big|\big[2^{\frac{n}{2}}\big](\lambda_m,\mu_m),S=0\big\rangle_{\rho=1} \quad (40)$$

TABLE III. The reduced matrix elements for the two-particle transfer operator

$$\left\langle [2^{\frac{n}{2}}](\lambda_f,\mu_f); S=0 \left|\left|\left| \left\{ \tilde{a}_{(0,\eta),\frac{1}{2}} \tilde{a}_{(0,\eta),\frac{1}{2}} \right\}^{(\lambda,\mu),S} \right|\right|\right| [2^{\frac{n}{2}+1}](\lambda_i,\mu_i); S=0 \right\rangle_\rho .$$

The first two columns show η and the irrep $[2^{\frac{n}{2}}]$, the following three columns give the (λ,μ)'s and the final columns exhibit the reduced matrix elements for the different values of the external multiplicity ρ.

η	$[2^{\frac{n}{2}}]$	(λ_f,μ_f)	(λ_i,μ_i)	(λ,μ)	$\rho=1$	$\rho=2$	$\rho=3$
3	$[2^2]$	$(8,2)$	$(12,0)$	$(0,6)$	-2.10051		
3	$[2^2]$	$(8,2)$	$(12,0)$	$(2,2)$	1.16110		
4	$[2^1]$	$(8,0)$	$(12,2)$	$(0,8)$	-5.45108		
4	$[2^1]$	$(8,0)$	$(12,2)$	$(2,4)$	2.43780		
4	$[2^2]$	$(12,2)$	$(18,0)$	$(0,8)$	-2.14322		
4	$[2^2]$	$(12,2)$	$(18,0)$	$(2,4)$	1.25137		
4	$[2^9]$	$(0,24)$	$(4,20)$	$(0,8)$	-3.33789		
4	$[2^9]$	$(0,24)$	$(4,20)$	$(2,4)$	1.89737		
4	$[2^9]$	$(0,24)$	$(4,20)$	$(4,0)$	-1.41421		
4	$[2^{10}]$	$(4,20)$	$(4,18)$	$(0,8)$	-2.43504		
4	$[2^{10}]$	$(4,20)$	$(4,18)$	$(2,4)$	-0.37391	-0.12955	-4.05720
4	$[2^{10}]$	$(4,20)$	$(4,18)$	$(4,0)$	-0.96609		
5	$[2^5]$	$(30,4)$	$(36,0)$	$(0,10)$	-1.60155		
5	$[2^5]$	$(30,4)$	$(36,0)$	$(2,6)$	0.93853		
5	$[2^5]$	$(30,4)$	$(36,0)$	$(4,2)$	-0.80494		

and

$$\left\langle [2^{\frac{n}{2}}](\lambda_m,\mu_m)KLM; S=0 | Q\cdot Q | [2^{\frac{n}{2}}](\lambda_m,\mu_m)KLM; S=0 \right\rangle$$

$$= (-1)^\eta \sqrt{2} \sum_l \sqrt{2l+1}\left[4C_2(\eta,0) - 3l(l+1)\right] \sum_{(\lambda,\mu)} \left\{ \left\langle (\eta,0)1l, (0,\eta)1\,l || (\lambda,\mu)10 \right\rangle_{\rho'=1} \right.$$

$$\sum_\rho \left\langle (\lambda_m,\mu_m)KL, (\lambda,\mu)10 || (\lambda_m,\mu_m)KL \right\rangle_\rho$$

$$\left. \left\langle (\lambda_m,\mu_m), S=0 \left|\left|\left| \left[a^\dagger_{(\eta,0),\frac{1}{2}} \tilde{a}_{(0,\eta),\frac{1}{2}} \right]^{(\lambda,\mu);0} \right|\right|\right| (\lambda_m,\mu_m), S=0 \right\rangle_\rho \right\}$$

$$- \frac{\sqrt{5}}{2} \frac{(\eta+3)!}{(\eta-1)!} \sum_{(\lambda,\mu)\rho} \left\{ \left\langle (1,1)12, (1,2)12 || (\lambda,\mu)10 \right\rangle_\rho \right.$$

$$\sum_{(\lambda_1,\mu_1)(\lambda_2,\mu_2)\rho_1} \left\{ \begin{array}{cccc} (\eta,0) & (0,\eta) & (1,1) & 1 \\ (\eta,0) & (0,\eta) & (1,1) & 1 \\ (\lambda_1,\mu_1) & (\lambda_2,\mu_2) & (\lambda,\mu) & \rho_1 \\ 1 & 1 & \rho & \end{array} \right\}$$

$$\sum_S \frac{\sqrt{2S+1}}{2} \sum_{\rho_2} \langle (\lambda_m,\mu_m)KL, (\lambda,\mu)10 \| (\lambda_m,\mu_m)KL \rangle_{\rho_2}$$

$$\langle [2^{\frac{n}{2}}](\lambda_m,\mu_m), S=0 \||| \left\{ \left\{ a^\dagger_{(\eta,0),\frac{1}{2}}\, a^\dagger_{(\eta,0),\frac{1}{2}} \right\}^{(\lambda_1,\mu_1),S} \left\{ a_{(0,\eta),\frac{1}{2}}\, a_{(0,\eta),\frac{1}{2}} \right\}^{(\lambda_2,\mu_2),S} \right\}^{\rho_1(\lambda,\mu);0}$$

$$\||| [2^{\frac{n}{2}}](\lambda_m,\mu_m), S=0 \rangle_{\rho_2} \Bigg\}. \quad (41)$$

In Table IV numerical results are shown for $n = 4$ particles in the shell $\eta = 4$, and intermediate spins $S = 0$ or 1, for the reduced matrix elements of the two-body operators, coupled to $(\lambda_0, \mu_0) = (0,0)$ and $(2,2)$, the only ones needed in the evaluation of N^2 and $Q \cdot Q$. Tables V, VI and VII exhibit the same matrix elements for $n = 6, 8$ and 10 particles respectively. Only nonzero values are exhibited. The first columns show the coupled (λ, μ) for the pair of creation operators and the pair of annihilation operators, respectively, the intermediate spin S their final coupling (λ_0, μ_0) and the multiplicity label ρ_0 of this coupling. The other columns depict the reduced matrix elements for different values of the external multiplicity ρ.

6. Conclusions

In this contribution we gave an explicit, analytic construction of the reduced matrix elements of the one-body, two-body operators and two-particle transfer operators within the leading SU(3) irrep in one oscillator shell. Partly we used already known results as SU(3) coupling and recoupling coefficients and transformation brackets from the spherical to the cylindrical basis. A *Mathematica* code was developed and used in order to evaluate the reduced matrix elements, and it is available under request. We gave several numerical examples and tabulated various coefficients.

Our contribution has also to be seen as a useful reference for people not being familiar with the group theoretical manipulation but which want to use these results in order to treat many particle systems in the SU(3) or pseudo SU(3) models.

Appendix A: A short review of SU(3)

We follow the formalism introduced by Moshinsky [15], which we briefly review here. We start with the creation operators $a_i^\dagger$ and annihilation operators a_i which satisfy anti-commutation relations,

$$\{a_i^\dagger, a^j\} \equiv a_i^\dagger a^j + a^j a_i^\dagger = \delta_i^j, \qquad \{a_i^\dagger, a_j^\dagger\} = \{a^i, a^j\} = 0, \quad (42)$$

TABLE IV. Reduced matrix elements for states with $n = 4$ particles in the $\eta = 4$ shell

$$\left\langle [2^2](12,2); S=0 \right\| \left\{ \left\{ a^{\dagger}_{(4,0),\frac{1}{2}} \, a^{\dagger}_{(4,0),\frac{1}{2}} \right\}^{(\lambda,\mu),S} \left\{ \tilde{a}_{(0,4),\frac{1}{2}} \, \tilde{a}_{(0,4),\frac{1}{2}} \right\}^{(\lambda',\mu'),S} \right\}^{\rho_0(\lambda_0,\mu_0);S_0} \left\| [2^2](12,2); S=0 \right\rangle_{\rho}.$$

The first five columns give the coupled (λ,μ)'s for the pair of creation operators and the pair of annihilation operators, the intermediate spin S, their final coupling (λ_0,μ_0) and their multiplicity label ρ_0, respectively. The others columns exhibit the reduced matrix elements for the different values of the external multiplicity ρ.

(λ,μ)	(λ',μ')	S	(λ_0,μ_0)	ρ_0	$\rho=1$	$\rho=2$	$\rho=3$
(4,2)	(0,8)	0	(2,2)	1	0.1512867	−0.2464468	0.1664138
(4,2)	(2,4)	0	(2,2)	1	−0.3482455	−0.2757678	−0.1008070
				2	−0.1834020	0.2288736	0.3718715
				3	−0.0964501	0.2364177	−0.3443055
(6,1)	(1,6)	1	(0,0)	1	−0.4364358		
(6,1)	(1,6)	1	(2,2)	1	−1.5681394	−0.6823224	0.1815724
				2	−0.8573655	1.3578570	−0.2644187
(8,0)	(0,8)	0	(0,0)	1	−0.7666519		
(8,0)	(0,8)	0	(2,2)	1	−2.2391842	−0.4562323	0.0972367
(8,0)	(2,4)	0	(2,2)	1	0.1512867	−0.2464468	0.1664138

where the indexes i and j indicate the spherical single-particle states available for the nucleons, corresponding to the total quantum number η_i of the three dimensional harmonic oscillator potential, the angular momentum l_i and its projection m_i, and the spin and isospin s_i and t_i with their projections m_{si} and m_{ti}. The upper and lower indices are introduced in order to distinguish different transformation properties others than cartesian. The range of allowed orbital angular momenta is given by

$$l = \eta, \eta - 2, \ldots, 1 \quad \text{or} \quad 0. \tag{43}$$

The number of orbital states r, restricted to a single shell is just

$$r = \sum_{l=0 \text{ or } 1}^{\eta} (2l+1) = \tfrac{1}{2}(\eta+1)(\eta+2). \tag{44}$$

Including spin we have $\Omega = 2r$, and using that the U($2r$) group contains the direct product of U(r) $\times$ U(2), we will construct the raising, lowering, and weight operators in U(r) and relate them with the SU(3) group, following the chain introduced in the introduction. In the following discussion we will refer only to one generic type of nucleon since we are working with separate proton and neutron subspaces, which later will be

TABLE V. The same as Table IV, for $n = 6$ particles

$$\left\langle [2^3](18,0); S=0 \right\| \left\{ \left\{ a^\dagger_{(4,0),\frac{1}{2}} a^\dagger_{(4,0),\frac{1}{2}} \right\}^{(\lambda,\mu),S} \left\{ \tilde{a}_{(0,4),\frac{1}{2}} \tilde{a}_{(0,4),\frac{1}{2}} \right\}^{(\lambda',\mu'),S} \right\}^{\rho_0(\lambda_0,\mu_0);S_0} \left\| [2^3](18,0); S=0 \right\rangle_\rho .$$

(λ,μ)	(λ',μ')	S	(λ_0,μ_0)	ρ_0	$\rho=1$
(4,2)	(0,8)	0	(2,2)	1	0.5629625
(4,2)	(2,4)	0	(0,0)	1	−0.3319700
(4,2)	(2,4)	0	(2,2)	1	−0.7957149
				2	−0.6107404
				3	−0.4175957
(6,1)	(1,6)	1	(0,0)	1	−1.3093073
(6,1)	(1,6)	1	(2,2)	1	−2.5798182
				2	−3.3046949
(8,0)	(0,8)	0	(0,0)	1	−1.4055284
(8,0)	(0,8)	0	(2,2)	1	−2.7327545
(8,0)	(2,4)	0	(2,2)	1	0.5629625

strongly coupled, and we will differentiate the spatial $\mu = (\eta, l, m)$ and spinorial (s) labels.

The generators of the U($2r$) group are

$$C^{\mu' s'}_{\mu s} \equiv a^\dagger_{\mu s} a^{\mu' s'}, \tag{45}$$

and from these we built those associated with the U(r) group by adding over the spinorial indices

$$C^{\mu'}_{\mu} = \sum_s a^\dagger_{\mu s} a^{\mu' s} = \sum_s C^{\mu' s}_{\mu s} . \tag{46}$$

There are r states, *i.e.*, r different spatial labels for η fixed.

We will now introduce the cylindrical SU(3) classification scheme of these states, which is alternative to the (η, l, m) defined above. We start by reviewing some properties of the three dimensional harmonic oscillator. We shall use units in which $\hbar$, the mass of the nucleon m, and the oscillator frequency ω, are taken as 1, *i.e.*,

$$\hbar = m = \omega = 1.$$

Denoting by $\vec{r}$ the coordinate and by $\vec{p}$ the momentum vectors, the creation $\vec{\eta}$ and annihilation $\vec{\xi}$ phonon operators are

$$\vec{\eta} = \tfrac{1}{\sqrt{2}}(\vec{r} - i\vec{p}), \qquad \vec{\xi} = \tfrac{1}{\sqrt{2}}(\vec{r} + i\vec{p}). \tag{48}$$

TABLE VI. The same as Table IV, for $n = 8$ particles

$$\left\langle \left[2^4\right](18,4); S=0 \right\| \left\{ \left\{ a^\dagger_{(4,0),\frac{1}{2}} \, a^\dagger_{(4,0),\frac{1}{2}} \right\}^{(\lambda,\mu),S} \left\{ \tilde{a}_{(0,4),\frac{1}{2}} \tilde{a}_{(0,4),\frac{1}{2}} \right\}^{(\lambda',\mu'),S} \right\}^{\rho_0(\lambda_0,\mu_0);S_0} \left\| \left[2^4\right](18,4); S=0 \right\rangle_\rho .$$

(λ,μ)	(λ',μ')	S	(λ_0,μ_0)	ρ_0	$\rho=1$	$\rho=2$	$\rho=3$
(0,4)	(2,4)	0	(2,2)	1	0.1633815	−0.2488914	0.1492112
(0,4)	(4,0)	0	(0,0)	1	−0.1032796		
(0,4)	(4,0)	0	(2,2)	1	−0.1576147	−0.2568638	−0.2985258
(2,3)	(1,6)	1	(2,2)	1	0.4823643	−0.5169284	0.1007830
(2,3)	(3,2)	1	(0,0)	1	−0.4276180		
(2,3)	(3,2)	1	(2,2)	1	−1.0915135	−0.5261541	0.0163226
				2	−0.1993259	0.7211192	0.2988647
				3	−0.0183487	0.1908910	−0.6693222
(4,2)	(0,8)	0	(2,2)	1	0.7783691	−0.3410843	−0.0347611
(4,2)	(2,4)	0	(0,0)	1	−0.7745967		
(4,2)	(2,4)	0	(2,2)	1	−1.2909502	−0.4042324	−0.1744572
				2	−0.9502132	0.1601879	0.2404646
				3	−0.6032437	0.5083000	−0.1099279
(4,2)	(4,0)	0	(2,2)	1	0.1633815	−0.2488914	0.1492112
(6,1)	(1,6)	1	(0,0)	1	−2.2694661		
(6,1)	(1,6)	1	(2,2)	1	−3.3034587	−0.7984070	−0.2097556
				2	−3.4395178	0.6700114	0.6297066
(6,1)	(3,2)	1	(2,2)	1	0.4823643	−0.5169284	0.1007830
(8,0)	(0,8)	0	(0,0)	1	−2.0273683		
(8,0)	(0,8)	0	(2,2)	1	−2.6416542	−0.7782408	0.0366824
(8,0)	(2,4)	0	(2,2)	1	0.7783691	−0.3410843	−0.0347611

In cylindrical coordinates $q = 1, 0, -1$ we have the metric $g_{qq'} = (-1)^q \delta_{q,-q'}$, and the rule for raising and lowering indices is

$$\eta^q = (-1)^q \eta_{-q}, \qquad \xi^q = (-1)^q \xi_{-q}, \qquad (\xi^q)^\dagger = \eta_q \,. \tag{49}$$

This operators satisfy the commutation rules

$$\begin{aligned} [\xi^q, \eta_{q'}] &= \xi^q \eta_{q'} - \eta_{q'} \xi^q = \delta^q_{q'}, \\ [\xi^q, \xi^{q'}] &= [\eta_q, \eta_{q'}] = 0. \end{aligned} \tag{50}$$

We define the operators

$$c_q^{q'} = \eta_q \xi^{q'}, \tag{51}$$

TABLE VII. The same as Table IV, for $n = 10$ particles

$$\left\langle \left[2^5\right](20,4); S=0 \right\| \left\{ \left\{ a^\dagger_{(4,0),\frac{1}{2}}\, a^\dagger_{(4,0),\frac{1}{2}} \right\}^{(\lambda,\mu),S} \left\{ \tilde{a}_{(0,4),\frac{1}{2}}\, \tilde{a}_{(0,4),\frac{1}{2}} \right\}^{(\lambda',\mu'),S} \right\}^{\rho_0(\lambda_0,\mu_0);S_0} \left\| \left[2^5\right](20,4); S=0 \right\rangle_\rho .$$

(λ,μ)	(λ',μ')	S	(λ_0,μ_0)	ρ_0	$\rho=1$	$\rho=2$	$\rho=3$
(0, 4)	(2, 4)	0	(2, 2)	1	0.4948643	−0.4491177	0.1427133
(0, 4)	(4, 0)	0	(0, 0)	1	−0.2754121		
(0, 4)	(4, 0)	0	(2, 2)	1	−0.3198218	−0.2037963	0.1254624
(2, 3)	(1, 6)	1	(2, 2)	1	1.1988845	−0.9512515	0.4457706
(2, 3)	(3, 2)	1	(0, 0)	1	−1.0690450		
(2, 3)	(3, 2)	1	(2, 2)	1	−1.8505548	−0.3025016	0.2631156
		1		2	−0.8018603	0.6491143	−0.4464200
		1		3	−0.1280897	0.7779431	0.1465049
(4, 2)	(0, 8)	0	(2, 2)	1	1.1711899	−0.4161026	0.4338491
(4, 2)	(2, 4)	0	(0, 0)	1	−1.3524704		
(4, 2)	(2, 4)	0	(2, 2)	1	−1.4874251	−0.3263928	0.2820416
		0		2	−1.2393744	0.1292542	0.0348390
		0		3	−1.0135904	0.4485234	−0.4103062
(4, 2)	(4, 0)	0	(2, 2)	1	0.4948643	−0.4491177	0.1427133
(6, 1)	(1, 6)	1	(0, 0)	1	−3.4914862		
(6, 1)	(1, 6)	1	(2, 2)	1	−3.2557850	−0.9292884	0.9034818
		1		2	−3.9900043	−0.1614718	−0.6614311
(6, 1)	(3, 2)	1	(2, 2)	1	1.1988845	−0.9512515	0.4457706
(8, 0)	(0, 8)	0	(0, 0)	1	−2.7514284		
(8, 0)	(0, 8)	0	(2, 2)	1	−2.2028964	−1.0208818	0.4796497
(8, 0)	(2, 4)	0	(2, 2)	1	1.1711899	−0.4161026	0.4338491

which, from the above commutation relations, satisfy

$$\left[c_q^{q'}, c_{q''}^{q'''}\right] = c_q^{q'''}\delta_{q''}^{q'} - c_{q''}^{q'}\delta_q^{q'''}, \tag{52}$$

exhibiting these operators as generators of U(3).

The states of the harmonic oscillator could be expressed in terms of creation operators acting on the ground state $|0\rangle$ (state of no excitation) defined by

$$\xi_q|0\rangle = 0, \qquad q = 1, 0, -1. \tag{53}$$

The states can be characterized by the three commuting integrals of motion c_q^q, for which the states become

$$|n_1 n_0 n_{-1}\rangle = [n_1!\, n_0!\, n_{-1}!]^{-1/2}(\eta_1)^{n_1}(\eta_0)^{n_0}(\eta_{-1})^{n_{-1}}|0\rangle, \tag{54}$$

where the n_q are the eigenvalues of c_q^q and the numerical factor is for normalization. The sum of the n_q must be equal to η [15].

These generators can be divided into three sets:

$$(1)\ c_1^0, c_1^{-1}, c_0^{-1}; \qquad (2)\ c_1^1, c_0^0, c_{-1}^{-1}; \qquad (3)\ c_0^1, c_{-1}^1, c_{-1}^0\ , \tag{55}$$

which respectively raise, give and lower the weight of the state.

The weight of the one-particle state (54) is then given by $(n_1 n_0 n_{-1})$, and with the help of the raising generators c_1^0, c_1^{-1} we can transform it into one and only one state of maximum weight $(\eta\ 0\ 0)$, which characterize the irrep. The state (54) corresponds to a single-particle basis for an irrep of U(3) in the chain U(3) $\supset$ U(2) $\supset$ U(1).

Both the spherical and cylindrical sets are complete and therefore can be expanded one in terms of one another using the transformation brackets $\langle n_1 n_0 n_{-1} | \eta l m\rangle$.

APPENDIX B: THE TRANSFORMATION BRACKETS BETWEEN THE SPHERICAL AND CYLINDRICAL BASIS

These brackets will be zero unless

$$n_1 + n_0 + n_{-1} = \eta, \qquad n_1 - n_{-1} = m \tag{56}$$

and they are real. They can be evaluated using the formula [16]:

$$\begin{aligned}\big\langle n_1 n_0 n_{-1} | \eta l m\big\rangle &= 2^{n_{-1}-l}\sqrt{\frac{2^m(2l+1)(l-m)!n_1!n_0!}{(\eta+l+1)!!(\eta-l)!!(l+m)!n_{-1}!}}\\ &\times \delta_{n_0,\eta-2n_{-1}-m}\,\delta_{n_1,n_{-1}+m}\sum_{s=0}^{l}\frac{(-1)^s\left(\frac{\eta-l}{2}+s\right)!\,(2l-2s)!}{\left(\frac{\eta-l}{2}+s-n_{-1}\right)!\,s!\,(l-s)!\,(l-2s-m)!}\end{aligned} \tag{57}$$

which is valid for $m \geq 0$. For the case $m < 0$ we have

$$\big\langle n_1 n_0 n_{-1} | \eta l - m\big\rangle = \big\langle n_{-1} n_0 n_1 | \eta l m\big\rangle. \tag{58}$$

APPENDIX C: THE ANGULAR MOMENTUM MULTIPLICITY

The maximum value of the label K, in the $(\lambda,\mu)KLM$ notation is the total number of occurrences $d(\lambda,\mu;L)$ of L in the irrep, given by [9]

$$d(\lambda,\mu:L) = \left[\tfrac{1}{2}(\lambda+\mu+2-L)\right] - \left[\tfrac{1}{2}(\lambda+1-L)\right] - \left[\tfrac{1}{2}(\mu+1-L)\right],$$

where [—] means the largest integer contained in the argument and is to be interpreted as zero if the argument is negative. Note that K, as used here, has *not* the same meaning as the one introduced by Elliott [2] but is just a running multiplicity label (see Ref. [10]) starting from 1.

This is a very useful formula for computational purposes. It also shows that for $L = 0$ there is only one state for each irrep (*i.e.* $K = 1$) if λ and μ are both even, and there is no $L = 0$ state if at least one of them is odd. This results implies that if the operators are coupled to $L_0 = S_0 = 0$ only even values for λ and μ are allowed. These are the only ones listed in Tables IV, V, VI and VII.

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