Lagrangian and Hamiltonian formulations of geometrical anisotropic optics

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ABSTRACT. A Lagrangian function for the time evolution of light rays in an anisotropic medium is obtained, the corresponding hamiltonian function is also given and it is shown that the resulting evolution equations are equivalent to impose the Fermat principle.

RESUMEN. Se obtiene una función lagrangiana para la evolución temporal de los rayos de luz en un medio anisótropo, se da también la función hamiltoniana correspondiente y se muestra que las ecuaciones de evolución resultantes equivalen a imponer el principio de Fermat.

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1. INTRODUCTION

In a recent paper [1], the Lie-Hamilton optics has been extended to anisotropic media starting from Fermat's principle of least time, which determines the paths followed by the light rays also in the case of anisotropic media (see, *e.g.*, Ref. [2]). The Lagrangian and Hamiltonian functions obtained in Ref. [1] give the evolution of the light rays with the coordinate along the optical axis of the system as parameter.

In this paper we obtain Lagrangian and Hamiltonian functions for the light rays in anisotropic media, using the time as evolution parameter. In Sect. 2, we start from some known relations valid in the case of isotropic media and we find a Lagrangian function for the light rays in an anisotropic medium. We show that the evolution determined by this Lagrangian satisfies the Fermat principle. In Sect. 3 we obtain the Hamiltonian function that generates the evolution of the light rays parametrized by one of the coordinates, thus reproducing some results of Ref. [1].

2. VARIATIONAL PRINCIPLES

The evolution of the geometrical optical rays satisfies Fermat's principle of least time

$$\delta \int_A^B n\, ds = 0, \tag{1}$$

where n is the refractive index of the optical medium and ds is the line element of three-dimensional space. In the case of an isotropic medium, n may be a function of position, $n = n(q^i)$, where the q^i are cartesian coordinates ($i = 1, 2, 3$); while in an anisotropic medium, the refractive index depends on the direction of the light ray. Despite the resemblance of Eq. (1) to Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \tag{2}$$

of classical mechanics, the refractive index n, appearing in Eq. (1), *cannot* be taken as a Lagrangian for the light rays since the arc length s cannot be used as an independent variable in the variational problem (1) (see, *e.g.*, Ref. [3]). Whereas all curves considered in the variation (2) have the same endpoints at the same times t_1 and t_2, the curves considered in the variation (1) may not have the same arc length limits (unless they have the same length). (Note also that in the case of Eq. (1) the variable s is restricted by the condition $|d\mathbf{q}/ds| = 1$.) Rather, Fermat's principle is the analogue of Jacobi's principle (or, equivalently, of the principle of least action) (see, *e.g.*, Refs. [3,4]).

In the case of an isotropic medium, one can define a momentum three-vector $\mathbf{p}$ of length n tangent to the light ray (see, *e.g.*, Refs. [5,6]). Since the velocity of light is given by c/n, where c is the velocity of light in vacuum, the norm of the velocity three-vector

$$\mathbf{v} \equiv \frac{d\mathbf{q}}{dt} \tag{3}$$

is equal to c/n. Therefore,

$$\mathbf{p} = \frac{n^2}{c}\mathbf{v}. \tag{4}$$

Recalling that the canonical momentum is defined by

$$p_i = \frac{\partial L}{\partial v^i}, \tag{5}$$

where L is the Lagrangian function of the system, and taking into account that in the present case n may be a function of q^i only, we can recover Eq. (4) by choosing

$$L = \frac{n^2}{2c}\mathbf{v}\cdot\mathbf{v}. \tag{6}$$

Using Eqs. (4) and (6) we can obtain the Hamiltonian function in the usual manner:

$$\begin{aligned} H &= \mathbf{p}\cdot\mathbf{v} - L \\ &= \frac{n^2}{2c}\mathbf{v}\cdot\mathbf{v} \\ &= \frac{c}{2n^2}\mathbf{p}\cdot\mathbf{p}, \end{aligned} \tag{7}$$

which, except for an inessential additive constant, agrees with the Hamiltonian given in Ref. [6] (see also Ref. [7]).

As we shall show, expression (6) also applies to the case of anisotropic media. The refractive index of an anisotropic medium depends on the direction of $\mathbf{v}$, but not on its magnitude; this implies that

$$v^i \frac{\partial n}{\partial v^i} = 0, \tag{8}$$

as can be seen by writing the left-hand side in terms of the spherical coordinates of $\mathbf{v}$ or using the Euler theorem for homogeneous functions. From Eqs. (5–6) we now obtain

$$p_i = \frac{n^2}{c} v^i + \frac{n}{c} \mathbf{v} \cdot \mathbf{v} \frac{\partial n}{\partial v^i}, \tag{9}$$

which reduces to Eq. (4) if n does not depend on v^i. Following Ref. [1], we introduce the *anisotropy vector*

$$A_i \equiv \frac{n}{c} \mathbf{v} \cdot \mathbf{v} \frac{\partial n}{\partial v^i}, \tag{10}$$

which, owing to Eq. (8), is orthogonal to $\mathbf{v}$:

$$\mathbf{A} \cdot \mathbf{v} = 0. \tag{11}$$

Thus,

$$\mathbf{p} = \frac{n^2}{c} \mathbf{v} + \mathbf{A}, \tag{12}$$

and, using Eqs. (6) and (11–12), one finds that the Hamiltonian is given by

$$\begin{aligned} H &= \left(\frac{n^2}{c} \mathbf{v} + \mathbf{A} \right) \cdot \mathbf{v} - \frac{n^2}{2c} \mathbf{v} \cdot \mathbf{v} \\ &= \frac{n^2}{2c} \mathbf{v} \cdot \mathbf{v} \\ &= \frac{c}{2n^2} (\mathbf{p} - \mathbf{A}) \cdot (\mathbf{p} - \mathbf{A}). \end{aligned} \tag{13}$$

Since H does not depend explicitly on t, H is a conserved quantity. In fact, from Eq. (12) it follows that

$$|\mathbf{p} - \mathbf{A}|^2 = n^2, \tag{14}$$

hence, H has the constant value $c/2$ for all light rays. As is well known, for a conservative system, Hamilton's principle

$$\delta \int (\mathbf{p} \cdot \mathbf{v} - H)\, dt = 0 \tag{15}$$

is equivalent to the principle of least action

$$\delta \int \mathbf{p} \cdot \mathbf{v}\, dt = 0, \tag{16}$$

where the variation is restricted to paths on the hypersurface $H =$ const., *i.e.*, such that Eq. (14) holds (see, *e.g.*, Refs. [3,4]). Substituting Eq. (12) into Eq. (16), making use of Eq. (11), we find the condition

$$\delta \int \frac{n^2}{c} \mathbf{v} \cdot \mathbf{v}\, dt = \delta \int 2H\, dt = 2H\, \delta \int dt = 0, \tag{17}$$

which is just Fermat's principle [Eq. (1)], thus showing that the Lagrangian (6) leads to the right evolution equations in isotropic or anisotropic media.

Substituting Eq. (16) into Lagrange's equations one obtains

$$\frac{dp_i}{dt} = \frac{n}{c} \mathbf{v} \cdot \mathbf{v} \frac{\partial n(q^j, v^j)}{\partial q^i} = \frac{c}{n} \frac{\partial n(q^j, v^j)}{\partial q^i}, \tag{18}$$

or, equivalently,

$$\frac{dp_i}{ds} = \frac{\partial n(q^j, v^j)}{\partial q^i}, \tag{19}$$

which mean that the change of $\mathbf{p}$ is parallel to the ordinary gradient of the refractive index expressed as a function of q^i and v^i. (Note that $\partial n(q^j, v^j)/\partial q^i$ are, in general, different from $\partial n(q^j, p_j)/\partial q^i$.) In the case of an isotropic medium, Eq. (18) leads to Snell's law in its usual form. (Recall that in an anisotropic medium, in general, $\mathbf{p}$ is not tangent to the light rays [Eq. (12)].)

3. Alternative parametrization

Instead of the time, we can use any coordinate as a parameter for the evolution of the light rays. For instance, considering the coordinate z as the independent variable, we can write Eq. (16) as

$$\delta \int \left(p_x \frac{dx}{dz} + p_y \frac{dy}{dz} - (-p_z) \right) dz = 0, \tag{20}$$

where, in order to take into account the constraint $H = \text{const.}$, p_z is the function of p_x, p_y, x, y, z determined by Eq. (14). By comparing Eqs. (20) and (15), we see that, in the present parametrization,

$$h \equiv -p_z = -\sqrt{n^2 - |\mathbf{p}_\perp - \mathbf{A}_\perp|^2} - A_z, \tag{21}$$

where $\mathbf{p}_\perp \equiv (p_x, p_y)$ and $\mathbf{A}_\perp \equiv (A_x, A_y)$, plays the role of the Hamiltonian function. Therefore, from Eq. (20) we obtain the evolution equations (see also Refs. [4,8])

$$\frac{dq^\alpha}{dz} = \frac{\partial h}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{dz} = -\frac{\partial h}{\partial q^\alpha}, \qquad (\alpha = 1, 2). \tag{22}$$

Making use of Eqs. (11–12) and (14) one finds that

$$A_z = -\frac{(\mathbf{p}_\perp - \mathbf{A}_\perp) \cdot \mathbf{A}_\perp}{\sqrt{n^2 - |\mathbf{p}_\perp - \mathbf{A}_\perp|^2}},$$

hence, the Hamiltonian function (21) is also given by

$$h = -\sqrt{n^2 - |\mathbf{p}_\perp - \mathbf{A}_\perp|^2} + \frac{(\mathbf{p}_\perp - \mathbf{A}_\perp) \cdot \mathbf{A}_\perp}{\sqrt{n^2 - |\mathbf{p}_\perp - \mathbf{A}_\perp|^2}}, \tag{23}$$

which agrees with the expression found in Ref. [1] (Eq. (3.4)).

4. Concluding remarks

As we have seen, in order to find the evolution equations for the light rays in an anisotropic medium, it is convenient to use the Lagrangian formalism since one can obtain the Lagrangian function for anisotropic media by simply replacing $n(q^i)$ by $n(q^i, v^i)$ in the Lagrangian function for isotropic media.

As stressed in Ref. [1], an anisotropic medium is characterized by the fact that, in general, the momentum $\mathbf{p}$ is not tangent to the light rays, owing to the presence of the anisotropy vector $\mathbf{A}$ [Eq. (12)]. At a given point of space, the anisotropy vector $\mathbf{A}$ is a function of $\mathbf{v}/|\mathbf{v}|$ only, which ranges over a sphere of unit radius; thus, owing to Eq. (11), $\mathbf{A}$ can be regarded as a vector field on the sphere and, therefore, has at least one zero (a critical point of the restriction of n to this sphere), where $\mathbf{p}$ is parallel to $\mathbf{v}$.

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REFERENCES

1. A.L. Rivera, S.M. Chumakov and K.B. Wolf, "Hamiltonian foundations of geometrical anisotropic optics", preprint IIMAS-UNAM, No. 19 (1994).
2. H.A. Buchdahl, *An Introduction to Hamiltonian Optics*, Cambridge University Press, Cambridge (1970), reprinted by Dover, New York (1993), Chap. 11.
3. C. Lanczos, *The variational principles of mechanics*, 4th ed., University of Toronto Press, Toronto (1970), reprinted by Dover, New York (1986), Chap. V, Sec. 6.
4. G.F. Torres del Castillo, *Rev. Mex. Fís.* **35** (1989) 691.
5. E. López Moreno y K.B. Wolf, *Rev. Mex. Fís.* **35** (1989) 291.
6. G.F. Torres del Castillo, *Rev. Mex. Fís.* **35** (1989) 301.
7. G. Krötzsch y K.B. Wolf, *Rev. Mex. Fís.* **37** (1991) 136.
8. G.F. Torres del Castillo, *Rev. Mex. Fís.* **36** (1990) 478.