 Revista Mexicana de Física 41, No. 2 (1995) 256–271

Nonlinear stability analysis of dissipative plasmas

J.A. ALMAGUER-ANDRADE
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Apartado postal 70-543, 04510 México, D.F., México

Recibido el 6 de julio de 1994; aceptado el 18 de enero de 1995

ABSTRACT. The problem of the stability of static, resistive, thermal conductive and viscous plasmas is investigated by means of an asymptotic expansion in terms of a small positive parameter ϵ which relates the Alfvén transit time to the characteristic time for dissipative effects. This twofold time-scale allows to separate the dynamics in a convenient way. It is found, assuming weak dissipation, that, at high frequencies, to first-order one recovers the well-known solutions of the ideal magnetohydrodynamics (MHD), whereas at second-order one gets oscillations respect to the long time-scale t which are modulated by a growing (or decaying) exponential behavior [see Eqs. (19.2) and (24.1,2)] respect to the short time-scale t^*. On the other hand, for low frequencies we found that the leading term is that of second order. This gives place to a situation in which the corresponding first-order solutions constitute, in such regime, a subclass of the known solutions from the ideal MHD (at first-order approximation). At second-order the solutions exhibit oscillation respect to the t time-scale and the behavior of their corresponding amplitudes is determined by considering the third-order approximation. It results that such amplitudes grow (or decay) linearly respect to the t^* time-scale [see, for instance, Eqs. (31)–(33) and (35)–(36)]. Finally, we discuss somewhat interesting points relevant to further investigations.

RESUMEN. El problema de la estabilidad de plasmas estáticos, resistivos, térmicamente conductores y viscosos es investigado por medio de una expansión asintótica en términos de un parámetro positivo pequeño ϵ, el cual relaciona el tiempo de tránsito de Alfvén t con el tiempo característico para los efectos disipativos. Esta escala temporal doble permite separar la dinámica en una forma conveniente. Se encuentra, suponiendo una disipación débil, que en altas frecuencias, a primer orden, uno recupera las soluciones conocidas de la magnetohidrodinámica (MHD) ideal, en tanto que a segundo orden uno obtiene oscilaciones respecto a la escala temporal larga t, que, a su vez, están moduladas por comportamiento exponencial creciente (o decreciente) [cf., Ecs. (19.2) y (24.1,2)] respecto a la escala temporal corta t^*. Por otra parte, para bajas frecuencias encontramos que el término lider es el de segundo orden. Esto da lugar a una situación en la cual las soluciones para primer orden constituyen, en tal régimen, una subclase de las soluciones conocidas de la MHD ideal para primer orden. A segundo orden las soluciones exhiben oscilaciones respecto a la escala t y el comportamiento de sus amplitudes correspondientes es determinado teniendo en cuenta la aproximación de tercer orden. Resulta que esas amplitudes crecen (o decrecen) linealmente respecto a la escala de tiempo t^* [cf., por ejemplo, Ecs. (31)–(33) y (35)–(36)]. Finalmente, discutimos algunos puntos de interés relevantes en vista a estudios ulteriores.

PACS: 59.55.Dy; 52.35.-g

1. INTRODUCTION

The study of plasma stability is one of the most important and far-reaching problems in plasma theory. Needless to say that the problem of stability is by no means a closed book.

It is fundamental in connection with the relaxation processes occurring in magnetically confined plasmas. To obtain an equilibrium solution as well as its stability conditions one might use a differential treatment or a variational formulation. Both approaches coincide when the plasma is conservative. The variational treatment provides an adequate tool to examine stability conditions in the case of non-dissipative plasmas [1] (it is the "energy principle"). At present, the variational formulation for the dissipative plasma, although it is an insufficiently supported framework, it allows us, with adequate modifications, to obtain a global description through a set of temperature profiles [2,3] and recover the so-called "profile consistency" condition. In this latter case the variational principle is built upon the physical assumption that the rate of entropy production reaches its minimum value for physically admissible relaxated configurations.

There is a vast literature dealing with the problem of the stability of resistive modes. Let us review several related works. For example, Park *et al.* [4] studied the equilibrium of a three-dimensional stellarator using a time-dependent relaxation method in a numerical fashion assuming that dissipative terms are small and including a density source. Supposing that the viscosity, μ, the resistivity, η, and the thermal conductivity, κ, are constant they obtained stable free-boundary equilibria which satisfy the zero field-line-averaged current condition. On the other hand, Gomberoff and Hernández [5] examined analytically a dissipative cylindrical plasma column, showing that nonideal effects such as viscosity and thermal conductivity allow to obtain a linear solution for the MHD equations even if the plasma is assumed incompressible, while there is no linear solution for the ideal MHD equations for a cylindrical current-carrying plasma limited by fixed boundaries if the plasma is assumed incompressible. Also they found that perpendicular viscosity removes the singularity of the ideal case at $m = nq$, but the boundary conditions can only be satisfied for $B_\theta/B_z \gg 1$ (m denotes the azimuthal number, q is the constant rotational transform, and $n = -kL/2\pi$ where k is a real wavenumber along the z-direction and L the length of the cylinder). Under that condition, the resultant spectrum is $\omega = 0$ (ω denotes the frequency related to the normal modes) at $m = nq$ and this spectrum persists for arbitrary μ providing that the Reynolds number, R, is greater than some critical Reynolds number, R_c, consequently there exists steady convection in the plasma and the mode with $\omega = 0$ at $k_\parallel = 0$ satisfies the whole set of linearized nonideal MHD equations with finite γ (γ stands for the specific heats ratio). Their work has shown that the viscosity does not change the range of the unstable modes rather changes appreciably the shape of the spectrum. Moreover, for $R = R_c$ the state of the plasma is not only marginally stable but also stationary, while for $R > R_c$ the complete nonlinear equations possess a stationary convective solution which bifurcates from the equilibrium solution.

Thermal instabilities in a cylindrical plasma column as well as steady-state solutions to the transport equations in a number of limiting cases were obtained by Dobrott *et al.* [6] and they also investigated marginal stability by using a linearization procedure. In their study were assumed the Spitzer resistivity and the classical perpendicular thermal conductivity, obtaining that bifurcations depend on the applied electric field and finding that the same solutions are unique functions of total current, whereas the steady-state solution is not recoverable by the approximation used by Fürth *et al.* [7].

Kerner *et al.* [8] studied the resistive Alfvén spectrum in straight cylindrical geometry making use of a spectral code, finding that the ideal continua disappear, the normal modes are damped, and the eigenvalues lie on specific curves that become independent of resistivity for small values of η. On the other hand, Minardi [9] has examined the resistive relaxation of a plasma confined in a Tokamak which is heated by an external auxiliary power, but restricting his study to the case $\omega = 0$ in order to explain the existence of the bifurcating states in the framework of the ordinary resistive effects in a Tokamak discharge subject to transport losses which depend anomalously on temperature. Such steady states are that with maximum probability for the plasma magnetic configuration (*i.e.*, states with maximum entropy). Likewise, Paris [10] used a dissipative MHD model to analyze stationary convection in plasmas under the scheme of the normal-mode decomposition, but he did not include resistivity and imposed some additional constraints upon both the velocity and magnetic field. In his work, however, he shows that the onset of instability is via a marginal stationary mode with $\omega = 0$.

The influence of resistivity on plasma stability was first investigated by Fürt *et al.* [11] for the plane slab model. They separate the resistive and ideal behavior of the plasma into two spatial regions: a thin layer where the resistivity is important, outside of which the motion follows the ideal-MHD model with zero growth rate. Next they matched the solutions of both regions. They studied, in particular, the rippling-, tearing-, and gravitational interchange-modes, finding that the growth rates go as $S^{2/5}$, $S^{2/5}$, and $S^{2/3}$, respectively, S being the Lundquist number (By passing, we notice here that the philosophy of the present work is rather close to previously cited works than those of Fürth *et al.* [11]).

It should be mentioned that the usual stability analysis is worked out as a normal-mode decomposition and this, naturally, is a habit coming from having dealed with linear theories. There resides the problem: the separability of the processes arising at different time scales remain out of consideration because of the 'temporal homogeneity' in the dynamical evolution under the standard normal-mode analysis. Despite that limitation, the normal-mode treatment to certain extent is very useful to get global criteria for stability and it also allows to separate into three regions: the stable dominia, the unstable region, and the transition zone between them. Such transition frequently corresponds to a steady solution of the linearized equations. However by using appropriate scalings it is possible to modify the standard normal-mode treatment in order to include several time-scales as we will show in this work.

Our aim in this work is a modest one: to explore the conditions of stability dynamics of a dissipative magnetofluid when a twofold temporal evolution is considered for some very particular and simple cases. Roughly speaking, the main idea consists in performing an asymptotic expansion in power series of a small positive dimensionless parameter, ϵ, of every physical variable. In fact, we might split the temporal dependence into a twofold time-scale in terms of the parameter ϵ. These two characteristic times, say the 'long' time, t, and the 'short' time, t^*, give place to two essentially different regimes for the dynamical evolution. As we will show later, at the 'long' time-scale every mode evolves, to first order in ϵ, in the well known way from the ideal MHD for linear stability analysis, thus at the first-order approximation we recover a known result. To second-order the system exhibits a more elaborate dynamics owed, basically, to the presence of a weak dissipation although every mode evolves in an uncoupled manner. On the other hand, at the 'slow' time-scale every mode evolves in a rather too simple way: to second-order we get modified solutions of that due to Bernstein *et al.* [1], while to third-order the amplitudes are linear functions of the 'short' time-scale.

This work is organized as follows. In Sect. 2 we present the dissipative MHD equations, discussing possible equilibria and our ordering scheme as well. In Sect. 3 it is performed the expansion, to n^{th} order in ϵ, of the system of differential equations governing the dynamical evolution at high frequencies, obtaining as result a differential equation of second order for the perturbed velocity to order ϵ^n. Next we pass to examine the time evolution of the velocity in both cases to first- and second-order, and the corresponding stability conditions are determined. Sect. 4 is devoted to a similar analysis for low frequencies but this time to second- and third-order. Finally, Sect. 5 contains a summary and conclusions.

2. Formulation of the problem

As was stressed in the Introduction we are considering a resistive, viscous, and thermal conducting magnetofluid. The dynamics of such plasma is governed by the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p = \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla \times (\eta \mathbf{J}) = 0, \tag{3}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = (\gamma - 1)[\eta J^2 + \nabla \cdot (\kappa \nabla T) - \mathbf{\Pi} : \nabla \mathbf{v}], \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

where $\mathbf{J} = \nabla \times \mathbf{B}$ and J is its magnitude. On the other hand, $\mathbf{\Pi}$ stands for the dissipative part of the stress tensor. The remaining symbols in these equations have their usual meaning. Restricting our study to the limit of very large fields (which is equivalent to assume $\omega_{\text{ci}}\tau_{\text{i}} > 1$, where τ_{i} and ω_{ci} denote the collision time, *i.e.*, the collision frequency is τ_{i}^{-1}, and the ion cyclotron frequency, respectively) the last term on the right-hand side of Eq. (2) will then involve only parallel viscosity [12]. In this approximation the dissipative part of the stress tensor becomes

$$\mathbf{\Pi} = -\frac{\mu(T)}{2}\mathbf{s}, \tag{6}$$

where $\mathbf{s}$ is related with the rate-of-strain tensor $\mathbf{W}$ defined by (see Appendix A)

$$W_{ij} = \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \tfrac{2}{3}\delta_{ij} \nabla \cdot \mathbf{v}, \tag{7}$$

δ_{ij} being the Kronecker delta.

In what follows we take the Spitzer resistivity, $\eta \propto T^{-3/2}$, and the classical dependence on temperature for both dynamic viscosity and thermal conductivity, namely $\mu, \kappa \propto T^{5/2}$ [13].

Here we are concerned with the case of a plasma filling wholly a cylindrical vessel whose wall is rigid and perfectly conducting. At such a boundary the appropriate conditions are

$$\mathbf{B} \cdot \hat{\mathbf{n}} = 0, \qquad \mathbf{v} \cdot \hat{\mathbf{n}} = 0, \qquad \mathbf{E} \cdot \hat{\mathbf{n}} = 0, \tag{8}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the wall.

The static equilibria states come from taking $\partial/\partial t = 0$ together with $\mathbf{v} = 0$ in the set (1)–(5); nevertheless, we undertake the present study from a general point of view, that is, we shall obtain the stability conditions for arbitrary equilibria.

In order to determine the stability conditions for the system under study we begin determining a twofold time-scale: the ratio of the involved time scales goes as a power of the small parameter ϵ which will be used to expand in power series every physical variable. In principle, one could have as many time-scales as transport coefficients besides the Alfvén time, $t_{\mathrm{A}} = a_0/v_{\mathrm{A}}$; v_{A} is the Alfvén speed defined using for the magnetic strength the typical value $\mathbf{B}_0$, $v_{\mathrm{A}} = |\mathbf{B}_0|/\sqrt{\rho_0}$, and a_0 is the length scale for magnetic variations. If one chooses the orderings for dissipative processes as $\eta \sim \epsilon^{j_1}$, $\kappa \sim \epsilon^{j_2}$, and $\mu \sim \epsilon^{j_3}$ (j_1, j_2, and j_3 are greater than unity), there exist three characteristic times: the resistive decay time, $t_{\mathrm{R}} = a_0^2/\eta$, the thermal diffusion time, $t_{\mathrm{T}} = \rho_0 a_0^3 C_p/\kappa v_{\mathrm{A}}$, and the viscous decay time, $t_{\mathrm{V}} = \rho_0 a_0^2/\mu$ (C_p and ρ_0 denote the specific heat at constant pressure and the characteristic value for the mass density, respectively). As is well known, the possible combinations of ratios of these characteristic times are useful to get a global physical picture of the dynamics. Particularly interesting are the Lundquist number, $S = t_{\mathrm{R}}/t_{\mathrm{A}} = a_0 v_{\mathrm{A}}/\eta \sim \epsilon^{-j_1}$; the magnetic Reynolds number, $R_{\mathrm{M}} = t_{\mathrm{V}}/t_{\mathrm{A}} = \rho_0 a_0 v_{\mathrm{A}}/\mu \sim \epsilon^{-j_3}$; the standard Reynolds number, $R = t_{\mathrm{V}}/t_0 = \rho_0 a_0 v_0/\mu \sim \epsilon^{-j_3}$; and $N = t_{\mathrm{T}}/t_{\mathrm{A}} = \rho_0 a_0^2 C_p/\kappa \sim \epsilon^{-j_2}$. v_0 is a typical value of the plasma flow velocity and t_0 the corresponding time of transit connected to a_0. By comparing the Lundquist number with the remaining parameters one obtains the following orderings: $S/R_{\mathrm{M}} \sim \epsilon^{j_3-j_1}$, $S/R \sim (v_{\mathrm{A}}/v_0)\epsilon^{j_3-j_1}$, and $S/N \sim (v_{\mathrm{A}}/v_0)\epsilon^{j_2-j_1}$. Of course there are several combinations but, for simplicity's sake, here we restrict ourselves to the case in which each transport process contributes equally to the dynamics (viz., $S \sim R_{\mathrm{M}} \sim R \sim N$), *i.e.* we assume $j_1 = j_2 = j_3 = j$ together with $v_0 \sim v_{\mathrm{A}}$. As a consequence, it suffices to consider two time scales, one related to the Alfvén speed and the other to dissipative processes. Then we may take t as the longer time-scale and t^* as the shorter, *i.e.*,

$$t^* = \epsilon^j t, \qquad j \geq 1; \tag{9}$$

with j being integer. For the remaining variables we set: (equilibria values) $-$ 1, and (n^{th}-order perturbations) $\sim \epsilon^n$.

We notice that the nonideal effects here considered are small (they go as ϵ^i, i being a positive integer) but even so they play a role of paramount importance in the present study as we shall see later. For the transport coefficients we will use the above mentioned dependence on temperature T normalized to its corresponding equilibrium value $T_e(r)$, namely

$$\eta = \eta_0 (T_e/T)^{3/2}, \qquad \mu = \mu_0 (T/T_e)^{5/2}, \qquad \kappa = \kappa_0 (T/T_e)^{5/2}, \tag{10}$$

where η_0, μ_0, and κ_0 are constant. By the way, the first equality in (10) is the well-known expression for Spitzer's resistivity.

3. STABILITY CONDITIONS FOR HIGH FREQUENCIES

In this section we will obtain the stability conditions at the long time-scale, accordingly we require the governing equations describing the dynamics in t. By expanding in power series of ϵ the system (1)–(5) one obtains to n^{th} order the following set of equations:

$$\frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_e \mathbf{v}_n) = D_n, \tag{11.1}$$

$$\rho_e \frac{\partial \mathbf{v}_n}{\partial t} + \nabla p_n - \mathbf{J}_e \times \mathbf{B}_n - \mathbf{J}_n \times \mathbf{B}_e = \mathbf{M}_n, \tag{11.2}$$

$$\frac{\partial \mathbf{B}_n}{\partial t} - \nabla \times (\mathbf{v}_n \times \mathbf{B}_e) = \mathbf{I}_n, \tag{11.3}$$

$$\frac{\partial p_n}{\partial t} + \mathbf{v}_n \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{v}_n = P_n, \tag{11.4}$$

$$\nabla \cdot \mathbf{B}_n = 0. \tag{11.5}$$

Here the subscript n denotes the order of approximation and e stands for denoting equilibria values. The terms on the right-hand sides involve physical variables at order less than $n-1$ and time derivatives with respect to t^*. Their explicit form will be shown at the moment of examining each particular case. Without loss of generality, we may restrict our examination to the case $i = 1$ (*i.e.*, the dissipative effects go as ϵ) and consider only static equilibria, $\mathbf{v}_e = 0$.

Taking the time derivative of the momentum Eq. (11.2) and using Eq. (11.3) together with the remaining expressions we obtain the corresponding equation for the temporal evolution of $\mathbf{v}_n$ as a function of the physical variables at order lower than n,

$$\rho_e \frac{\partial^2 \mathbf{v}_n}{\partial t^2} - \mathbf{F}(\mathbf{v}_n) = \mathbf{H}_n, \tag{12}$$

where $\mathbf{F}$ is the familiar operator from the linear stability analysis in the ideal-MHD model [1] defined as

$$\mathbf{F}(\mathbf{v}) = \nabla(\gamma p_e \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla p_e) + \mathbf{J}_e \times \nabla \times (\mathbf{v} \times \mathbf{B}_e) - \mathbf{B}_e \times \nabla \times \nabla \times (\mathbf{v} \times \mathbf{B}_e), \tag{13}$$

while $\mathbf{H}_n$ is given by

$$\mathbf{H}_n = -\nabla P_n + \mathbf{J}_\mathrm{e} \times \mathbf{I}_n - \mathbf{B}_\mathrm{e} \times \nabla \times \mathbf{I}_n + \frac{\partial \mathbf{M}_n}{\partial t}. \tag{14}$$

3.1. First-order approximation

To first-order Eq. (12) becomes

$$\rho_\mathrm{e} \frac{\partial^2 \mathbf{v}_1}{\partial t^2} - \mathbf{F}(\mathbf{v}_1) = 0, \tag{15}$$

seeing that $\mathbf{H}_1 \equiv 0$. Hence, the first-order solutions will be like that of the ideal MHD stability problem (see Appendix B). Indeed, by assuming a discrete spectrum given by the following relationship:

$$\mathbf{F}(\mathbf{u}_k) = -\rho_\mathrm{e} \omega_k^2 \mathbf{u}_k, \tag{16}$$

it is possible to recover the familiar solutions obtained by Bernstein *et al* [1]. Here $\mathbf{u}_k$ and ω_k denote the eigenvector and its eigenvalue, respectively, related to $\mathbf{F}$ for the mode k. It should be noted that $\mathbf{F}$ is a hermitian operator making use of the standard definition for the inner product

$$\langle \mathbf{u},\, \mathbf{v} \rangle = \int_V \mathbf{u} \cdot \mathbf{v}^* \, d^3x, \tag{17}$$

where $\mathbf{u}$ and $\mathbf{v}$ are two arbitrary vector functions and V is the volume of the region of interest, taking into account the boundary conditions (8). Obviously, the hermiticity of $\mathbf{F}$ ensures that every eigenvalue ω_k^2 is real. Under present circumstances the set $\{\mathbf{u}_k\}$ constitutes a complete basis introducing the following orthonormality condition:

$$\langle \mathbf{u}_k,\, \rho_\mathrm{e} \mathbf{u}_{k'} \rangle = \delta_{kk'}, \tag{18}$$

with $\delta_{kk'}$ denoting the Kronecker delta. The inner product of Eq. (12) with $\mathbf{u}_k$, making use of expressions (16) and (18), yields the known ideal solutions (see Appendix B) providing [14]

$$\mathbf{v}_1 = \sum_k \left\{ a_k^{(1)} \cos(\omega_k t) + b_k^{(1)} \sin(\omega_k t) \right\} \mathbf{u}_k. \tag{19.1}$$

Precedent result implies there is not any change with respect to the ideal case at the long time scale (*viz.*, high frequencies) to first-order approximation, although this situation does not exclude the possibility that $\mathbf{v}_1$ depends on t^* through both $a_k^{(1)}$ and $b_k^{(1)}$. We will show below that the relevant differences start off the second-order approximation. Nevertheless it should be noticed that the use of a multiply periodic expansion like that given by (19.1) complicates the present study because it drives a strong coupling among all modes [15]. For simplicity, we restrict our attention to the case of a *monochromatic* expansion [15]. In other words, from now on we only consider just one mode (*viz.*, one frequency) for the first-order approximation. Equation (19.1) then becomes

$$\mathbf{v}_1 = \left\{ a^{(1)} \cos(\omega t) + b^{(1)} \sin(\omega t) \right\} \mathbf{u}. \tag{19.2}$$

Then we will use for the rest of the high frequency analyses the solutions to first-order given in Appendix B but considering only one mode and, consequently, one frequency.

3.2. Second-order approximation

To second-order, Eq. (12) becomes

$$\rho_e \frac{\partial^2 \mathbf{v}_2}{\partial t^2} - \mathbf{F}(\mathbf{v}_2) = \mathbf{H}_2. \tag{20}$$

$\mathbf{H}_2$ is given explicitly in Appendix C (see Eq. (C2)).

For the purposes of the procedure outlined before, we will take for solving Eq. (20) the simplest case $j = 1$ (*i.e.* $t^* = \epsilon t$). Specifically, we look for solutions in the form $\mathbf{v}_2 = \sum_k \langle \mathbf{u}_k, \rho_e \mathbf{v}_2 \rangle \mathbf{u}_k$, thus Eq. (20) takes the form

$$\frac{d^2}{dt^2} \langle \mathbf{u}_k, \rho_e \mathbf{v}_2 \rangle + \omega_k^2 \langle \mathbf{u}_k, \rho_e \mathbf{v}_2 \rangle = \langle \mathbf{u}_k, \mathbf{H}_2 \rangle, \tag{21}$$

with the expression for the somewhat complicated term $\mathbf{H}_2$ given by (C3) in Appendix C. Of course $\mathbf{u}_k$ is one of the eigenvectors of $\mathbf{F}$.

It is clear that a particular solution to Eq. (21) is

$$\langle \mathbf{u}_k, \rho_e \mathbf{v}_2 \rangle = \int_0^t \left(\frac{e^{\lambda_1^{(k)}(t-t')} - e^{\lambda_2^{(k)}(t-t')}}{\lambda_1^{(k)} - \lambda_2^{(k)}} \right) \langle \mathbf{u}_k, \mathbf{H}_2 \rangle \, dt', \tag{22.1}$$

where $\lambda_i^{(k)}$ denotes a solution for the algebraic equation $\lambda^2 + \omega_k^2 = 0$. It should be observed that Eq. (22.1) could be rewritten as

$$\begin{aligned} \langle \mathbf{u}_k, \rho_e \mathbf{v}_2 \rangle = {} & \frac{\sin(\omega_k t)}{\omega_k} \int_0^t \cos(\omega_k t') \langle \mathbf{u}_k, \mathbf{H}_2 \rangle \, dt' \\ & - \frac{\cos(\omega_k t)}{\omega_k} \int_0^t \cos(\omega_k t') \langle \mathbf{u}_k, \mathbf{H}_2 \rangle \, dt'. \end{aligned} \tag{22.2}$$

This solution is adequate for the case when, as natural, $\langle \mathbf{u}_k, \rho_e \mathbf{v}_2 \rangle = 0$ at $t = 0$.

The solution for the homogeneous equation related to Eq. (21) is, of course, a superposition of sines and cosines of $\omega_k t$ when $\omega_k^2 > 0$, and does not give place to instabilities; however when $\omega_k^2 < 0$ we have solutions which grow or decay exponentially, but by using an appropriate initial condition we can eliminate the growing terms. Thus it appears that the only potentially unstable contribution is that related to the particular solution (22.2). Hence, for ensuring bounded solutions we have to impose that the amplitudes in Eq. (22.2) satisfy a non-resonance condition, that is

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle \mathbf{u}_k,\, \mathbf{H}_2\rangle \begin{pmatrix}\cos(\omega_k t)\\ \sin(\omega_k t)\end{pmatrix} dt = 0. \tag{23}$$

The application of (23) to (22.2) leads to a system of ordinary differential equations for $a^{(1)}(t^*)$ and $b^{(1)}(t^*)$, namely

$$2\omega^2\frac{d}{dt^*}a^{(1)} - \Omega a^{(1)} = 0, \tag{24.1}$$

$$2\omega^2\frac{d}{dt^*}b^{(1)} - \Omega b^{(1)} = 0. \tag{24.2}$$

In these equations Ω is defined by

$$\begin{aligned}\Omega := \langle \mathbf{u},\, \mathcal{D}_2\rangle = (1-\gamma)\left\langle \mathbf{u},\, \nabla\left[2\eta_0\mathbf{J}_\mathrm{e}\cdot\tilde{\mathbf{J}}_1 - \tfrac{3}{2}\eta_0 J_\mathrm{e}^2\frac{\tilde{T}_1}{T_\mathrm{e}} + \kappa_0\nabla^2\tilde{T}_1 + \tfrac{5}{2}\kappa_0\nabla\cdot\left(\frac{\tilde{T}_1}{T_\mathrm{e}}\nabla T_\mathrm{e}\right)\right]\right\rangle \\ - \eta_0\left\langle \mathbf{u},\, \mathbf{J}_\mathrm{e}\times\nabla\times\tilde{\mathbf{J}}_1 - \tfrac{3}{2}\mathbf{J}_\mathrm{e}\times\nabla\times\left(\frac{\tilde{T}_1}{T_\mathrm{e}}\mathbf{J}_\mathrm{e}\right) - \mathbf{B}_\mathrm{e}\times\nabla\times\nabla\times\tilde{\mathbf{J}}_1\right. \\ \left. + \tfrac{3}{2}\mathbf{B}_\mathrm{e}\times\nabla\times\nabla\times\left(\frac{\tilde{T}_1}{T_\mathrm{e}}\mathbf{J}_\mathrm{e}\right)\right\rangle - \frac{\mu_0}{2}\omega^2\langle \mathbf{u},\, \nabla\cdot\tilde{\mathbf{s}}_1\rangle.\end{aligned} \tag{25}$$

Here the quantities with tilde correspond to the spatial part of the first-order solutions for one mode (see Eqs. (B1)–(B2) in Appendix B).

Therefore it is immediate that when $\Omega < 0$ we have a decreasing amplitude while $\Omega > 0$ gives place to a positive growing rate. These conditions constitute the corresponding stability criteria for the present case.

4. Stability at low frequencies

4.1. Second-order approximation

As was pointed out in the last section, because of the functional form of $\mathbf{H}_n$ there is not any substantial change at high frequencies comparing with the ideal case in what concerns to stability conditions, at least for cases here considered, although the resistive contributions appear in Eqs. (23) and (24) through the term Ω. Let us choose now a low-frequency regime (*viz.*, 'short' time-scale). To start with, we shall examine the temporal evolution espect to t^* according to

$$\left|\frac{\partial}{\partial t^*}\right| \sim \epsilon^j, \tag{26}$$

and expand again in power series of ϵ the system (1)–(5) but this time we set $i = 2$. We get therefore instead of (11.1)–(11.4), in the simplest case $j = 1$,

$$\frac{\partial \rho_n}{\partial t^*} + \nabla \cdot (\rho_e \mathbf{v}_{n+1}) = D'_{n+1}, \tag{27.1}$$

$$\nabla p_n - \mathbf{J}_n \times \mathbf{B}_e - \mathbf{J}_e \times \mathbf{B}_n = \mathbf{M}'_n, \tag{27.2}$$

$$\frac{\partial \mathbf{B}_n}{\partial t^*} - \nabla \times (\mathbf{v}_{n+1} \times \mathbf{B}_e) = \mathbf{I}'_{n+1}, \tag{27.3}$$

$$\frac{\partial p_n}{\partial t^*} + \mathbf{v}_{n+1} \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{v}_{n+1} = P'_{n+1}, \tag{27.4}$$

together with Eq. (11.5). The terms on the right-hand side of (27.1)–(27.4) are analogous to that appearing in Eqs. (11.1)–(11.5).

The derivative of the momentum Eq. (27.2), with respect to t^*, making use of the remaining equations, leads to

$$\mathbf{F}(\mathbf{v}_{n+1}) + \mathbf{H}'_{n+1} = 0, \tag{28}$$

where

$$\mathbf{H}'_{n+1} = -\nabla P'_{n+1} - \mathbf{B}_e \times \nabla \times \mathbf{I}'_{n+1} + \mathbf{J}_e \times \mathbf{I}'_{n+1} + \frac{\partial \mathbf{M}'_n}{\partial t^*}. \tag{29}$$

It is worth to notice that $\mathbf{H}'_1 \equiv 0$ and as a consequence $\mathbf{F}(\mathbf{v}_1) \equiv 0$ for arbitrary equilibria. This implies that $\mathbf{v}_1 = 0$ and consequently the leading order system related to (28) will be those corresponding to the second-order approximation,

$$\mathbf{F}(\mathbf{v}_2) = 0, \tag{30}$$

for the case under consideration.

Now, we take for $\mathbf{v}_2$ a similar expression to that given for $\mathbf{v}_1$ in (19.2) with a slight modification

$$\mathbf{v}_2 = \left\{\frac{da^{(2)}}{dt^*}\cos(\omega t) + \frac{db^{(2)}}{dt^*}\sin(\omega t)\right\}\mathbf{u}', \tag{31}$$

where $\mathbf{u}'$ belongs to a subspace of that space containing the vectors $\mathbf{u}$. Former vector has the property of satisfying $\mathbf{F}(\mathbf{u}') = 0$ in virtue of Eq. (30).

Substituting (31) into the system (27.1)–(27.4) with $n = 1$ we might obtain a set of equations for ρ_1, p_1, T_1, and $\mathbf{B}_1$, by using $p = \rho T$. These solutions are similar to those from the ideal MHD, however they depend only on the time t^* through both $a^{(2)}(t^*)$ and $b^{(2)}(t^*)$ while their spatial dependence appear through both $\mathbf{u}'$ and the equilibrium values. It is to say, we get similar solutions to that of the ideal-MHD but they are a more restricted ones because of the replacement of $\mathbf{u}$ by $\mathbf{u}'$. That solutions are:

$$\rho_1 = -g^{(2)}(t^*, t)\, \nabla \cdot (\rho_e \mathbf{u}'_k), \tag{32.1}$$

$$\mathbf{B}_1 = \quad g^{(2)}(t^*, t)\, \nabla \times (\mathbf{u}'_k \times \mathbf{B}_e), \tag{32.2}$$

$$p_1 = -g^{(2)}(t^*, t)\, (\mathbf{u}' \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u}'_k), \tag{32.3}$$

$$T_1 = \quad g^{(2)}(t^*, t) \left[\frac{1}{\rho_e} \mathbf{u}' \cdot \nabla p_e + \frac{\gamma p_e}{\rho_e} \nabla \cdot \mathbf{u}' + \frac{T_e}{\rho_e} \nabla \cdot (\rho_e \mathbf{u}') \right], \tag{32.4}$$

where

$$g^{(2)}(t^*, t) = a_k^{(2)}(t^*) \cos(\omega t) + b^{(2)}(t^*) \sin(\omega t). \tag{33}$$

Note that the momentum balance equation is fulfilled when $\mathbf{u}'$ satisfies

$$\nabla(\mathbf{u}' \cdot \nabla p_e + \gamma p_e \nabla \cdot \mathbf{u}') = \mathbf{J}_e \times \nabla \times (\mathbf{u}' \times \mathbf{B}_e) - \mathbf{B}_e \times \nabla \times \nabla \times (\mathbf{u}' \times \mathbf{B}_e). \tag{34}$$

From this relationship one gets a particular structure for the eigenvectors $\mathbf{u}'$ for a given equilibrium state.

As remarked previously, the solutions for the present range of frequencies constitute a more restrictive class respect to those from the ideal MHD. From (34) we can determine $\mathbf{u}'$ which is required for further analyses, though, we do not perform any numerical evaluation. Now we will proceed to examine the stability to the next order.

4.2. Third-order approximation

To examine the case concerning to the third-order approximation, we use for the velocity to third order a similar expression like that of the second order, (31), namely

$$\mathbf{v}_3 = \left\{ \frac{da^{(3)}}{dt^*} \cos(\omega t) + \frac{db^{(3)}}{dt^*} \sin(\omega t) \right\} \mathbf{u}'. \tag{35}$$

Substituting (35) and $\mathbf{v}_1 = 0$ into (28), setting $n = 2$ and making use of (32)–(34), we get, after taking its inner product with $\mathbf{u}'$,

$$\frac{d}{dt^*} \left\{ a^{(2)} \cos(\omega t) + b^{(2)} \sin(\omega t) \right\} = - \frac{\langle \mathbf{u}', \mathcal{D}_2(\mathbf{u}'; \omega = 0) \rangle}{\langle \mathbf{u}', \mathcal{N}_2(\mathbf{u}'; \omega = 0) \rangle}. \tag{36}$$

$\mathcal{D}_2$ and $\mathcal{N}_2$ are familiar to reader, they arised when we analyzed the case of high frequencies to second-order approximation. There is a difference, however, because we use here $\mathbf{u}'$ instead $\mathbf{u}$ and they were, for the case at hand, evaluated at $\omega = 0$ (see Appendix C).

Of course, Eq. (36) is valid just when $\langle \mathbf{u}', \mathcal{N}_2(\mathbf{u}'; \omega = 0)\rangle \neq 0$; then we have that $\langle \mathbf{u}', \mathcal{D}_2(\mathbf{u}'; \omega = 0)\rangle / \langle \mathbf{u}', \mathcal{N}_2(\mathbf{u}'; \omega = 0)\rangle > 0$ implies that the amplitude decays linearly, whereas if that quotient is negative the amplitude grows linearly; both occurring at the 'short' time-scale.

5. CONCLUSIONS AND FINAL REMARKS

In this section we summarize the points treated in this work. In order to study the stability of a dissipative plasma (*i.e.*, there are present resistivity, thermal conductivity, and viscosity) against small disturbances from a static equilibrium initial state, we separate the dynamics by introducing two time scales. One of them is related to Alfvén transit time whereas the other is concerned with the dissipation in the sense that it is the characteristic time for such a processes. Basically, we have performed a decomposition in oscillating modes but including two time-scales, thus this treatment appears in some extent as a kind of generalization of the conventional normal-mode analysis. It worth mentioning that one might to set up, at least, three different regimes (*i.e.*, three characteristic time-scales) corresponding to resistivity, thermal conductivity, and viscosity. However, in the present paper we restrict ourselves to consider only two regimes, that defined by the resistive diffusion and the Alfvén transit time.

Starting from a fluid model for a dissipative plasma contained within a rigid and perfectly conducting cylindrical vessel we have discussed sufficient conditions for stability for the three lowest orders after performing an asymptotic expansion in terms of a positive small parameter ϵ. This parameter is related to the Lundquist number (in the present case, also to the Reynolds number) as was discussed in Sec. 2.2. The time scales and the expansion in power series of ϵ were restricted to positive integer powers. The main advantage of the approach shown in this work seems to be, in the author's opinion, that it allows us to determine, in a precise manner, how the dissipation affects the stability conditions at different characteristic times.

In particular, for the present case, we found that at high frequencies and to first-order in ϵ one recovers the well known solution for an ideal plasma whilst to second-order one gets an oscillatory behavior (respect to t) and their corresponding amplitudes depend on the first-order quantities in such way that the stability criterio (say, bounded solutions) gives place to an exponential dependence upon t^* [see Eqs. (24.1) and (24.2)].

On the other hand, for the low frequencies regime, we found that the second-order is the leading order. This latter implies that at this time-scale the first-order quantities possess a similar behavior to that of the ideal-MHD but they are a more restrictive class of solutions. When it is considered the third order approximation we found solutions which oscillate respect to t with an amplitude which grows (or decays) linearly respect to t^* [see Eq. (36)].

It is apparent that a nonlinear evolution arises as one increases the order of approximation, however, such case deserve a further study.

It is worth to notice a missing point in this work. The conventional device for studying plasmas is an externally forced system. Thus it is required in order to get a more realistic description to include in this kind of studies, for example, the presence of a source heating whose role could be to sustain the plasma dynamics, through thermal waves, against the dissipation.

Finally, we should like to add we believe that results discussed here are justified enough to explore further extension; for example, to examine the effect of anomalous transport and anisotropy upon the stability conditions.

ACKNOWLEDGEMENTS

The author is sincerely grateful to the anonymous referee for his kind comments and suggestions which were helpful to improve this paper.

APPENDIX A. ION STRESS TENSOR

In terms of cartesian coordinates the Braginskii ion stress is given [13], assuming in particular that the equilibrium magnetic field goes along the z–direction, by

$$\Pi_{xx} = -\frac{\mu}{2}(W_{xx} + W_{yy}) - \frac{\mu_1}{2}(W_{xx} - W_{yy}) - \mu_3 W_{xy}, \tag{A.1}$$

$$\Pi_{yy} = -\frac{\mu}{2}(W_{xx} + W_{yy}) + \frac{\mu_1}{2}(W_{xx} - W_{yy}) + \mu_3 W_{xy}, \tag{A.2}$$

$$\Pi_{xy} = \Pi_{yx} = -\frac{\mu_1}{2} W_{xy} + \frac{\mu_3}{2}(W_{xx} - W_{yy}), \tag{A.3}$$

$$\Pi_{xz} = \Pi_{zx} = -\mu_2 W_{xz} - \mu_4 W_{yz}, \tag{A.4}$$

$$\Pi_{yz} = \Pi_{zy} = -\mu_2 W_{yz} + \mu_4 W_{xz}, \tag{A.5}$$

$$\Pi_{zz} = -\mu W_{||||}, \tag{A.6}$$

with W_{ij} defined by (7). The viscosity coefficients have the following functional form:

$$\mu = 0.96\,\frac{\rho}{m_{\rm i}} T_i \tau_i, \tag{A.7}$$

$$\mu_1 = \frac{\rho}{m_{\rm i}} T_i \tau_i \left(\frac{\frac{24}{5}\zeta^2 + 2.23}{16\,\zeta^4 + 16.12\,\zeta^2 + 2.33} \right), \tag{A.8}$$

$$\mu_2 = \frac{\rho}{m_{\rm i}} T_i \tau_i \left(\frac{\frac{6}{5}\zeta^2 + 2.23}{\zeta^4 + 4.03\,\zeta^2 + 2.33} \right), \tag{A.9}$$

$$\mu_3 = 2\frac{\rho}{m_\mathrm{i}} T_i \zeta \left(\frac{4\zeta^4 + 2.38}{16\,\zeta^4 + 16.12\,\zeta^2 + 2.33} \right), \tag{A.10}$$

$$\mu_4 = \frac{\rho}{m_\mathrm{i}} T_i \tau_i \zeta \left(\frac{\zeta^2 + 2.38}{\zeta^4 + 4.03\,\zeta^2 + 2.33} \right), \tag{A.11}$$

where $\zeta = \omega_i \tau_i$ and m_i is the ion mass. In the limit of very large fields $\zeta \gg 1$, $\mu_i = 0$ for $i = 1, \ldots, 4$, therefore only μ contributes to $\mathbf{\Pi}$. On the other hand, $\mathbf{s}$ is given by

$$\mathbf{s} = -(\mathbf{1} - 3\mathbf{e}_\parallel \mathbf{e}_\parallel) W_{\parallel\parallel}, \tag{A.12}$$

in view that $\mathrm{tr}(\mathbf{W}) \equiv 0$, $\mathrm{tr}(\mathbf{W})$ being the trace of $\mathbf{W}$. Here $\mathbf{1}$ is the unit tensor of second-rank, the subscript '$\parallel$' refers to the parallel direction with respect to the equilibrium magnetic field $\mathbf{B}_\mathrm{e}$, and $\mathbf{e}_\parallel = \mathbf{B}_\mathrm{e}/B_\mathrm{e}$ with $B_\mathrm{e} = |\mathbf{B}_\mathrm{e}|$. On the other hand,

$$\tau_i = \frac{3}{4} \frac{(m_\mathrm{i} T_i)^{3/2}}{\sqrt{\pi} \rho \lambda e^4} \tag{A.13}$$

with λ the Coulomb logarithm, and e the electron charge. Thereby (A.7), together with (A.13), leads to

$$\mu = \frac{0.72}{\sqrt{\pi/m_\mathrm{i}}\,\lambda e^4} T_i^{5/2} \sim T^{5/2}. \tag{A.14}$$

Thus the stress tensor becomes that given by the expression (6).

APPENDIX B. IDEAL-MHD SOLUTIONS

The ideal solutions for linear stability, like those obtained by Bernstein [1] using an energy principle can be written as

$$\rho_1 = \sum_k f_k^{(1)}(t^*, t)\, \nabla \cdot (\rho_\mathrm{e} \mathbf{u}_k) \equiv \sum_k f_k^{(1)}(t^*, t)\, \tilde{\rho}_1^{(k)}(\mathbf{x}), \tag{B.1}$$

$$\mathbf{B}_1 = \sum_k f_k^{(1)}(t^*, t)\, \nabla \times (\mathbf{B}_\mathrm{e} \times \mathbf{u}_k) \equiv \sum_k f_k^{(1)}(t^*, t)\, \tilde{\mathbf{B}}_1^{(k)}(\mathbf{x}), \tag{B.2}$$

$$p_1 = \sum_k f_k^{(1)}(t^*, t)\, [\mathbf{u}_k \cdot \nabla p_\mathrm{e} + \gamma p_\mathrm{e} \nabla \cdot \mathbf{u}_k] \equiv -\sum_k f_k^{(1)}(t^*, t)\, \tilde{p}_1^{(k)}, \tag{B.3}$$

where

$$f_k^{(1)}(t^*, t) = \omega_k^{-1} [b_k \cos(\omega_k t) - a_k \sin(\omega_k t)]. \tag{B.4}$$

For the perturbed temperature we have

$$T_1 = \sum_k f_k^{(1)}(t^*, t) \left[\frac{1}{\rho_e} \mathbf{u}_k \cdot \nabla p_e - (1-\gamma) T_e \nabla \cdot \mathbf{u}_k - \frac{T_e}{\rho_e} \mathbf{u}_k \cdot \nabla \rho_e \right]$$

$$\equiv \sum_k f_k^{(1)}(t^*, t)\, \tilde{T}_1^{(k)}. \tag{B.5}$$

APPENDIX C. THE EXPRESSION FOR $\mathbf{H}_2$

Here we provide the detailed functional form of $\mathbf{H}_2$. From (30) we have

$$\mathbf{H}_2 = -\nabla P_2 + \mathbf{J}_e \times \mathbf{I}_2 - \mathbf{B}_e \times \nabla \times \mathbf{I}_2 + \frac{\partial \mathbf{M}_2}{\partial t}, \tag{C.1}$$

which, making use of (24)–(27) with $n = 2$, leads to

$$\begin{aligned}
\mathbf{H}_2 = \nabla & \left[\frac{\partial p_1}{\partial t^*} + \mathbf{v}_1 \cdot \nabla p_1 + \gamma p_1 \nabla \cdot \mathbf{v}_1 + (1-\gamma)(2\eta_0 \mathbf{J}_e \cdot \mathbf{J}_1 \right. \\
& \left. + \eta_1 J_e^2 + \nabla \cdot (\kappa_0 \nabla T_1 + \kappa_1 \nabla T_e)) \right] \\
& + \mathbf{J}_e \times \left[-\frac{\partial \mathbf{B}_1}{\partial t^*} + \nabla \times (\mathbf{v}_1 \times \mathbf{B}_1 - \eta_0 \mathbf{J}_1 - \eta_1 \mathbf{J}_e) \right] \\
& - \mathbf{B}_e \times \nabla \times \left[-\frac{\partial \mathbf{B}_1}{\partial t^*} + \nabla \times (\mathbf{v}_1 \times \mathbf{B}_1 - \eta_0 \mathbf{J}_1 - \eta_1 \mathbf{J}_e) \right] \\
& - \frac{\partial \rho_1}{\partial t} \frac{\partial \mathbf{v}_1}{\partial t} - \rho_1 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} - \rho_e \left(\frac{\partial \mathbf{v}_1}{\partial t} \right) \cdot \nabla \mathbf{v}_1 - \rho_e \mathbf{v}_1 \cdot \nabla \left(\frac{\partial \mathbf{v}_1}{\partial t} \right) \\
& + \frac{\partial \mathbf{J}_1}{\partial t} \times \mathbf{B}_1 - \rho_e \frac{\partial^2 \mathbf{v}_1}{\partial t \partial t^*} + \mathbf{J}_1 \times \frac{\partial \mathbf{B}_1}{\partial t} - \frac{\mu_0}{2} \frac{\partial}{\partial t} \nabla \cdot \mathbf{s}_1,
\end{aligned} \tag{C.2}$$

where we have taken $j = 1$. This expression may be written as

$$\begin{aligned}
\mathbf{H}_2 = {} & [\dot{a} \sin(\omega t) - \dot{b} \cos(\omega t)](2\rho_e \omega\, \mathbf{u}) \\
& + \frac{1}{2\omega} [(b^2 - a^2) \sin(2\omega t) + 2ab \cos(2\omega t)] \mathcal{N}_2 \\
& + \frac{1}{\omega} [b \cos(\omega t) - a \sin(\omega t)] \mathcal{D}_2,
\end{aligned} \tag{C.3}$$

where the dot denotes derivative with respect to t^*, and

$$\begin{aligned}\mathcal{N}_2(\mathbf{u};\,\omega) = {} & \nabla(\mathbf{u}\cdot\nabla\tilde{p}_1 + \gamma\tilde{p}_1\nabla\cdot\mathbf{u}) + \mathbf{J}_e \times \nabla \times (\mathbf{u}\times\tilde{\mathbf{B}}_1) \\ & - \mathbf{B}_e \times \nabla \times \nabla \times (\mathbf{u}\times\tilde{\mathbf{B}}_1) + 2\tilde{\mathbf{J}}_1 \times \tilde{\mathbf{B}}_1 \\ & + 2\omega^2\tilde{\rho}\mathbf{u} - 2\rho_e\omega^2\mathbf{u}\cdot\nabla\mathbf{u}, \end{aligned} \tag{C.4}$$

$$\begin{aligned}\mathcal{D}_2(\mathbf{u};\,\omega) = {} & (1-\gamma)\nabla\left[2\eta_0\mathbf{J}_e\cdot\tilde{\mathbf{J}}_1 - \tfrac{3}{2}\eta_0 J_e^2\frac{\tilde{T}_1}{T_e} + \kappa_0\nabla^2\tilde{T}_1\right. \\ & \left. + \tfrac{5}{2}\kappa_0\nabla\cdot\left(\frac{\tilde{T}_1}{T_e}\nabla T_e\right)\right] - \eta_0\mathbf{J}_e\times\nabla\times\tilde{\mathbf{J}}_1 \\ & + \tfrac{3}{2}\eta_0\mathbf{J}_e\times\nabla\times\left(\frac{\tilde{T}_1}{T_e}\mathbf{J}_e\right) + \mathbf{B}_e\times\nabla\times\nabla\times\tilde{\mathbf{J}}_1 \\ & - \tfrac{3}{2}\eta_0\mathbf{B}_e\times\nabla\times\nabla\times\left(\frac{\tilde{T}_1}{T_e}\mathbf{J}_e\right) - \frac{\mu_0}{2}\omega^2\nabla\cdot\tilde{\mathbf{s}}_1. \end{aligned} \tag{C.5}$$

The spatial dependence is introduced through $\mathbf{u}_k$ and equilibria quantities.

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