Symmetry Lie algebra of the two body system with a Dirac oscillator interaction

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ABSTRACT. A few years ago Moshinsky and Szczepaniak introduced a Dirac equation linear not only in the momentum but also in the coordinate, which they called the Dirac oscillator, as for the large component of the eigenstate with positive energy, it reduces to a normal oscillator with a strong spin-orbit term. This problem has interesting degeneracies that were shown by Quesne and Moshinsky to be due to an $o(4) \oplus o(3,1)$ symmetry Lie algebra. The equation was then generalized to a two particle system with a Dirac oscillator interaction, for which the degeneracy disappears for states of parity $-(-1)^j$, with j being the total angular momentum, but remains for states of parity $(-1)^j$. We show that for the latter, the degeneracy is due to a u(3) symmetry Lie algebra if we take states of spin 0 and 1 separately or to an o(4) symmetry Lie algebra if we take them together. Furthermore we consider the nonrelativistic limit of our problem which reduces it to an operator $\hat{N} - \mathbf{L}\cdot\mathbf{S}$ where $\hat{N}$ is the total number of quanta, $\mathbf{L}$ the orbital angular momentum and $\mathbf{S}$ the total spin, whose eigenvalues are now $s = 0$ or 1. In this case the symmetry Lie algebra for the states of parity $(-1)^j$ remains the one discussed above, but there is now degeneracy also for states of parity $-(-1)^j$, which is explained, by a reasoning similar to that for the single particle Dirac oscillator by the symmetry Lie algebra $o(4) \oplus o(3,1)$ but now with a spin $s = 1$ instead of $s = 1/2$.

RESUMEN. Algunos años atrás, Moshinsky y Szczepaniak introdujeron una ecuación de Dirac lineal no sólo en el momento sino también en la coordenada, la cual llamaron el oscilador de Dirac, ya que para la componente grande del eigenestado con energía positiva, éste se reduce a un oscilador normal con un término de un fuerte acoplamiento espín-órbita. Este problema tiene degeneraciones interesantes que, como ha sido indicado por Quesne y Moshinsky, se deben al álgebra de simetría $o(4) \oplus o(3,1)$. La ecuación fue generalizada para un sistema de dos partículas con una interacción del tipo de oscilador de Dirac para el cual la degeneración desaparece para los estados de paridad $-(-1)^j$, siendo j el momento angular total, pero se mantiene para los estados de paridad $(-1)^j$. Nosotros demostramos que para este último, la degeneración es debida al álgebra u(3) si tomamos los estados de espín 0 y 1 separados, o al álgebra o(4) si los tomamos juntos. Además consideramos el límite no-relativista de nuestro problema, el cual se reduce al operaror $\hat{N} - \mathbf{L}\cdot\mathbf{S}$, donde $\hat{N}$ es el número total de cuantas, $\mathbf{L}$ el momento angular orbital y $\mathbf{S}$ el espín total, cuyos eigenvalores son 0 o 1. En este caso el álgebra de simetría para los estados de paridad $(-1)^j$ es igual a la discutida anteriormente, pero existe ahora una degeneración para los estados de paridad $-(-1)^j$, la cual es explicada por un razonamiento similar al del oscilador de Dirac de una sola partícula, es decir, según el álgebra $o(4) \oplus o(3,1)$, pero ahora con espín $s = 1$ en vez de $s = 1/2$.

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1. Introduction and summary

When Dirac [1] introduced his equation, his starting point was the relativistic relation between energy and momentum, *i.e.*

$$E^2 = p^2c^2 + m^2c^4. \tag{1}$$

Instead of proceeding to convert it into a quantum mechanical equation by replacing E and $\mathbf{p}$ by the operators $i\hbar\partial/\partial t$ and $-i\hbar\nabla$ and thus getting what is known now as the Klein-Gordon equation, he linearized it and got the equation that bears his name.

If instead of the relation given above for a free particle, we would have an added term quadratic in the coordinates, *i.e.*, an oscillator interaction, we could think of the possibility of linearizing it. This was done by Moshinsky and Szczepaniak [2] and it leads to the replacement of the momentum $\mathbf{p}$ in the Dirac equation by

$$\mathbf{p} \rightarrow \mathbf{p} - im\omega\mathbf{r}\beta, \tag{2}$$

where m is the mass of the particle, ω the frequency of the oscillator, $\mathbf{r}$ the position vector and β the matrix appearing with that name in the ordinary Dirac equation. The substitution (2) gives rise to what the authors called a Dirac oscillator, as for the large component it reduces to a standard oscillator with a very strong spin-orbit coupling term.

Actually Dirac equations linear in both momentum and coordinates, had been proposed before [3] the one discussed in Ref. [2]. But these attempts could be compared with Viking, Polynesian and Chinese predecessors of Coulombs discovery of America, in the sense that only the latter led to continuous and ever increasing contacts. By now there are several dozen papers on the Dirac oscillator, some dealing with the one body [4] case mentioned above, but others concerned with many body systems interacting through Dirac oscillators [5].

The one body problem presents interesting degeneracies that were explained group theoretically by Quesne and Moshinsky [6]. Most of these degeneracies disappear in many body cases [5], which furthermore may require the diagonalization of *finite* matrices. One problem that can be solved exactly [7], and maintains, at least in part, interesting degeneracies, is the system of two particles interacting through a Dirac oscillator, and the object of this paper is to analyze them from the standpoint of the corresponding Lie algebras.

The article is divided in essentially two parts, the first one is the relativistic problem and the second its nonrelativistic limit.

In the first part we note the difference of the solutions of the problem when the parity is $(-1)^j$, with j being the total angular momentum, and those for parity $-(-1)^j$. Only the former have accidental degeneracy and thus we separate the two cases with the help of appropriate projection operators.

For parity $(-1)^j$ there are symmetry Lie algebras of the u(3) type for the cases of spin 0 and 1 separately, but more interestingly there is a o(4) symmetry Lie algebra when they are considered together. The generators of all these Lie algebras are obtained explicitly.

In the second part, the nonrelativistic limit gives a Hamiltonian of the type $\hat{N} - \mathbf{L}\cdot\mathbf{S}$ where $\hat{N}$ is the operator that gives the total number of quanta, $\mathbf{L}$ is the orbital angular momentum and $\mathbf{S}$ the total spin in this case of eigenvalue 0 or 1. For parity $(-1)^j$ the symmetry Lie algebras are the same as for the relativistic case, but now there is also degeneracy for parity $-(-1)^j$ and these are explained by the direct sum Lie algebra $o(4) \oplus o(3,1)$.

Finally in the concluding section we mention some different approaches to the problem, including that of supersymmetry, though the detailed discussion of the latter is left for another article.

Our starting point will be the Eq. (4.1I) given in Ref. [7], and in the following discussion we refer to this paper by a roman number I, and add it to all the equations quoted.

2. THE RELATIVISTIC PROBLEM

In reference I we considered a Poincaré invariant equation for a two body system with a Dirac oscillator interaction. As discussed in Section 4 of this paper [7], in the center of mass frame, the equation for our system becomes

$$\left[\tfrac{1}{\sqrt{2}}(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\cdot(\mathbf{p} - i\omega\mathbf{r}B) + \beta_1 + \beta_2\right]\psi = E\psi, \tag{3}$$

corresponding to Eq. (4.1I) but where now we use relativistic units $\hbar = m = c = 1$, so that the frequency ω appears explicitly and we replace $i\partial/\partial X^0$ by its eigenvalue, that we shall denote as E.

The $\mathbf{r}$, $\mathbf{p}$ are relative position and momentum respectively and

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \boldsymbol{\alpha}_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 \end{pmatrix}, \tag{4a, b}$$

$$\beta_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4c, d}$$

$$B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{4e}$$

It is convenient to express all matrices of Eq. (4) in a 4×4 type as indicated in (4.4I). They act on state ψ of 4 components

$$\psi = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix}, \tag{5}$$

as indicated in (4.3I).

Introducing creation and annihilation operators by the definitions

$$\boldsymbol{\eta} = \tfrac{1}{\sqrt{2}}(\omega^{1/2}\mathbf{r} - i\omega^{-1/2}\mathbf{p}), \qquad \boldsymbol{\xi} = \tfrac{1}{\sqrt{2}}(\omega^{1/2}\mathbf{r} + i\omega^{-1/2}\mathbf{p}), \tag{6}$$

we can use Eq. (3) to express the components ψ_{21}, ψ_{12} as operators in $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ acting on ψ_{11}, ψ_{22}, and finally writing the latter in terms of ϕ_+, ϕ_- by the relation

$$\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix}, \tag{7}$$

we see that $(\phi_+,\, \phi_-)$ satisfy the matrix operator equation [7]

$$\mathcal{O} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \equiv \begin{bmatrix} 4\omega(\boldsymbol{\eta}\cdot\mathbf{S})(\boldsymbol{\xi}\cdot\mathbf{S}) - E^2 & 2E \\ 2E & 4\omega[\boldsymbol{\eta}\cdot\boldsymbol{\xi} - \mathbf{L}\cdot\mathbf{S} - (\boldsymbol{\eta}\cdot\mathbf{S})(\boldsymbol{\xi}\cdot\mathbf{S})] - E^2 \end{bmatrix} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = 0, \tag{8}$$

which, in the present notation, corresponds to (4.8I).

The states $(\phi_+,\, \phi_-)$ can be characterized by the eigenvalues of the operators

$$\hat{N}, \quad \mathbf{J}^2, \quad J_3, \quad \mathbf{S}^2, \quad P, \tag{9}$$

which we denote, respectively, as

$$N, \quad j(j+1), \quad m, \quad s(s+1), \quad \pm, \tag{10}$$

and where

$$\hat{N} = \boldsymbol{\eta}\cdot\boldsymbol{\xi}, \qquad \mathbf{J} = \mathbf{L} + \mathbf{S}, \qquad \mathbf{L} = -i(\boldsymbol{\eta}\times\boldsymbol{\xi}), \qquad \mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \tag{11}$$

with P being the parity operator. These operators commute with $\mathcal{O}$ and among themselves.

The functions $\phi_\pm$ can then be expressed in terms of the kets [7]

$$|N(\ell, s)jm\rangle = \sum_{\mu,\sigma} \langle \ell\mu, s\sigma | jm\rangle R_{N\ell}(r)\, Y_{\ell\mu}(\theta,\varphi)\, X_{s\sigma}, \tag{12}$$

where r, θ, φ are spherical coordinates associated with $\mathbf{r}$, the $\langle\ \ |\ \ \rangle$ is a Clebsch-Gordan coefficient, $R_{N\ell}(r)$ is the radial function of a three dimensional oscillator for N quanta and orbital angular momentum ℓ, $Y_{\ell\mu}(\theta,\varphi)$, $\mu = \ell, \ell-1, \ldots, -\ell$, a spherical harmonic and $X_{s\sigma}$ a spin state with its projection σ, where $s = 0, 1$. The parity of the ket is given by $(-1)^\ell$.

2.1. Eigenstates and eigenvalues of Eq. (3), and projection operators for states of different parities

We note that for definite (N, j) and spin $s = 0$ we have a single state

$$|N(j,0)jm\rangle, \tag{13}$$

whose parity is $(-1)^j$. For spin $s = 1$ and parity $(-1)^j$ we also have a single state

$$|N(j,1)jm\rangle, \tag{14}$$

while for parity $-(-1)^j$ we have two states

$$|N(j \pm 1, 1) jm\rangle. \tag{15}$$

The calculation of the eigenvalues E in Eq. (8) was done in Refs. [7,8] and in our units it is given for $s = 0$ by

$$E^2 = 0 \quad \text{or} \quad E^2 = 4 + 4\omega N, \tag{16}$$

and for $s = 1$ and parity $(-1)^j$ is

$$E^2 = 0 \quad \text{or} \quad E^2 = 4 + 4\omega(N + 1), \tag{17}$$

while for parity $-(-1)^j$, E^2 is the solution of the fourth order algebraic equation

$$E^2\left\{E^2[E^2 - 4 - 4\omega(N+1)][E^2 - 4 - 4\omega(N+2)] - 64\omega^2 j(j+1)\right\} = 0. \tag{18}$$

The corresponding eigenstates are given by Eqs. (4.21, 4.22, 4.23I), with the coefficients obtained in the Appendix A of Ref. [7].

We shall restrict ourselves to values of $E > 0$, as the case $E = 0$ has been discussed separately in a recent article, while $E < 0$ will be mentioned in the concluding section, in relation with some aspects of supersymmetry of the problem that will be discussed in a future paper. We then see that, except when $j = 0$, the states of parity $-(-1)^j$, have no accidental degeneracy, so we will concentrate in this section on those of parity $(-1)^j$. Thus what we require first are projection operators that allow to separate the states of different parities. This is easily achieved with the help of the operators $\hat{L}$, $\hat{J}$ defined by

$$\hat{L} = \left(\mathbf{L}^2 + \tfrac{1}{4}\right)^{\frac{1}{2}} - \tfrac{1}{2}, \qquad \hat{J} = \left(\mathbf{J}^2 + \tfrac{1}{4}\right)^{\frac{1}{2}} - \tfrac{1}{2}, \tag{19}$$

whose eigenvalues are, respectively,

$$\ell, j. \tag{20}$$

If we now consider the projection operator

$$\mathcal{P} = (\hat{L} - \hat{J} - 1)(\hat{J} - \hat{L} - 1), \tag{21a}$$

we see that its value is 1 if $\ell = j$ and 0 if $\ell = j \pm 1$. Thus applying it to the operator $\mathcal{O}$ of (8) we see that

$$\mathcal{P}\mathcal{O}\mathcal{P} \tag{21b}$$

allows only eigenvalues E corresponding to states of parity $(-1)^j$.

From now on we shall refer to the symmetry Lie algebra of the two particle system with Dirac oscillator interactions, as the one whose generators commute with $\mathcal{P}\mathcal{O}\mathcal{P}$ of (21b).

2.2. The u(3) *symmetry Lie algebra of the states with parity* $(-1)^j$ *and spin* $s = 0$ *or* 1

The operator $\mathcal{O}$ of (8) with spin $s = 0$ reduces to

$$\mathcal{O}\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix} = \begin{bmatrix}-E^2 & 2E\\ 2E & 4\omega\boldsymbol{\eta}\cdot\boldsymbol{\xi} - E^2\end{bmatrix}\begin{bmatrix}\phi_+\\ \phi_-\end{bmatrix}, \tag{22}$$

and it is immediately clear that the operators

$$C_q^r = \eta_q\xi^r, \quad q, \qquad r = 1, 0, -1, \tag{23}$$

commute with $\mathcal{O}$ and that, furthermore, satisfy the commutation relations

$$[C_q^r, C_s^t] = C_q^t\delta_s^r - C_s^r\delta_q^t, \tag{24}$$

so that they are the generators of u(3) Lie algebra and in the projected notation used in (21) these generators become

$$\big[1 - (\mathbf{S}^2/2)\big]\mathcal{P}\,\eta_q\,\xi^r\,\mathcal{P}\big[1 - (\mathbf{S}^2/2)\big], \tag{25}$$

where $[1 - (\mathbf{S}^2/2)]$ guarantees that for spin 1 the bracket vanishes while for spin 0 it becomes 1.

The creation η_q and annihilation ξ^r operators are the spherical components of those defined in (6) and their matrix elements with respect to the states $|N(j,0)jm\rangle$ are given by [9]

$$\begin{aligned}\big\langle N+1(\ell,0)\ell m+q\big|\eta_q\big|N(j,0)jm\big\rangle &= \big\langle N+1(\ell,0)\ell\big\|\eta\big\|N(j,0)j\big\rangle\big\langle jm,1q\big|\ell m+q\big\rangle\\ &= \left\{\left[\frac{(N+j+3)(j+1)}{(2j+3)}\right]^{\frac{1}{2}}\delta_{\ell,j+1}\right.\\ &\quad\left.+\left[\frac{(N-j+2)j}{(2j-1)}\right]^{\frac{1}{2}}\delta_{\ell,j-1}\right\}\big\langle jm,1q\big|\ell m+q\big\rangle,\end{aligned} \tag{26}$$

while that of ξ^q can be obtained from (26) by Hermitian conjugation. In (26) the double vertical lines indicated reduced matrix elements while $\langle\ |\ \rangle$ is a Clebsch-Gordan coefficient.

If we now turn our attention to the states $|N(j,1)jm\rangle$ whose parity is $(-1)^j$ but the spin is 1, then $\mathcal{O}$ has the full form (8) multiplied by the projection operators as in (21b). To get the generators of the u(3) symmetry Lie algebra we need to find a creation operator η'_q whose reduced matrix elements with respect to the states $|N(j,1)jm\rangle$ in the ket, and $\langle N+1(j\pm 1,1)j\pm 1m'|$ in the bra, are the same as those of in (26).

We shall achieve this objective by first using standard Racah algebra and results in the curly bracket in (26), to obtain for the *ordinary* creation operator η_q the reduced matrix element [10]

$$\langle N+1(\ell,1)\ell \| \eta \| N(j,1)j \rangle = \left\{ \left[\frac{(N+j+3)(j+1)}{(2j+3)}\right]^{\frac{1}{2}} \frac{[j(j+2)]^{\frac{1}{2}}}{(j+1)} \delta_{\ell,j+1} + \left[\frac{(N-j+2)j}{(2j-1)}\right]^{\frac{1}{2}} \frac{[(j+1)(j-1)]^{\frac{1}{2}}}{j} \delta_{\ell,j-1} \right\}. \tag{27}$$

We now note that the operator $\hat{J}$ defined in (19) is diagonal with respect to all the quantum numbers in the ket $|N(\ell,s)jm\rangle$ and with eigenvalue j. Thus the operator $\hat{J}\eta_q - \eta_q\hat{J}$ has reduced matrix elements of the form

$$\langle N+1(\ell,1)\ell \| \hat{J}\eta - \eta\hat{J} \| N(j,1)j \rangle = \left\{ \left[\frac{(N+j+3)(j+1)}{(2j+3)}\right]^{\frac{1}{2}} \frac{[j(j+2)]^{\frac{1}{2}}}{(j+1)} \delta_{\ell,j+1} - \left[\frac{(N-j+2)j}{(2j-1)}\right]^{\frac{1}{2}} \frac{[(j+1)(j-1)]^{\frac{1}{2}}}{j} \delta_{\ell,j-1} \right\}. \tag{28}$$

Thus we see that

$$\eta_q \pm (\hat{J}\eta_q - \eta_q\hat{J}) \tag{29}$$

will have only matrix elements with $\ell = j+1$ if we use the + sign or $\ell = j-1$, if we use the − sign, so long as we are restricted to states of parity $(-1)^j$. It is clear then that the new creation operator η'_q will have its reduced matrix elements

$$\langle N+1(\ell,1)\ell \| \eta' \| N(j,1)j \rangle, \tag{30}$$

in exactly the same form as (26) for η_q if we write

$$\eta'_q = \mathcal{P}\left\{\tfrac{1}{2}\hat{J}(\hat{J}-1)^{-\frac{1}{2}}(\hat{J}+1)^{-\frac{1}{2}}[\eta_q + \hat{J}\eta_q - \eta_q\hat{J}] + \tfrac{1}{2}[\eta_q + \eta_q\hat{J} - \hat{J}\eta_q](\hat{J}+1)^{-\frac{1}{2}}(\hat{J}-1)^{-\frac{1}{2}}\hat{J}\right\}\mathcal{P}. \tag{31}$$

The corresponding annihilation operator is $\xi^{q'} = (\eta'_q)^\dagger$ and thus it can be written immediately if we recall that $\hat{J}$ is Hermitian and $\eta_q^\dagger = \xi^q$.

The generators of the u(3) symmetry Lie algebra are then given by

$$(\mathbf{S}^2/2)\,\eta'_q\,\xi^{r'}\,(\mathbf{S}^2/2), \tag{32}$$

as $\mathcal{P}$ in (31) limit us to states with parity $(-1)^j$, while $(\mathbf{S}^2/2)$ is 1 for spin $s = 1$ and 0 for $s = 0$.

It is important to note that in the states

$$|N(j,1)jm\rangle \tag{33}$$

the one with $j = 0$ does not exist. Thus the dimensionality of the number states corresponding to the given N is not the one that is normally associated with u(3), which is $(1/2)(N+1)(N+2)$. This situation was discussed extensively in Ref. [9], where it was shown that in quantum mechanics u(3) does not explain completely the accidental degeneracy for $s = 1$ though u(3) is a symmetry Lie algebra of the corresponding classical problem. This can also be seen in Eq. (31) where clearly η'_q does not exist if the eigenvalue j of $\hat{J}$ is $j = 0$.

To find the full symmetry algebra for parity $(-1)^j$ we now turn to considering the states with spin 0 and 1 together, and will find a o(4) symmetry Lie algebra.

2.3. The o(4) *symmetry Lie algebra for states of parity* $(-1)^j$ *and both spins*

We start by giving in Fig. 1 the levels for parity $(-1)^j$ and both spins $s = 0$ and $s = 1$ indicating on the right of the levels the total angular momentum and parity as j^π and above the levels the values (N, ℓ). On the ordinate we give the number $\nu = N + s$ with N being the total number of quanta, which, from Eqs. (16), (17), is related to the square of the total energy by

$$\nu \equiv (N + s) = (4\omega)^{-1}(E^2 - 4). \tag{34}$$

We note that when ν is even we have states with $j^\pi = 0^+, 1^-, 2^+, 3^-, \ldots, \nu^+$ while when ν is odd $j^\pi = 1^-, 2^+, 3^-, \ldots, \nu^-$. This immediately suggests [11] that the symmetry Lie algebra is o(4) with the representation $[\nu, 0]$ for ν even and $[\nu, 1]$ for ν odd.

The set of states we have to deal with, in the notation (12), now have the form

$$|\nu - s(j,s)jm\rangle \equiv |\nu jm)_s, \tag{35}$$

where from (34), $\nu - s$ is the number of quanta, $s = 0, 1$ and $j = \nu - s,\ \nu - s - 2, \ldots, 0$ or 1 depending on whether $\nu - s$ is even or odd.

Our objective now will be to find the generators of an o(4) Lie algebra that connect all states (35) of fixed ν. For this purpose we remember that we have at our disposal only combinations of operators

$$\eta_q, \quad \xi^q = (-1)^q \xi_{-q}, \quad S_q = \tfrac{1}{2}(\sigma_1 + \sigma_2)_q, \quad S'_q = \tfrac{1}{2}(\sigma_1 - \sigma_2)_q, \qquad q = 1, 0, -1. \tag{36}$$

We would like first to find operators that relate $|\nu jm)_s$ with $|\nu j \pm 1m')_{s'}$, in terms of those in (36). For this purpose we note that we have the reduced matrix elements of η_q with respect to the states (35) in (26) and (27) for $s = 0$ or 1, where we need to replace N by ν and $\nu - 1$ respectively.

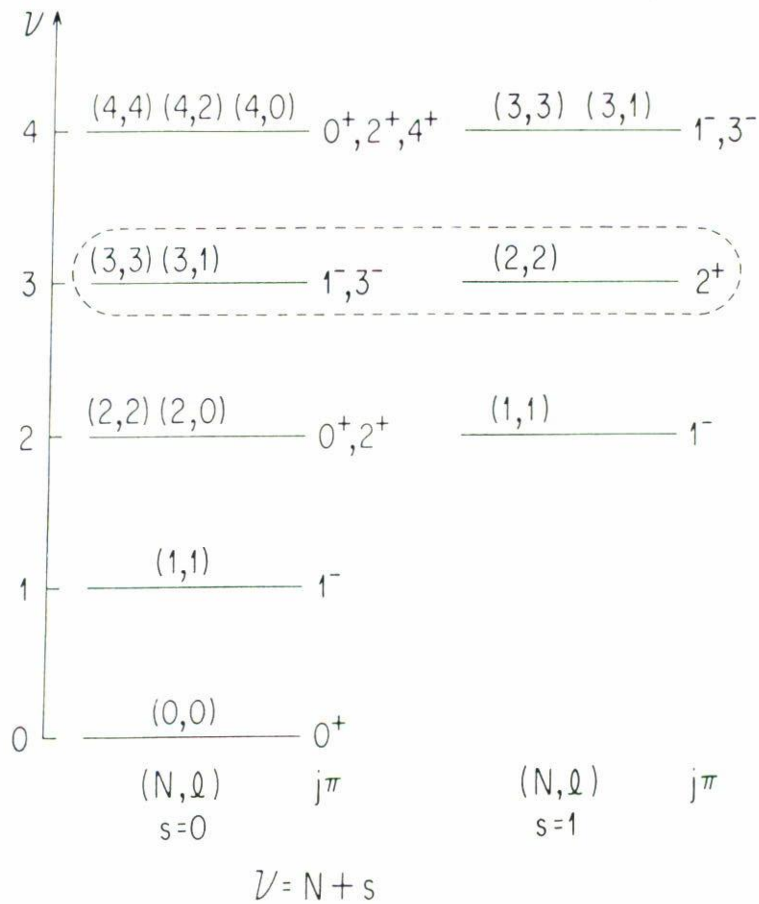

FIGURE 1. Levels of the spectra of the relativistic problem for the cases when $s = 0$ and $s = 1$, where in both $j = \ell$. The dashed line surrounds an example of the kind of degeneracy we want to explain.

We shall start by noting that the difference of the spin of the two particles, given by the vector of components $S'_q = \frac{1}{2}(\sigma_1 - \sigma_2)_q$ can only connect a state of total spin $s = 0$ with $s = 1$ or viceversa. This is due to the fact that the matrix element

$$\left\langle \tfrac{1}{2}\tfrac{1}{2}sm \middle| \tfrac{1}{2}(\sigma_1 - \sigma_2)_q \middle| \tfrac{1}{2}\tfrac{1}{2}s'm' \right\rangle \tag{37}$$

(where the 1/2 corresponds to the spins of each individual particle of the system) changes by a phase factor

$$(-1)^{1+s+s'} \tag{38}$$

when we interchange 1 and 2 of the two particles. Thus (37) vanishes if $s = s'$.

We now introduce new vector creation $\tilde{\boldsymbol{\eta}}$ and annihilation $\tilde{\boldsymbol{\xi}}$ operators given by previous ones $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ multiplied vectorially with $\mathbf{S}'$, *i.e.*,

$$\tilde{\eta}_q = \left[\boldsymbol{\eta} \times \mathbf{S}'\right]^1_q, \qquad \tilde{\xi}_q = \left[\boldsymbol{\xi} \times \mathbf{S}'\right]^1_q \tag{39}$$

and which have the hermiticity property

$$\left\{ \left[\boldsymbol{\eta} \times \mathbf{S}'\right]^1_q \right\}^\dagger = -(-1)^q \left[\boldsymbol{\xi} \times \mathbf{S}'\right]^1_{-q}. \tag{40}$$

Using standard Racah algebra [10], as well as the reduced matrix elements of $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ given in (26), we obtain for $\tilde{\eta}_q$, $\tilde{\xi}_q$ the following expressions, in which bra and ket are restricted to states of parity $(-1)^j$:

$$\left\langle N+1(j',1)j' \right\| \left[\boldsymbol{\eta} \times \mathbf{S}'\right]^1 \left\| N(j,0)j \right\rangle = \left[\frac{(j+2)(N+j+3)}{2(2j+3)}\right]^{\frac{1}{2}} \delta_{j',j+1}$$

$$- \left[\frac{(j-1)(N-j+2)}{2(2j-1)}\right]^{\frac{1}{2}} \delta_{j',j-1}. \tag{41}$$

The reduced matrix element of $[\boldsymbol{\xi} \times \mathbf{S}']^1_q$ can be obtained from (41) with the help of the hermiticity condition (40).

We note that in the reduced matrix elements of $\tilde{\eta}_q$, $\tilde{\xi}_q$, where the ket has orbital and total angular momentum j, the corresponding values for the bra are either $j+1$ or $j-1$. We would like though to have operators in which these two possibilities can be separated. We can achieve our purpose by defining, as was done in Eq. (29) for the case of spin 1, the operators

$$\tilde{\eta}^\pm_q = \tfrac{1}{2}\left\{\tilde{\eta}_q \pm (\hat{J}\tilde{\eta}_q - \tilde{\eta}_q\hat{J})\right\}, \tag{42}$$

$$\tilde{\xi}^\pm_q = \tfrac{1}{2}\left\{\tilde{\xi}_q \pm (\hat{J}\tilde{\xi}_q - \tilde{\xi}_q\hat{J})\right\} \tag{43}$$

with the hermiticity relation

$$\left(\tilde{\eta}^\pm_q\right)^\dagger = -\tilde{\xi}^{q\mp} = -(-1)^q \tilde{\xi}^\mp_{-q}. \tag{44}$$

We now note from Fig. 1 that, for example, if we have ν even the states (N,ℓ,s) corresponding to it, where N is the total number of quanta, ℓ the orbital angular momentum and s the spin, go as follows:

$$(\nu,0,0),\quad (\nu-1,1,1),\quad (\nu,2,0),\quad (\nu-1,3,1),\quad \ldots,\quad (\nu,\nu,0), \tag{45a}$$

$$0^+,\qquad 1^-,\qquad 2^+,\qquad 3^-,\qquad \ldots,\qquad \nu^+, \tag{45b}$$

where below (45a) we put the j^π of (45b). We see then that alternatively we decrease or increase the number of quanta by 1, and at the same time increase or decrease the spin between 0 and 1. The $\ell = j$ increases monotonically from 0 to ν.

From Ref. [9] we see that the type of operators we require are of the form

$$F'_q = \mathcal{P}\hat{S}\tilde{\xi}^+_q\mathcal{P}, \tag{46a}$$

$$F''_q = \mathcal{P}\tilde{\eta}^+_q\hat{S}\mathcal{P}, \tag{46b}$$

if we want to move from left to right in the sequence of states (45a). In (46a,b) the operator $\hat{S}$ is defined by

$$\hat{S} = \left(\mathbf{S}^2 + \tfrac{1}{4}\right)^{\frac{1}{2}} - \tfrac{1}{2}, \tag{47}$$

with eigenvalues 0 or 1, and is introduced in (46a,b) to guarantee that the bra for F'_q has spin 1 and thus, from (37), the ket has spin 0, while F''_q has bra of spin 0 and ket of spin 1.

If we want now to move from right to left in the sequence of states (45a) we need the operators

$$G'_q = \mathcal{P}\tilde{\eta}_q^- \hat{S}\mathcal{P}, \qquad G''_q = \mathcal{P}\hat{S}\tilde{\xi}_q^- \mathcal{P}, \tag{48a, b}$$

where the $\hat{S}$ appears for similar reasons to those of the previous paragraph.

The presence of the projection operators $\mathcal{P}$ of (21a) guarantees that F'_q, G'_q, F''_q, G''_q, act only on states of parity $(-1)^j$.

From Eq. (41) we see that the only non vanishing reduced matrix elements of the operators (46), (48) are the following

$$\left\langle N-1(j+1,1)j+1\middle\|F'\middle\|N(j,0)j\right\rangle \equiv {}_1(\nu j+1\|F'\|\nu j)_0 = \left[\frac{(j+2)(\nu-j)}{2(2j+3)}\right]^{\frac{1}{2}}, \tag{49a}$$

$$\left\langle N+1(j+1,0)j+1\middle\|F''\middle\|N(j,1)j\right\rangle = {}_0(\nu j+1\|F''\|\nu j)_1 = \left[\frac{j(\nu+j+2)}{2(2j+3)}\right]^{\frac{1}{2}}, \tag{49b}$$

$$\left\langle N+1(j-1,0)j-1\middle\|G'\middle\|N(j,1)j\right\rangle = {}_0(\nu j-1\|G'\|\nu j)_1 = -\left[\frac{(j+1)(\nu-j+1)}{2(2j-1)}\right]^{\frac{1}{2}}, \tag{49c}$$

$$\left\langle N-1(j-1,1)j-1\middle\|G''\middle\|N(j,0)j\right\rangle = {}_1(\nu j-1\|G''\|\nu j)_0 = -\left[\frac{(j-1)(\nu+j+1)}{2(2j-1)}\right]^{\frac{1}{2}}, \tag{49d}$$

where we used the relation (34).

With the help of Eqs. (46), (48), (49) we may write explicitly the generators of the o(4) symmetry Lie group, when we have the representations $[\nu,0]$ for ν even and $[\nu,1]$ for ν odd.

To proceed we first recall that for the o(4) $\supset$ o(3) chain, the generators are [11] the angular momentum $\mathbf{L}$, which we denote by L_i, $i=1,2,3$ and the Runge-Lenz vector A_i, $i=1,2,3$ which have the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, A_j] = i\epsilon_{ijk}A_k, \quad [A_i, A_j] = i\epsilon_{ijk}L_k. \tag{50a, b, c}$$

The basis for the representation is a ket of the form

$$|[p,q]\ell m\rangle, \tag{51}$$

where $[p,q], \ell, m$ are respectively the irreps of o(4), o(3), o(2); with p,q being integers giving respectively the maximum and minimum angular momenta in the representations of o(4).

The reduced matrix element of L_i with respect to the states (51) is diagonal in ℓ and has the value $[\ell(\ell+1)]^{\frac{1}{2}}$, while that of A_i is given by [11]

$$\begin{aligned}\langle [p,q]\ell'' \| A \| [p,q]\ell \rangle = &\left[\frac{(p+\ell+2)(p-\ell)(\ell+1-q)(\ell+1+q)}{(\ell+1)(2\ell+3)}\right]^{\frac{1}{2}} \delta_{\ell'',\ell+1} \\ &+ \frac{q(p+1)}{[\ell(\ell+1)]^{\frac{1}{2}}}\delta_{\ell'',\ell} \\ &- \left[\frac{(p+\ell+1)(p+1-\ell)(\ell+q)(\ell-q)}{\ell(2\ell-1)}\right]^{\frac{1}{2}} \delta_{\ell'',\ell-1}. \end{aligned} \tag{52}$$

We need now to write A_q in terms of the operators F'_q, F''_q, G'_q, G''_q, $\hat{N}$, $\hat{J}$, $\hat{S}$, in such a way that the matrix elements with respect to the states of the form (51) are the same as those in (52).

If ν is even $[p,q] = [\nu, 0]$, and thus from (49) and the fact that the matrix elements with respect to the states (51) of $\hat{N}$, $\hat{J}$, $\hat{S}$ are diagonal with eigenvalues N, j, s respectively, gives us the following generators for the o(4) Lie algebra:

$$J_q = L_q + S_q, \quad L_q = -i\big[\boldsymbol{\eta} \times \boldsymbol{\xi}\big]^1_q, \quad S_q = \tfrac{1}{2}(\sigma_1 + \sigma_2)_q, \tag{53}$$

$$\begin{aligned} A_q = &F'_q \left[\frac{2(\hat{N}+\hat{S}+\hat{J}+2)(\hat{J}+1)}{(\hat{J}+2)}\right]^{\frac{1}{2}} + F''_q \left[\frac{2(\hat{N}+\hat{S}-\hat{J})(\hat{J}+1)}{\hat{J}}\right]^{\frac{1}{2}} \\ &+ \left[\frac{2(\hat{N}+\hat{S}+\hat{J}+2)(\hat{J}+1)}{(\hat{J}+2)}\right]^{\frac{1}{2}} G'_q + \left[\frac{2(\hat{N}+\hat{S}-\hat{J})(\hat{J}+1)}{\hat{J}}\right]^{\frac{1}{2}} G''_q, \end{aligned} \tag{54}$$

with $q = 1, 0, -1$ and F'_q, F''_q, given by (46) and G'_q, G''_q, by (48).

If ν is odd the representation $[p,q] = [\nu, 1]$ and so from Eqs. (49), (52) the generators of the subgroup o(3) of o(4) remain the J_q of (53), but now the operator A_q is given by

$$\begin{aligned} A_q = &F'_q \left[\frac{2(\hat{N}+\hat{S}+\hat{J}+2)\hat{J}}{(\hat{J}+1)}\right]^{\frac{1}{2}} + F''_q \left[\frac{2(\hat{N}+\hat{S}-\hat{J})(\hat{J}+2)}{(\hat{J}+1)}\right]^{\frac{1}{2}} \\ &+ (\hat{N}+\hat{S}+1)[\hat{J}(\hat{J}+1)]^{-1} J_q \\ &+ \left[\frac{2(\hat{N}+\hat{S}+\hat{J}+2)\hat{J}}{(\hat{J}+1)}\right]^{\frac{1}{2}} G'_q + \left[\frac{2(\hat{N}+\hat{S}-\hat{J})(\hat{J}+2)}{(\hat{J}+1)}\right]^{\frac{1}{2}} G''_q. \end{aligned} \tag{55}$$

We have thus achieved our objective of writing explicitly the generators of the o(4) symmetry Lie algebra, in the form J_q, A_q, $q = 1, 0, -1$ of Eqs. (53), (54) for all states with ν even and of Eqs. (53), (55) for all odd ν.

As a last point we note that o(4), in the notation (50), has the two Casimir operators [11]

$$C_1 = \mathbf{L}^2 + \mathbf{A}^2, \tag{56a}$$

$$C_2 = \mathbf{L} \cdot \mathbf{A}, \tag{56b}$$

and if the representation is $[p, q]$, then the eigenvalues of these operators, which we denote by $\langle C_1 \rangle$, $\langle C_2 \rangle$ have the form [11]

$$\langle C_1 \rangle = (p^2 + q^2 + 2p), \tag{57a}$$

$$\langle C_2 \rangle = 2(p+1)q. \tag{57b}$$

For the representation $[p, q] = [\nu, 0]$ corresponding to even ν, the value of these Casimir operators are

$$\langle C_1 \rangle = \nu(\nu + 2), \qquad \langle C_2 \rangle = 0, \tag{58a, b}$$

while for ν odd the representation is $[\nu, 1]$ and

$$\langle C_1 \rangle = (\nu + 1)^2, \qquad 2\langle C_2 \rangle = (\nu + 1). \tag{59a, b}$$

In the next section we turn our attention to the nonrelativistic approximation of our problem.

3. SYMMETRY LIE ALGEBRA FOR THE NON RELATIVISTIC LIMIT OF OUR PROBLEM

In the previous section we discussed the relativistic problem for the system of two particles with a Dirac oscillator interaction, which led us to the operator of Eq. (8) acting on a two component wave function (ϕ_+, ϕ_-). If we write Eq. (8) explicitly as a system of two equations in ϕ_+, ϕ_-, and eliminate ϕ_- between them, we get for ϕ_+ (which from now on we designate simple as ϕ) the equation

$$\left\{E^4 - 4E^2 - 4\omega E^2[\hat{N} - (\mathbf{L}\cdot\mathbf{S})] + 16\omega^2[\hat{N} - (\mathbf{L}\cdot\mathbf{S}) - (\boldsymbol{\eta}\cdot\mathbf{S})(\boldsymbol{\xi}\cdot\mathbf{S})](\boldsymbol{\eta}\cdot\mathbf{S})(\boldsymbol{\xi}\cdot\mathbf{S})\right\}\phi = 0, \tag{60}$$

where $\hat{N} = \boldsymbol{\eta}\cdot\boldsymbol{\xi}$ and we continue using the relativistic units $\hbar = m = c = 1$.

Equation (60) still corresponds to the full relativistic problem but now in terms of a single wave function ϕ. To find the nonrelativistic approximation we note that, in normal units, the energy associated with the oscillator interaction is $\hbar\omega$ while the rest energy in mc^2, and thus we obtain the non relativistic case if $\hbar\omega \ll mc^2$. In units where $\hbar = m = c = 1$, the last inequality becomes $\omega \ll 1$ and, in that case, we can disregard the term with ω^2 in (60). We have then states ϕ with $E = 0$, which were dealt with in another article [12] plus those satisfying the equation

$$\epsilon\phi \equiv \tfrac{1}{4}[E^2 - 4]\phi = \omega(\hat{N} - \mathbf{L}\cdot\mathbf{S})\phi, \tag{61}$$

where ϵ is the nonrelativistic energy, as E is close to the rest mass 2 for the two particle system so that $\frac{1}{4}(E-2)(E+2) \simeq E - 2 = \epsilon$.

We now proceed to discuss the accidental degeneracy of the eigenstates of the operator in the right hand side of (61). Again the parity is a good quantum number as the operators N, $\mathbf{L}$, $\mathbf{S}$ are invariant under reflection. Thus we can consider separately the states of parity $(-1)^j$ and $-(-1)^j$. The former have $\ell = j$ and so the eigenvalues ν of $\hat{N} - \mathbf{L}\cdot\mathbf{S}$ are

$$\nu \equiv N + 1 \text{ if } s = 1, \qquad \nu = N \text{ if } s = 0, \tag{62}$$

and thus we have exactly the same situation as for the states of parity $(-1)^j$ in the relativistic problem. The symmetry Lie algebra is then o(4) and has the generators given in Eqs. (53), (54) if ν is even and Eqs. (53), (55) if ν is odd.

For parity $-(-1)^j, \ell = j \pm 1$ and thus the situation is quite different from the relativistic case where, from Eq. (18), we saw that no accidental degeneracy was present. It is convenient now to write the states (15) in terms of the orbital angular momentum ℓ, in the form

$$|N(\ell, 1)\ell \pm 1m\rangle, \tag{63}$$

where

$$\ell = N, N-2, N-4, \ldots 1 \text{ or } 0, \tag{64}$$

so we can denote, as usual,

$$N = 2n + \ell; \quad \ell, n = 0, 1, 2, \ldots \tag{65}$$

The nonrelativistic energy in Eq. (61), divided by ω, *i.e.* (ϵ/ω), is then

$$(\epsilon/\omega) = N + \ell + 1 = 2(n + \ell) + 1 = 2(n + j + 1) + 1 \equiv 2\nu + 1, \quad \text{if } j = \ell - 1, \tag{66}$$

$$(\epsilon/\omega) = N - \ell = 2n, \quad \text{if } j = \ell + 1. \tag{67}$$

The states (63) with the $-$ sign *i.e.* $j = \ell - 1$, have from (66) the value $\nu = n + j + 1$, so that the degeneracy is ν as we can have $j = 0, 1, 2, \ldots, \nu - 1$, as seen in Fig. 2. The symmetry Lie algebra seems then to be o(4) with irrep $[\nu - 1, 0]$, and we shall derive its generators in the following subsection.

On the other hand the states (63) with the $+$ sign *i.e.* $j = \ell + 1$, are an infinite set as $(\epsilon/\omega) = 2n$ is independent of ℓ, so that j can take the values $j = 1, 2, 3, \ldots$, as seen in

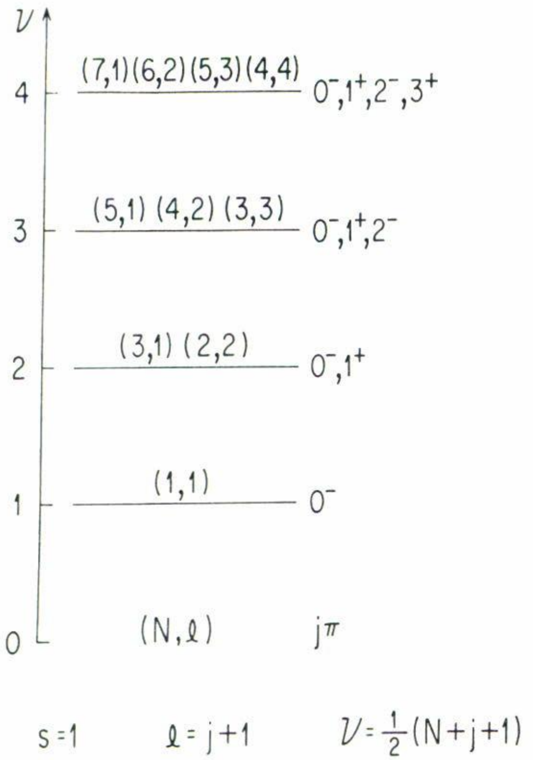

FIGURE 2. Levels of the operator $\hat{N} - \mathbf{L} \cdot \mathbf{S}$, with eigenvalue $2\nu + 1$ where $\nu = \frac{1}{2}(N + j + 1)$ for the case $s = 1$, $\ell = j + 1$.

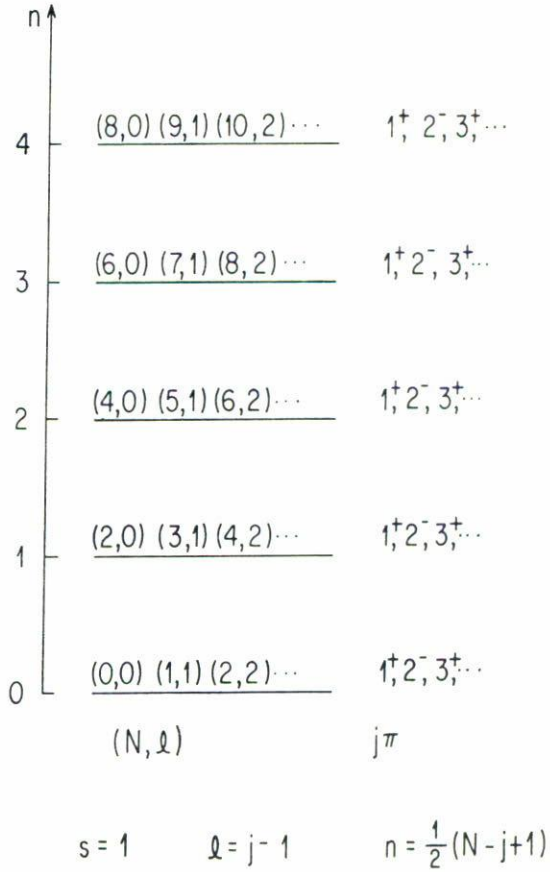

FIGURE 3. Levels of the operator $\hat{N} - \mathbf{L} \cdot \mathbf{S}$, with eigenvalue $2n$ where $n = \frac{1}{2}(N - j + 1)$ for the case $s = 1$, $\ell = j - 1$.

Fig. 3. The symmetry Lie algebra is then a non compact one, and we shall show in the last subsection that it is o(3, 1).

3.1. The o(4) *symmetry Lie algebra when* $\ell = j + 1$

In the states of Eq. (63) with a $-$ sign, we replace ℓ by $j + 1$ and N by $2\nu - j - 1$ to write

$$|2\nu - j - 1(j + 1, 1)jm\rangle \equiv |[\nu - 1, 0]jm]\!]. \tag{68}$$

The ket on the right hand side of (68) corresponds to the $|[p, q]\ell m\rangle$ appearing in (52) if we take

$$p = \nu - 1, \quad q = 0, \quad \ell = j, \tag{69}$$

and the reduced matrix elements

$$[\![\, [\nu - 1, 0]j'' \| A \| [\nu - 1, 0]j \,]\!] \tag{70}$$

of the Runge-Lenz vector **A** of Eq. (50) are given by (52) with the substitution (69).

To write $A_q, q = 1, 0, -1$ explicitly in terms of the creation η_q and annihilation ξ_q operators and $\hat{N}$, $\hat{J}$, $\hat{L}$, $\hat{S}$, we need combinations of η_q in the form of Eq. (29), as well as of its hermitian conjugate, to guarantee that there are contributions only when $j'' = j+1$ or $j-1$. Furthermore our operators should apply only to states of the form (68) where $\ell = j+1$ and ν is fixed, so we need the projection operator

$$\mathcal{P}^+ \equiv \tfrac{1}{2}(\hat{L} + 1 - \hat{J})(\hat{L} - \hat{J}) \tag{71}$$

which vanishes for $\ell = j$ or $\ell = j-1$ but becomes 1 for $\ell = j+1$. We also need to use the projection operator $\hat{S}$ to guarantee that our kets have only the spin $s = 1$.

We now introduce the operators

$$F_q \equiv \mathcal{P}^+ \hat{S} \tfrac{1}{2}[\eta_q - (\hat{J}\eta_q - \eta_q \hat{J})]\hat{S}\mathcal{P}^+, \tag{72a}$$

$$G_q \equiv \mathcal{P}^+ \hat{S} \tfrac{1}{2}[\xi_q + (\hat{J}\xi_q - \xi_q \hat{J})]\hat{S}\mathcal{P}^+, \tag{72b}$$

that have the hermiticity property

$$F_q^\dagger = G^q = (-1)^q G_{-q}. \tag{73}$$

The only reduced matrix elements of F_q, G_q that are different from 0 are the following:

$$\begin{aligned}\left\langle N+1(j,1)j-1 \middle\| F \middle\| N(j+1,1)j \right\rangle &\equiv [\![[\nu-1,0]j-1 \| F \| [\nu-1,0]j]\!] \\ &= \left[\frac{2(j+1)(\nu-j)}{(2j+1)}\right]^{\frac{1}{2}},\end{aligned} \tag{74}$$

$$\begin{aligned}\left\langle N-1(j+2,1)j+1 \middle\| G \middle\| N(j+1,1)j \right\rangle &\equiv [\![[\nu-1,0]j+1 \| G \| [\nu-1,0]j]\!] \\ &= -\frac{\left[(2j+1)(j+2)(2\nu-2j-2)\right]^{\frac{1}{2}}}{(2j+3)}\end{aligned} \tag{75}$$

as can be seen from (26).

Turning now our attention to the relation (52) with the notation (69), we see that the operator A_q can be written as

$$\begin{aligned}A_q = &-G_q \left[\frac{(\hat{N} + \hat{L} + 2\hat{J} + 2)(\hat{J}+1)(2\hat{J}+3)}{4(\hat{J}+2)(2\hat{J}+1)}\right]^{\frac{1}{2}} \\ &- \left[\frac{(\hat{N} + \hat{L} + 2\hat{J} + 2)(\hat{J}+1)(2\hat{J}+3)}{4(\hat{J}+2)(2\hat{J}+1)}\right]^{\frac{1}{2}} F_q\end{aligned} \tag{76}$$

and these operators for $q = 1, 0, -1$, together with the total angular momentum J_q, $q = 1, 0, -1$ of (53) constitute the six generators of o(4) symmetry Lie algebra satisfying, in cartesian components, the commutation rules of Eq. (50).

3.2. The o(3,1) *symmetry Lie algebra when* $\ell = j - 1$

In the states of Eq. (63) with a + sign we replace ℓ by $j-1$ and N by $2n+\ell = 2n+j-1$ to write

$$|2n+j-1(j-1,1)jm\rangle \equiv |[-1+in,1]jm\}. \tag{77}$$

We now proceed to explain the notation that we use for the ket in the curly bracket, which requires a discussion of the generators, reduced matrix elements and Casimir operators of the o(3,1) symmetry Lie algebra.

According to the discussion of Biedenharn [11], and the one in Ref. [9] for the one body Dirac oscillator we can pass from the generator A_q of o(4) to the a_q of o(3,1) by the substitution $A_q \to -ia_q$ and the reduced matrix elements of a_q with respect to the states (77), can be obtained by analytic continuation from those of A_q in Eq. (52).

We need though to characterize the irrep of o(3,1) to which the ket (77) belongs. For this purpose we note that for the irrep $[p,q]$ of o(4) the eigenvalue of the Casimir operator C_1 of (56a) is (p^2+q^2+2p). The corresponding Casimir operator of o(3,1) will be designated as c_1 and, from the analytic continuation indicated in the previous paragraph, it has the form

$$c_1 = \mathbf{J}^2 - \mathbf{a}^2, \tag{78}$$

where we replaced $\mathbf{L}$ in (56) by the total angular momentum $\mathbf{J}$ of the present problem.

For the eigenvalue $\langle c_1\rangle$ of these Casimir operators we have to replace p by a complex number [11]

$$k = a + ib, \qquad a, b \text{ real}, \tag{79}$$

as we want to deal with representation for the continuous series, while q remains equal to 1 as $j = 1$ is the lowest value of the angular momentum. We have then that, with the replacements indicated, we obtain

$$\langle c_1\rangle = k^2 + 1 + 2k = a^2 - b^2 + 2iab + 2a + 2ib + 1. \tag{80}$$

As c_1 is an hermitian operator $\langle c_1\rangle$ should be real and this requires that $a = -1$, so our irreps are characterized by $[k,q]$ with $k = -1+ib$, $q = 1$.

The second Casimir operator c_2 of (57b) translates into

$$c_2 = -i\mathbf{J}\cdot\mathbf{a} \tag{81}$$

and, as the eigenvalue $2(k+1)q$ becomes $2ib$, we see that with the factor $-i$ in (81) we get the eigenvalue

$$\langle c_2 \rangle = 2b. \tag{82}$$

On the other hand from Eq. (67) we see that $(\epsilon/\omega) = 2n$ so this immediately suggests that $b = n$, and as $q = 1$, the irrep of o(3,1) is characterized by $[-1+in, 1]$ as it appears in Eq. (77).

The reduced matrix elements of a_q, with respect to the states $|[-1+in,1]jm\}$ can then be obtained from Eq. (52) by analytic continuation when we replace p by $-1+in$ and q by 1, so that we get

$$\begin{aligned}\big\{[-1+in,1]j''\big\|a\big\|[-1+in,1]j\big\} = {}& \left\{\frac{[n^2+(j+1)^2]j(j+2)}{(j+1)(2j+3)}\right\}^{\frac{1}{2}} \delta_{j'',j+1} \\ & - \frac{n}{[j(j+1)]^{\frac{1}{2}}}\delta_{j'',j} \\ & - \left\{\frac{(n^2+j^2)(j^2-1)}{j(2j-1)}\right\}^{\frac{1}{2}} \delta_{j'',j-1}.\end{aligned} \tag{83}$$

To write a_q, $q = 1, 0, -1$ explicitly in terms of the creation η_q and annihilation ξ_q operators as well as of $\hat{N}$, $\hat{J}$, $\hat{L}$, $\hat{S}$, we need combinations of the η_q in the form of Eq. (29) as well as of their hermitian conjugates, to guarantee that they contribute only when $j'' = j+1$ or $j-1$. Furthermore the operator J_q, will connect states of the same j in bras and kets of the form (77).

As we need operators that apply only to states of the form (77), where $\ell = j-1$ and n is fixed, we require the projection operator

$$\mathcal{P}^- = \tfrac{1}{2}(\hat{J}+1-\hat{L})(\hat{J}-\hat{L}), \tag{84}$$

which vanishes for $\ell = j$ or $\ell = j+1$ but becomes 1 for $\ell = j-1$. We need also to use the projection operator $\hat{S}$ to guarantee that our kets have the spin $s = 1$.

We now introduce the operators

$$f_q = \mathcal{P}^-\hat{S}\tfrac{1}{2}\big\{\eta_q + (\hat{J}\eta_q - \eta_q\hat{J})\big\}\hat{S}\mathcal{P}^-, \tag{85}$$

$$g_q = \mathcal{P}^-\hat{S}\tfrac{1}{2}\big\{\xi_q - (\hat{J}\xi_q - \xi_q\hat{J})\big\}\hat{S}\mathcal{P}^-, \tag{86}$$

that have the hermiticity property

$$f_q^\dagger = g^q = (-1)^q g_{-q}. \tag{87}$$

The only reduced matrix elements of f_q, g_q that are different from 0 are the following

$$\langle N+1(j,1)j+1\|f\|N(j-1,1)j\rangle = \{[-1+in,1]j+1\|f\|[-1+in,1]j\}$$
$$= \left[\frac{j(2n+2j+1)}{(2j+1)}\right]^{\frac{1}{2}}, \tag{88}$$
$$\langle N-1(j-2,1)j-1\|g\|N(j-1,1)j\rangle = \{[-1+in,1]j-1\|g\|[-1+in,1]j\}$$
$$= -\frac{\left[(2j+1)(j-1)(2n+2j-1)\right]^{\frac{1}{2}}}{(2j-1)}, \tag{89}$$

as can be seen from (26).

Turning now our attention to the reduced matrix elements (83), (88), (89), we see that the operator a_q can be written as

$$a_q = \tfrac{1}{2}g_q\left\{\frac{(2\hat{J}-1)(\hat{J}+1)[(\hat{N}-\hat{L})^2+16\hat{J}^2]}{2(2\hat{J}+1)\hat{J}[\hat{N}-\hat{L}+4\hat{J}-2]}\right\}^{\frac{1}{2}} - \tfrac{1}{4}(\hat{N}-\hat{L})[\hat{J}(\hat{J}+1)]^{-1}J_q$$
$$+\tfrac{1}{2}\left\{\frac{(2\hat{J}-1)(\hat{J}+1)[(\hat{N}-\hat{L})^2+16\hat{J}^2]}{2(2\hat{J}+1)\hat{J}[\hat{N}-\hat{L}+4\hat{J}-2]}\right\}^{\frac{1}{2}} f_q, \tag{90}$$

and for $q = 1, 0, -1$ the operators a_q with the total angular momentum J_q, $q = 1, 0, -1$ of (53), constitute the six generators of o(3, 1).

Thus we have completed the determination of all the generators of the symmetry Lie algebras associated with the operator $\hat{N} - \mathbf{L}\cdot\mathbf{S}$.

We shall conclude by discussing some general aspects of the problems analyzed in this paper.

4. CONCLUSION

As we mentioned after Eq. (18) we only dealt, in the relativistic problem, with the positive energy levels. The ones of energy $E = 0$ have been discussed in another publication [12], but there remain those in which the energy is negative. It was shown by Quesne [13], when analyzing the single particle Dirac oscillator, that using the Dirac assumption that all negative energy levels are filled [1], those with $E < 0$, can be considered to have positive energy but for the antiparticle with frequency $-\omega$ instead of ω, as happens also for electrons, where the negative energies states are made positive for the antiparticle when the charge is changed to $-e$. With this type of approach Quesne was able to analyze together the states of positive and negative energy, using the formalism of supersymmetry. We plan to extend, in a future publication, this type of analysis to the *two* particle system with a Dirac oscillator interaction.

Finally we would like to point out that in a recent article Moshinsky, Quesne and Loyola [14], analyzed the problem of all possible symmetries for the two dimensional harmonic oscillator for Hamiltonians that are linear functions of the number operator $\hat{N}$ and orbital angular momentum $\hat{M}$. The results of the present paper indicate that a similar consideration could be carried out for the three dimensional oscillator with spin, where the latter could take the values $s = 0, 1/2, 1$ or $s = 0$ and 1 together. We intend to look also into this problem in a future publication.

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