Mechanics of self-affine cracks

ALEXANDER S. BALANKIN
Instituto Tecnológico y de Estudios Superiores de Monterrey
Campus Estado de México, México

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ABSTRACT. In this work mechanics of self-affine cracks is analyzed. The asymptotic behavior of stresses in the vicinity of the crack tip for cracks with self-affine surface (or/and profile) are derived for two and three dimensional problems. These asymptotics differ from stress field asymptotics for smooth crack as well as for monofractal cracks. Moreover, our results also differ from results which were derived for self-affine cracks in the works of Mosolov [15] and E. Bouchard and J.P. Bouchard [16]. The relations between micro and macro fracture toughness are also derived for ductile and brittle materials. The theoretical results are discussed with respect to recent experimental observations.

RESUMEN. En este trabajo se analiza la mecánica de las grietas auto-afines. Se deriva la asintótica de los tensiones en la vecindad de la punta de grieta con superficie y/o perfil auto-afín para problemas bi y tridimensionales. Estas asintóticas difieren de las asintóticas de campo de tensiones para grietas tan lisas como monofractales. Más aún, nuestros resultados difieren también de aquéllos obtenidos para grietas auto-afines en los trabajos de A. Mosolov [115] y de E. Bouchard y J.P. Bouchard [16]. También se derivan las relaciones entre la tenacidad a la micro y a la macro fractura para materiales frágiles, así como para dúctiles. Los resultados teóricos se discuten respecto a las observaciones experimentales recientes.

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The fracture process in real materials is characterized by the high extent of spatiotemporal non-uniformity which results in the complex morphology of fracture surfaces and crack shape [1]. Starting from the pioneer work of Mandelbrot *et al.* [2] there have been numerous works focusing on statistical characterization of the roughness of cracks (see, for example, Refs. [3–5] and references therein). Now it is well established experimentally that crack faces in solids are self-affine objects.[1] Many of these experiments aimed at establishing a correlation between fractal dimension of fracture surface and macroscopic fracture parameters, such as fracture toughness K_{IC}, the impact energy measured during Charpy tests, etc. Mecholsky and Mackin [6], and Mecholsky *et al.* [7] showed that the fracture toughness increases as the fractal dimension of fracture surface, D_{F}, increases. This result was explained by consideration of fractal crack growth in the number of

[1]Many different materials have been investigated with different fracture behavior, from ductile to brittle, at very different scales, from nanometric scale using atomic force or scanning tunneling microscopy, micrometer to centimeter scale using profilometry measurements on a variety of materials, image analysis technique, or other techniques, meter to kilometer scale for geological faults, and up to 1000 kilometer scale for geophysical phenomena [5].

works [8–11]. It was also shown [9–12][2] that the acceptance of the fractal structure of the crack leads to a change in the asymptotic behavior of the stress field σ_{ij} in the vicinity of the crack tip. Instead of the standard relationship $\sigma_{ij} \sim K_{\mathrm{I}}/\sqrt{r}$, the following equation is valid for a statistically self-similar (*mono*)fractal crack in linearly elastic materials:

$$\sigma_{ij} \sim K_{\mathrm{f}} X^{-\alpha} \phi_{ij}(\theta, \nu), \qquad X = \frac{r}{\ell_0}, \qquad \alpha = \frac{d - D_{\mathrm{F}}}{2}, \tag{1}$$

where ℓ_0 is a microscopic cutoff (for example, plastic zone size) below which the stress field saturates, r is the distance from the crack tip, K_{f} is the stress intensity factor for fractal crack (instead of common stress intensity factor K_{I}), $\phi_{ij}(\theta, \nu)$ is a dimensionless function of angle θ between directions of observation and crack propagation and Poisson's ratio ν; d is the dimension of the problem, and D_{F} is the metric (fractal) dimension of crack ($d - 1 \leq D_{\mathrm{F}} \leq d$). Moreover, in our works [10,13] it has been emphasized that change in the spatial distribution of the stress fields in the vicinity of fractal crack tip leads to a possibility of the fractal crack growth in the direction of uniaxial compression of the brittle solid (columnar fracture[3]).

However, it must be pointed out that the above-mentioned results were derived with the assumption that the crack face is isotropic monofractal which is characterized by unique fractal dimension.[4] In fact, as noted above, crack faces in real materials are characterized by self-affine geometry, and obey the property of statistical self-similarity only in the limiting range of length scales $L : L_0 \ll L \ll L_{\mathrm{m}}$.

The problem of self-affine crack propagation in brittle solids was recently analyzed by Mosolov [15], and E. and J.P. Bouchard [16]. Although in both works the same Griffith criterion of fracture was used,[5] the authors of these works have derived diametrically opposite results. According to the last work [16], the stress field singularity in the vicinity of crack tip involves an exponent

$$\alpha = \frac{D_{\mathrm{F}} - (d - 1)}{2} \tag{2}$$

instead of the exponent given by the same relation valid for monofractal cracks [*i.e.*, Eq. (1)], which was derived in Ref. [15] for the plane problem ($d = 2$) of self-affine crack propagation in a direction perpendicular to the applied longitudinal tensile stress.

Although we disagree with both calculations, it should be emphasized that the relation (2) is associated with an unphysical effect:[6] a more singular stress field (larger α) is related to a more rough crack (larger D_{F}).

[2] In Ref. [12] was realized invalid result $\alpha = 2 - D_{\mathrm{F}}$ for two dimensional problem ($1 \leq D_{\mathrm{F}} \leq 2$).

[3] Later, in works [14,15], the same result was derived for cracks with self-affine profile.

[4] In some cases this assumption is valid, but only for a limited range of spatial scales.

[5] It is pertinent to note that actually even in the case of brittle fracture the elastic energy release associated with removal of the load on rough surface [17] must be also taken into account.

[6] According to Eq. (7) in Ref. [16] the area of a rough crack face, $S_{\mathrm{rc}} \propto L^{\beta}$, increases with increasing average crack length L more slowly than the area of a smooth crack face, $S_{\mathrm{sc}} \propto L^{(d-1)}S$, *i.e.*, $\beta < d - 1$; this error is a consequence of the application of Eq. (5) in [17] which is not valid for non-differentiable self-affine surfaces.

The difficulties associated with consideration of self-affine crack propagation occur because of the compexity of self-affine structures. Instead of self-similarity of fractal structures, self-affine objects are statistically invariant under an affine transformation. For a surface with a mean plane parallel to plane (x, y) the affine transformation is written in terms of the horizontal distances x and y, and the vertical distance z: $x' \to \lambda_x x$, $y' \to \lambda_y y$, $z' \to \lambda_z z$. Requiring that such transformations be combined, a group structure is implied. As a consequence λ_y and λ_z have to be homogeneous functions of, say, λ_x; both scale as $\lambda_y = \lambda_x^{\nu_y}$, $\lambda_z = \lambda_x^{\nu_z}$. but the exponents ν_y and ν_z are in general different. If so, then $Z = Y^H$ where the Hurst exponent H is given by the relationship $H = \nu_y/\nu_z$. The fractal dimension of self-affine structures is not uniquely defined. Moreover, even in the simplest case a self-affine structure is characterized not only by the roughness exponent H, but also by a characteristic correlation length ξ. It was shown by Mandelbrot [18] that for self-affine fractal records we must distinguish between the local ($L \ll \xi$) and global ($L \gg \xi$) fractal dimensions. The last one is always equal to the topological dimension of self affine structures, *i.e.*, $D_{\mathrm{F}} = d - 1$, while in the local limit the fractal dimension $D_{\mathrm{f}} \geq d - 1$. Moreover, the relation between D_{f} and the Hurst exponent H depends on the definition of fractal dimension. We summarize the relationships between the Hurst exponent and various fractal dimensions discussed in literature in Table I.[7] It should be noted that different dimensions are related to different phenomena and different experimental techniques of estimation of the fractal dimension.

Unfortunately, we cannot even formulate the problem of crack propagation with self-affine (non-differentiable) profiles (surfaces) within the framework of the classical theory of elasticity, because the boundary conditions on the crack face are formulated for the stress components normal and tangential to the surface. To determine these components we must know the unit vector normal to the surface, which is in turn expressed by means of the spatial derivatives of the surface form. In the case of a non-differentiable self-affine surface it is impossible to introduce the vector normal to the surface because the spatial derivatives are undefined. Hence, before dealing with the problem of self-affine crack propagation within a framework of the elasticity theory we must first of all formulate new boundary conditions on a self-affine surface.[8] However, we can derive the asymptotic for the stress field distribution in the vicinity of the crack tip using balance energy analysis.

Energy conservation requires that the total kinetic and potential energy release associated with crack propagation should be spent partially on the formation of the new surface and partially dissipated:

$$\Delta W = \Delta U_E + \Delta T = -\Delta\Gamma + \Delta U_{\mathrm{D}}, \tag{3}$$

where ΔT and ΔU_E are, respectively, the kinetic and potential energies, $\Delta\Gamma = 2\Delta S\gamma$ is the incremental surface energy acquired, ΔS is the increment of the area of fracture surface, γ is the surface tension; and ΔU_{D} is the dissipation associated with irreversible processes (such as plastic deformations, etc.). The roughness of the crack faces leads to

[7] Notice that for self-similar fractals all these dimensions are equal to the metric (Hausdorff-Besicovitch) dimension.

[8] This topic will be the subject of forthcoming paper.

TABLE I. Relationships between the roughness (Hurst) exponent and various fractal dimensions of self-similar and self-affine fractals with topological dimensions equal to $d-1$ embedded in d–dimensional Euclidean space.

Dimension	Self-affine fractals: Local limit	Self-affine fractals: Global limit	Self-similar fractals
Similarity, D_S	—	—	$(d-1)/H$
Hausdorff-Besicovitch, D_H	$d-H$	$d-1$	$(d-1)/H$
Box-counting, D_B	$d-H$	$d-1$	$(d-1)/H$
Divider, D_D (comass, rule)	Latent fractal dimensions: $(d-1)/H$, if $H > (d-1)/d$; d, if $H \leq (d-1)/d$.	$d-1$	$(d-1)/H$
Contour ($d=2$), D_c (single coastline)	$2/(1+H)$[1] $1+(1-H)/(\nu+1)$[2]	$d-1$	$1/H$
Gap dimension,[3] D_G	$\log N/\log b^*$	—	$(d-1)/H$
Mass,[4] D_M	$\log_{b'}(Nb''/b')$	$\log_{b'}(Nb'/b'')$	$(d-1/H$

[1] Mean field approximation. [2] ν is the correlation length exponent from percolation theory. [3] N is the number of similar parts, r_i is the concern length of each part in i-direction, and $b^* = (r_1 r_2 \cdots r_d)^{1/d}$ is the effective base (for a self-similar fractal all r_i equal to r). [4] $b' = \max r_i$ and $b'' = \min r_i$ are the largest and smallest base of the self-affine transformation (for a self- similar transformation $b' \equiv b''$).

an increase in the area of the fracture surface and to re-radiating of path of the energy which flows toward the crack boundary. As a result, the effective surface energy of a plane cut model of real rough crack increases.

Let us first consider a problem of quasistaic ($\Delta T = 0$) crack growth with mean plane perpendicular to the longitudinal tensile stress applied to the elastic solid. Then, the elastic-energy release can be estimated by noting that the stress field is essentially relaxed on scales $r < \Delta L$ (ΔL is the crack length increment) and unperturbed on larger scales:

$$\Delta U_e \propto \int_{V(\Delta L)} U_e(x_i)\, d^d x_i, \tag{4}$$

where $U_e \simeq \langle \sigma^2(x_i) \rangle / 2E$ is the mean density of the elastic energy in the unloading volume $V(\Delta L)$ near the crack tip, E is the elastic modulus of the material; $i = 1, 2$, $d^d x_i = dx\, dy$ for a two dimensional problem, and $i = 1, 2, 3$, $d^3 x_i = dx\, dy\, dz$ for a three-dimensional problem. We assume that in the vicinity of the crack tip the stress field is characterized by a power law asymptotic behavior

$$\sigma(r) \simeq \sigma_\infty X^{-\alpha} \phi(\Theta), \quad \text{where } X = r/\ell_0.$$

Then the equation for the released elastic energy may be written in the form

$$\Delta U_{\rm E} \propto \frac{\phi^2\sigma^2}{2E}\int_{\ell_0}^{\Delta L} X^{-2\alpha} r^{d-1}\,dr \simeq \frac{\phi^2\sigma^2\ell_0^d}{(d-2\alpha)E}\left(\frac{\Delta L}{\ell_0}\right)^{d-2\alpha} \tag{5}$$

When the crack path is regular, the increment of the surface energy is proportional to $\Delta\Gamma \propto 2\gamma(\Delta L)^{d-1}$. At the onset of fracture, $\Delta\Gamma$ and $\Delta U_{\rm E}$ should be of the same order of magnitude. This implies that $\alpha = 1/2$, $K_{\rm I} \sim \sigma\sqrt{r}\phi(\theta)$, and $K_{\rm IC} \simeq \sqrt{\gamma_{\rm e}E}$, which are the classical results of fracture mechanics based on theory of elasticity [19].

In the case of self-affine cracks, the increment of the crack surface ΔS corresponds to a crack length increment ΔL as $\Delta S \propto (\Delta L)^{\beta_s}$. If $\Delta L \ll \xi$, then after almost literal repetition of the operations of Ref. [17] we derive the expression for exponent β_s in the following forms:

$$\beta_s = \begin{cases} \dfrac{d-1}{H}, & \text{if } \frac{d-1}{d} \le H \le 1, \\ d, & \text{if } 0 \le H < \frac{d-1}{d} \end{cases}, \tag{6}$$

i.e., the increment of the crack surface area is governed by the local latent fractal dimension of the crack face. Notice that we may measure this dimension by the divider (compass, or rule) method.

It immediately follows from Eqs. (5) and (6) that in the vicinity of the crack tip, $\ell_0 \ll r \ll \xi$, the exponent of the stress field singularity is

$$\alpha = \frac{1-d(1-H)}{2H} = \frac{1-d[D_{\rm B}-(d-1)]}{2(d-D_{\rm B})}, \tag{7}$$

if $H \ge (d-1)/d$ ($D_{\rm B} \le (d^2-d+1)/d$), or $\alpha = 0$, and

$$\sigma(r) = \sigma(\ell_0) = \sigma(\xi) = \text{const.}, \tag{8}$$

if $H < (d-1)/d$. Thus if the box-counting dimension[9] of the crack face is greater than or equal to its critical value ($D_{\rm B} \ge 1.5$ for a two dimensional problem, and $D_{\rm B} \ge 7/3$ for a three dimensional problem), the stress field does not depend on the distance from the crack tip in the interval $\ell_0 \le r \le \xi$, while at larger distances from the crack tip we always have the classical asymptotic behavior

$$\sigma(r) \simeq \frac{K_{\rm I}}{\sqrt{r}}, \qquad r \gg \xi, \tag{9}$$

where $K_{\rm I}$ is the standard stress intensity factor.

[9]It is precisely the box-counting dimension of crack faces the more commonly estimated dimension in metallographic experiments [1]; in the case of monofractal cracks we have $D_{\rm B} = 1/H$ and Eqs. (7), (8) are identical to Eq. (1).

These results are in good agreement with recent experimental observations by Ivanova *et al.* [20].

The case of $H < (d-1)/d$ is associated with ductile fracture [1] for which, making use of the nanofracture approach [21] and Eqs. (5)-(9) we derive the following equation for macroscopic fracture toughness:

$$K_{\mathrm{IC}} = 2\sqrt{Gb\tau_0}\sqrt{\frac{\xi}{\ell_0}}, \tag{10}$$

where G is the shear modulus, b is the absolute value of the Burgers vector of the elementary dislocation, and τ_0 is the friction force (Shmid's constant).

In the case of brittle (quasi-brittle) fracture, the box-counting dimension of the fracture surface $D_{\mathrm{B}} \le 2.33$, and it follows from Eqs. (5)-(9) that the macroscopic fracture toughness of the brittle material equals to

$$K_{\mathrm{IC}} = \sqrt{\gamma_{\mathrm{e}}E}\sqrt{\frac{D_{\mathrm{B}}-1}{d-D_{\mathrm{B}}}}\left(\frac{\xi}{\ell_0}\right)^{\epsilon}, \qquad \epsilon = \frac{(d-1)(1-H)}{2H} = \frac{(d-1)[D_{\mathrm{B}}-(d-1)]}{2(d-D_{\mathrm{B}})}. \tag{11}$$

The critical value $H = (d-1)/d$ corresponds to ductile-brittle transition, which was experimentally investigated in Ref. [22]. Equations (10) and (11) agree well with experimental observations in Refs. [6,7,20,22–24].

Experimental data provide an evidence that in compression a solid can undergo a brittle fracture. In such case, the nature of the fracture is columnar, and separation of the solid occurs into vertical columns produced by the crack growth in the direction of the uniaxial compression. For this case[10] we derive the following expression for the stress field asymptotic behavior in the vicinity of the tip of a self-affine crack:

$$\sigma(X) \propto \frac{\sqrt{(d-1)(1-H)}}{H}X^{-\alpha}, \quad X = \frac{r}{\ell_0}, \quad \alpha = \frac{(d-1)(H^2+H-1)}{2H^2}, \tag{12}$$

which also differs from the result of Ref. [15]. From this relation is follows that a self-affine crack can grow in the direction of uniaxial compression only if $(\sqrt{5}-1)/2 = \Phi^* \simeq 0.618 < H < 1$.[11]

We have thus shown that the morphology of the fracture surface governs the stress distribution in the vicinity of the crack tip and relations between micro and macro fracture toughness of the solid material.

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[10] In the case of columnar fracture, elastic energy is released only because of unloading of the rough surface [17].

[11] It is interesting to note that the critical value H_c is equal to the golden mean Φ^*.

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