Investigación *Revista Mexicana de Física* **41**, *No. 4 (1995) 480–492*

Generalization of the Papacostas-Xanthopoulos solution

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Recibido el 16 de noviembre de 1994; aceptado el 16 de febrero de 1995

ABSTRACT. A generalization of the Papacostas-Xanthopoulos solution is obtained by applying to it a Harrison transformation. It describes the collision of two plane gravitational waves with non-collinear polarization supporting an electromagnetic field. For a particular range of the parameters, no curvature singularities occur in the generalized solution.

RESUMEN. Aplicando una transformación de Harrison se obtiene una generalización de la solución de Papacostas-Xanthopoulos. Dicha solución describe la colisión de dos ondas planas gravitacionales con polarización no colineal portando un campo electromagnético. En la generalización obtenida no se presentan singularidades en la curvatura en un intervalo particular de los parámetros.

PACS: 04.20.Jb

1. INTRODUCTION

In recent years the study on colliding plane gravitational waves has increased greatly. For a complete review on the topic see the book by Griffiths [1]. The interest in these systems is related to the nonlinear effects that manifest when two plane-fronted gravitational waves collide. After the collision, it can occur both, a Cauchy horizon or a curvature singularity. In vacuum and for constant polarization the avoidance or evolution of the singularity depends on the relation between the amplitudes of the two incoming plane waves [2]. For electrovacuum the result is not so conclusive, because in this case the interaction between the electromagnetic and gravitational fields makes it more involved. In this last case, it is hard to establish results of general character. Thus to obtain an insight in what happens in the collision of two plane gravitational waves supporting electromagnetic fields one resorts to the analysis of concrete colliding gravitational waves possessing interesting peculiarities.

The spacetimes describing the interaction region produced after the collision of plane gravitational waves contain two commuting spacelike Killing vectors and there exist several generating techniques to obtain solutions with these symmetries. All the techniques that have been developed for stationary axisymmetric spacetimes (one spacelike and one timelike Killing vector fields) can be applied, with slight modifications, to generate cylindrically symmetric spacetimes, in particular, colliding plane waves.

In this paper we give the explicit expressions of the metric and electromagnetic field for a generalization of the colliding gravitational plane wave solution studied by Papacostas and Xanthopoulos [3]. The generalization is obtained via a Harrison transformation [4], which incorporates to the seed solution two electromagnetic field parameters. The seed solution is a five parameter family of solutions of the Einstein-Maxwell equations and exhibits no curvature singularities in the extended space-times for a particular range of the free parameters.

In Sect. 2 the new metric and its electromagnetic field are explicitly given. Section 3 is devoted to the analysis of the behaviour of the Weyl curvature coefficients in the colliding region. Section 4 deals with the extension of the spacetime regions where the collision takes place to regions before the collision event, and here also the characterization of the polarization in such regions is accomplished. Some concluding remarks are given in the last Sect. 5.

2. THE GENERALIZATION OF THE PAPACOSTAS-XANTHOPOULOS SOLUTION

Papacostas and Xanthopoulos [3] started their research from the metric (2.1) below, which occurs to be a special case of a Carter [5] solution with two commuting spacelike Killing fields, derived also by Plebański [6]. Papacostas and Xanthoupolos showed that this metric corresponds to an Einstein-Maxwell spacetime, which describes the interaction of two plane fronted gravitational and electromagnetic waves.

We shall briefly give the steps followed in generating a solution starting from the Papacostas-Xanthopoulos solution. The seed metric is given in the form

$$g = \Delta \left\{ \frac{dp^2}{P} - \frac{dq^2}{Q} \right\} + \frac{P}{\Delta}\left(d\varphi - q^2\, d\sigma\right)^2 + \frac{Q}{\Delta}\left(d\varphi + p^2\, d\sigma\right)^2, \tag{2.1}$$

where

$$\begin{aligned} \Delta &= p^2 + q^2, \\ P &= \gamma - \tfrac{1}{2}(e^2 + b^2) + 2np - \epsilon p^2, \\ Q &= -\gamma - \tfrac{1}{2}(e^2 + b^2) + 2mq - \epsilon q^2, \end{aligned} \tag{2.2}$$

with the additional conditions that $P \geq 0$, $Q \geq 0$, in order to fix the signature. The five parameters that characterize this solution are γ, n, m, ϵ and $2\nu^2 = e^2 + b^2$. The electromagnetic field is determined by the vector potential

$$A_\mu = \frac{1}{p^2 + q^2}(eq - bp)\delta^\varphi_\mu + \frac{pq}{p^2 + q^2}(ep + bq)\delta^\sigma_\mu. \tag{2.3}$$

This metric structure may develop the formation of horizons, with an external spacetime exhibiting two-dimensional spacelike or timelike curvature singularities, or no singularities at all. This fact is established in Ref. [3]. To arrive, from (2.1), at the metric in Ref. [3] one accomplishes the following correspondences:

$$\begin{aligned} &p \to t, \quad q \to z, \quad \varphi \to y, \quad \sigma \to x, \quad P \to E^2, \quad Q \to H^2, \\ &2\epsilon \to a, \quad 2n \to b, \quad \gamma - \nu^2 \to c, \quad 2m \to f, \quad -\gamma - \nu^2 \to g, \end{aligned} \tag{2.4}$$

with an additional change in the signature: from $(-,+,+,+)$ to $(+,-,-,-)$.

Choosing the Killing vector as a linear combination of the Killing directions ∂_φ and ∂_σ according to

$$K^\mu = \alpha\delta^\mu_\varphi + \beta\delta^\mu_\sigma, \tag{2.5}$$

the metric (2.1) can be rewritten in the form

$$g = \Delta\left\{\frac{dp^2}{P} - \frac{dq^2}{Q}\right\} + \frac{PQ}{f}\left(d\varphi'\right)^2 + f(d\sigma' - W\,d\varphi')^2, \tag{2.6}$$

with

$$f = K^\mu K_\mu = \frac{1}{\Delta}\left\{P\left(\alpha - \beta q^2\right)^2 + Q\left(\alpha + \beta p^2\right)^2\right\} =: \frac{D}{\Delta} > 0, \tag{2.7}$$

$$W = -\frac{1}{D(\alpha^2+\beta^2)}\left\{P(\alpha - \beta q^2)(\beta + \alpha q^2) + Q(\alpha + \beta p^2)(\beta - \alpha p^2)\right\}, \tag{2.8}$$

$$d\varphi' = \beta\,d\varphi - \alpha\,d\sigma, \qquad d\sigma' = \frac{1}{\alpha^2+\beta^2}[\alpha\,d\varphi + \beta\,d\sigma]. \tag{2.9}$$

In these coordinates the components of the electromagnetic scalar potential are

$$\phi = A_{\sigma'} + i\mathcal{A}, \tag{2.10}$$

where

$$\begin{aligned} A_{\sigma'} &= \frac{1}{\Delta}\left\{eq(\alpha + \beta p^2) - bp(\alpha - \beta q^2)\right\}, \\ \mathcal{A} &= \frac{p}{\Delta}\left\{\alpha(ep + bq) + \beta pq(bp - eq)\right\}. \end{aligned} \tag{2.11}$$

Einstein-Maxwell's equations for a spacetime with two Killing vectors can be formulated equivalently in terms of the Ernst potentials in the form of the well known Ernst equations. The Ernst equations possess an intrinsic SU(2, 1) symmetry which allows transformations that generate new solutions from known ones [7]. Li and Ernst [8] and also García [9] have proved that when SU(2, 1) transformations are executed, such transformations will always yield bona fide colliding wave solutions when bona fide colliding seed metrics are employed. In the present work, we have applied the so called Harrison charging transformation to the solution corresponding to the line element (2.6). The effect of this transformation is to incorporate into the seed metric two additional parameters, E and B, interpreted in terms of electromagnetic fields. This Harrison transformation occurs to be not quite relevant in the case of generation of new useful stationary axisymmetric fields because under such a transformation asymptotic flatness is not always preserved.

The SU(2, 1) transformations operate on the level of the Ernst potentials ϕ and ε. These potentials can be evaluated from the relations

$$d\phi = -iK\rfloor\omega, \qquad d\varepsilon = iK(dK + *dK) - 2\bar{\phi}\,d\phi, \tag{2.12}$$

where $K = K_\mu\, dx^\mu$, ω is the electromagnetic two-form, $\rfloor$ denotes the step product and $*$ is the Hodge's star operation (see details in Ref. [10]). The Ernst potentials corresponding to the seed metric (2.6) are

$$\phi = i\left[\frac{e+ib}{p+iq}\right](\alpha - i\beta pq), \tag{2.13}$$

and

$$\begin{aligned}\varepsilon = &-f - 2\nu^2\Delta^{-1}(\alpha^2+\beta^2p^2q^2)\\ &- 2i\big\{mp\big[\Delta^{-1}(\alpha+\beta p^2)^2 - \beta(3\alpha+\beta p^2)\big]\\ &- nq\big[\Delta^{-1}(\alpha-\beta q^2)^2 + \beta(3\alpha-\beta q^2)\big]\\ &+ \alpha\beta\epsilon pq + \beta^2\gamma pq + \beta\nu^2\Delta^{-1}pq\big[2\alpha+\beta(p^2-q^2)\big]\big\},\end{aligned} \tag{2.14}$$

where $2\nu^2 = e^2 + b^2$.

To "magnetize" a given metric one applies the Harrison transformation to the Ernst potentials according to the rules

$$\begin{aligned}&\tilde{\varepsilon} = \varepsilon\psi^{-1}, && \tilde{\phi} = \psi^{-1}[\phi + (E+iB)\varepsilon],\\ &\psi = 1 - 2(E-iB)\phi - \delta\varepsilon, && \delta = E^2 + B^2,\end{aligned} \tag{2.15}$$

where $E = \text{const.}$, $B = \text{const.}$ are the added electric and magnetic field parameters. The tilde is used to denote the new quantities.

To carry out this procedure for the metric (2.6) it is convenient to write it in the form

$$g = f^{-1}\big[fg_2 + \mathcal{K}^2(d\varphi')^2\big] + f\big[d\sigma' - W\,d\varphi'\big]^2, \tag{2.16}$$

with the definitions

$$g_2 = \Delta(d\rho^2 - d\tau^2) = e^{2\gamma}(d\rho^2 - d\tau^2) = \Delta\left(\frac{dp^2}{P} - \frac{dq^2}{Q}\right), \qquad \mathcal{K}^2 = PQ. \tag{2.17}$$

We can recognize the metric (2.16) as the one for a spacetime that may possess cylindrical symmetry. The result of the Harrison transformation in (2.16) amounts to transform [11] the functions f and $\tilde{W}$ as follows:

$$f \to \tilde{f}|\psi|^{-2}, \qquad W \to \tilde{W}. \tag{2.18}$$

Thus, the generated metric amounts to

$$g = |\psi|^2 f^{-1}\left\{f\Delta\left[\frac{dp^2}{P} - \frac{dq^2}{Q}\right] + \mathcal{K}^2(d\varphi')^2\right\} + |\psi|^{-2}f(d\sigma' - \tilde{W}\,d\varphi')^2, \tag{2.19}$$

with the electromagnetic field given by the two-form

$$-\omega = d\tilde{\phi} \wedge \left[d\sigma' + \tilde{W}\, d\varphi'\right] + *\left\{d\tilde{\phi} \wedge \left[d\sigma' + \tilde{W}\, d\varphi'\right]\right\}, \tag{2.20}$$

where the new $\tilde{W}$ function oughts to fulfill the equation

$$d\tilde{W} = \psi\tilde{\psi}\, dW + if^{-1}\left\{(\psi\tilde{\psi}_{,p} - \tilde{\psi}\psi_{,p})P\, dq + (\psi\tilde{\psi}_{,q} - \tilde{\psi}\psi_{,q})Q\, dp\right\}. \tag{2.21}$$

The structural functions P, Q and Δ are given by formulas (2.2), the function f is defined in (2.7). The complex factor function ψ is given in terms of the Ernst potential ϕ and ε in (2.15).

Integrating $\tilde{W}$ in (2.21), we obtain

$$\tilde{W}(E, B|\alpha, \beta) = W(0, 0|\alpha, \beta) + \Omega(E, B|\alpha, \beta), \tag{2.22}$$

where $W(0,0|\alpha,\beta) = W$ as in (2.8). The function $\Omega(E, B|\alpha, \beta)$ amounts to

$$\begin{aligned} D\Omega(E, B|\alpha, \beta) = &-4E(+)(\alpha - \beta q^2)qP - 4E(-)(\alpha + \beta p^2)pQ \\ &- 6\delta\nu^2\left[(\alpha^2 - \beta^2 q^4)P - (\alpha^2 - \beta^2 p^4)Q\right] \\ &- 4\delta E(+)\mathcal{M} + 4\delta E(-)\mathcal{N} + \delta^2\left\{\mathcal{A}P + \mathcal{B}Q + \mathcal{H}PQ\right\}, \end{aligned} \tag{2.23}$$

with $E(+) = Ee + Bg$, $E(-) = Eg - Be$ and where the polynomials $\mathcal{A}$, $\mathcal{B}$, $\mathcal{H}$, $\mathcal{M}$ and $\mathcal{N}$ are

$$\begin{aligned} \mathcal{A} = &-\alpha^4(2m^2 + 2n^2 - 4m\epsilon q + \epsilon^2 q^2) - \alpha^3\beta q^2(8n^2 + 4\epsilon\gamma + \epsilon^2 q^2) \\ &- 2\alpha^2\beta^2 q^2\left[(\gamma - \nu^2)(2(\gamma + \nu^2) - 6mq + \epsilon q^2) - (m^2 + 3n^2)q^2\right] \\ &+ \alpha\beta^3 q^4\left[2(\gamma - \nu^2)\nu^2 + 4n^2q^2 - (\gamma - \nu^2)(6m - \epsilon q)q\right] + \beta^4(\gamma - \nu^2)^2 q^6, \\ \mathcal{B} = &\ \alpha^4(2m^2 + 2n^2 - 4m\epsilon p + \epsilon^2 p^2) - \alpha^3\beta p^2(8n^2 - 4\epsilon\gamma + \epsilon^2 p^2) \\ &+ 2\alpha^2\beta^2 p^2\left[(\gamma + \nu^2)(2(\gamma - \nu^2) + 6np - \epsilon p^2) - (n^2 + 3m^2)p^2\right] \\ &- \alpha\beta^3 p^4\left[2(\gamma + \nu^2)\nu^2 - 4m^2p^2 - (\gamma + \nu^2)(6n - \epsilon p)p\right] - \beta^4(\gamma + \nu^2)^2 p^6, \\ \mathcal{M} = &\ (\alpha - \beta q^2)\left[\alpha m(\alpha + \beta q^2) - \beta^2(\gamma - \nu^2)q^3 - \alpha^2\epsilon q\right]P - \alpha m(\alpha + \beta p^2)(\alpha - 3\beta p^2)Q \\ &+ \beta q\left[2\alpha(\alpha + \beta p^2) + \beta p^2(\alpha - \beta q^2)\right]PQ, \\ \mathcal{N} = &-(\alpha + \beta p^2)\left[\alpha n(\alpha - \beta p^2) + \beta^2(\gamma + \nu^2)p^3 - \alpha^2\epsilon p\right]Q + \alpha n(\alpha - \beta q^2)(\alpha + 3\beta q^2)P \\ &+ \beta p\left[2\alpha(\alpha - \beta q^2) - \beta q^2(\alpha + \beta p^2)\right]PQ, \end{aligned}$$

$$\begin{aligned}\mathcal{H} = &-8\alpha^3\beta(mq+np) + 12\alpha^2\beta^2(nq-mp)pq\\ &-3\alpha\beta^3pq\big[q(-2pQ-4\epsilon pq^2-2nq^2+3\epsilon p^3) - p(-2qP+4\epsilon qp^2+2mp^2-3\epsilon q^3)\big]\\ &-\beta^4p^2q^2\big[3(\gamma+\nu^2)p^2+3(\gamma-\nu^2)q^2-2mqp^2-2npq^2\big].\end{aligned} \tag{2.24}$$

3. THE CURVATURE QUANTITIES IN REGION I

To establish the behaviour of the spacetime on the focussing surface, one evaluates the Weyl complex coefficients $C^{(a)}, a = 1, \ldots, 5$ in null coordinates (u, v). To carry out the transformation from (2.19) to null coordinates, one performs an intermediate coordinate transformation

$$\epsilon p - n = C\eta, \qquad \epsilon q - m = A\mu, \tag{3.1}$$

with $A^2 = 4m^2 - 4\epsilon(\gamma + \nu^2)$ and $C^2 = 4n^2 + 4\epsilon(\gamma - \nu^2)$, followed by

$$\begin{aligned}\eta &= u\sqrt{1-v^2} + v\sqrt{1-u^2} = uV + vU\\ \mu &= u\sqrt{1-v^2} - v\sqrt{1-u^2} = uV - vU.\end{aligned} \tag{3.2}$$

Under these transformations the line element (2.19) becomes

$$g = |\psi|^2\left\{\frac{\Delta}{\epsilon}4\frac{du\,dv}{UV} + \mathcal{K}^2 f^{-1}(d\varphi')^2\right\} + |\psi|^{-2}f(d\sigma' - \tilde{W}\,d\varphi')^2. \tag{3.3}$$

To calculate the Weyl coefficients $C^{(a)}$, we use the null tetrad formalism [12] and choose the null tetrad as

$$\begin{aligned}\sqrt{2}e^1 &= \big(\mathcal{K}|\psi|f^{-\frac{1}{2}}\big)d\sigma' + i|\psi|^{-1}f^{\frac{1}{2}}(d\sigma' - \tilde{W}\,d\varphi') = \sqrt{2}(e^2)^*,\\ e^3 &= \sqrt{2}|\psi|e^{\gamma}U^{-1}\,du, \qquad e^4 = \sqrt{2}|\psi|e^{\gamma}V^{-1}\,dv,\end{aligned} \tag{3.4}$$

With respect to this basis, the Weyl coefficients are [11]

$$\begin{aligned}C^{(1)} = &\ C_s^{(1)}N^{-1} + \tfrac{1}{2}e^{-2\gamma}N^{-2}\big\{N_u\big[u + U^2(2\gamma_u - f_uf^{-1})\big]\\ &- U^2\big[N_{uu} - 2N_u^2N^{-1} - J^2(u)N^{-1} + 2if\tilde{W}_uJ(u)\mathcal{K}^{-1}\big]\big\}\\ &+ \tfrac{i}{2}e^{-2\gamma}N^{-2}\big\{iJ(u)\big[u + U^2(2\gamma_u - f_uf^{-1} + 3N_uN^{-1})\big]\\ &+ U^2\big[2f\tilde{W}_uN_u\mathcal{K}^{-1} - i\partial_uJ(u)\big]\big\},\end{aligned} \tag{3.5.a}$$

$$C^{(3)} = C_s^{(3)} N^{-1} + \tfrac{1}{12} UVe^{-2\gamma} N^{-2} \big\{ 2N_{uv} - 4N_u N_v N^{-1} - 2J(u)J(v)N^{-1} + 2f^{-1}(N_u f_v + N_v f_u) + 4\mathcal{K}^{-1}(vN_u + uN_v) + 2if[\tilde{W}_u J(v) - \tilde{W}_v J(u)] \big\} - \tfrac{1}{4} UVe^{-2\gamma} N^{-2} \big\{ J(v)(f_u f^{-1} - N_u N^{-1}) + J(u)(f_v f^{-1} - N_v N^{-1}) + 2\mathcal{K}^{-1}[uJ(v) + vJ(u)] - if\mathcal{K}^{-1}(\tilde{W}_v N_u - \tilde{W}_u N_v) \big\}, \tag{3.5.b}$$

where $N = |\psi|^2$ and $J(x) = \psi \partial_x \psi^* - \psi^* \partial_x \psi, \quad x = u, v$. The subscript s labels the seed solution and the subscripts u and v denote ∂_u and ∂_v, respectively.

The expression for $C^{(5)}$ is obtained from the one for $C^{(1)}$ making $u \to v$. Besides,

$$C^{(2)} = 0 = C^{(4)}. \tag{3.6}$$

On the other hand for an electromagnetic field determined by a vector potential $(A_\mu) = (0, 0, A_{\varphi'}, A_{\sigma'})$, the nonvanishing null tetrad components of the electromagnetic tensor $F_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, are

$$\begin{aligned} 2F_{13} &= \frac{ie^{-\gamma}}{\sqrt{f}\psi^2} U \big\{ \phi_u + (E + iB)\varepsilon_u + \delta^2(\varepsilon\phi_u - \phi\varepsilon_u) \big\} = 2F_{23}^*, \\ 2F_{14} &= \frac{ie^{-\gamma}}{\sqrt{f}\psi^2} V \big\{ \phi_v + (E + iB)\varepsilon_v + \delta^2(\varepsilon\phi_v - \phi\varepsilon_v) \big\} = 2F_{24}^*, \end{aligned} \tag{3.7}$$

where ϕ and ε are the complex Ernst potentials of the seed solution, given by expressions (2.13) and (2.14).

From the definition of the traceless Ricci tensor one gets the nonvanishing components,

$$fC_{11} = e^{-2\gamma} UV \phi_u \phi_v^* = fC_{22}^*, \quad fC_{33} = -e^{-2\gamma} U^2 \phi_u \phi_u^*, \quad fC_{44} = -e^{-2\gamma} V^2 \phi_v \phi_v^*, \tag{3.8}$$

In the interaction region the seed metric used (2.1) is of Petrov type D (see Refs. [3,6], with only $C_s^{(3)} \neq 0$. With respect to the null tetrad (E^1, E^2, E^3, E^4)

$$\begin{aligned} \left\{ \begin{matrix} E^1 \\ E^2 \end{matrix} \right. &= \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{\Delta}{P}}\, dp \pm i \sqrt{\frac{P}{\Delta}}\, (d\phi - q^2\, d\sigma) \right\}, \\ \left\{ \begin{matrix} E^3 \\ E^4 \end{matrix} \right. &= \frac{1}{\sqrt{2}} \left\{ \sqrt{\frac{Q}{\Delta}}\, (d\phi + p^2\, d\sigma) \pm \sqrt{\frac{\Delta}{Q}}\, dq \right\}, \end{aligned} \tag{3.9}$$

the expression for $C_s^{(3)}$ is

$$3C_s^{(3)} = -\frac{3}{\Delta^2} \{ pP_p - qQ_q \} - \frac{2Q}{\Delta^3} \big\{ 2(2q^2 - p^2) - \Delta^{-1} \big\} + \frac{2P}{\Delta^3} \big\{ 2(2p^2 - q^2) - \Delta^{-1} \big\} + \frac{3i}{\Delta^3} \big\{ (Q_q \Delta^{-1} - 4qQ) + (P_p \Delta^{-1} - 4pP) \big\}, \tag{3.10}$$

while the invariant of the electromagnetic field is

$$F^{\mu\nu}F_{\mu\nu} = \frac{e^2+b^2}{(p^2+q^2)^2}. \tag{3.11}$$

In addition to the singularities in the seed metric (2.1), analyzed in detail in Ref. [3], from expressions (3.5) we note that new singularities could arise when $N = |\psi|^2 = 0$ which amounts to $\psi\psi^* = 0$, more precisely to

$$\psi\psi^* = \mathrm{Re}^2\,\psi + \mathrm{Im}^2\,\psi = 0, \tag{3.12}$$

however, this quantity can only be zero if $\mathrm{Re}\,\psi = 0 = \mathrm{Im}\,\psi$, but in this case the transformation (2.15) is meaningless. The seed metric (2.1), when $P = 0$ or $Q = 0$, seems to be singular. Nevertheless, Papacostas and Xanthopoulos [3] have proved that the surfaces $P = 0$ and $Q = 0$ correspond to regular surfaces. There arises the question, Has the metric (2.19) or (3.5) singularities on the surfaces $P = 0$ or $Q = 0$ (or equivalently $\mathcal{K} = 0$)? The analysis shows that the answer is negative —there are no curvature singularities. For $C^{(1)}$ the suspicious terms are $\tilde{W}_u\mathcal{K}^{-1}$ and $\tilde{W}_v\mathcal{K}^{-1}$; these terms are well-behaved since $\tilde{W}_v$ and $\tilde{W}_u$ are proportional to $\mathcal{K}$ (see Eq. (12) in Ref. [9]). On the other hand, in $C^{(3)}$, we have the terms $\frac{UVe^{-2\gamma}}{3N^2}\mathcal{K}^{-1}(vN_u + uN_v)$ and $\frac{-UVe^{-2\gamma}}{2N^2}\mathcal{K}^{-1}[vJ(u) + uJ(v)]$, which may be gathered as

$$\frac{UVe^{-2\gamma}}{N^2}\mathcal{K}^{-1}\left\{\tfrac{1}{6}v[5\psi^*\psi_u - \psi\psi_u^*] + \tfrac{1}{6}u[5\psi_v\psi^* - \psi_v^*\psi]\right\}. \tag{3.13}$$

The expression in brackets is

$$\begin{aligned}
&(u\phi_v + v\phi_u)\tfrac{10}{6}\left\{-E + iB + 2\delta\phi^* + (E - iB)\delta\varepsilon^*\right\} + \\
&(u\phi_v^* + v\phi_u^*)\tfrac{2}{6}\left\{E + iB - 2\delta\phi - (E + iB)\delta\varepsilon\right\} + \\
&(u\varepsilon_v + v\varepsilon_u)\tfrac{5}{6}\left\{-\delta + 2(E + iB)\delta\phi^* + \delta^2\varepsilon^*\right\} + \\
&\qquad\qquad (u\varepsilon_v^* + v\varepsilon_u^*)\tfrac{1}{6}\left\{\delta - 2(E - iB)\delta\phi - \delta^2\varepsilon\right\},
\end{aligned} \tag{3.14}$$

where E, B and δ are the parameters introduced by (2.15), while ε and ϕ are given in (2.13) and (2.14). Each term possesses the generic factor $\Xi := uX_v + vX_u$. If this generic factor is proportional to $\mathcal{K}$, i.e., $\Xi = \mathcal{K}\Sigma$, where Σ is a nondiverging term, then a solution generated from a bona fide non singular solution by a Harrison transformation, does not possesses new singularities at least at the level of the Weyl curvature quantities. In our particular case, with $X = X(p(u,v), q(u,v))$, the evaluation of the generic term Ξ yields

$$\begin{aligned}
\Xi &= u(X_p p_v + X_q q_v) + v(X_p p_u + X_q q_u) \\
&= (up_v + vp_u)X_p + (uq_v + vq_u)X_q \\
&= (1 - u^2 - v^2)\left\{\frac{C}{\epsilon UV}(uV + vU)X_p + \frac{A}{\epsilon UV}(vU - uV)X_q\right\}.
\end{aligned} \tag{3.15}$$

From the last expression it is apparent that for each term in (3.14) the factor $(1-u^2-v^2)$ arises. Since

$$\mathcal{K}^2 = PQ = \frac{AC}{\epsilon}(1-u^2-v^2)^2, \tag{3.16}$$

the $\mathcal{K}$ in the denominator in $C^{(3)}$ cancels out, therefore, the whole expression does not diverge. So, it turns out that no singularity appears at $P=0$ neither at $Q=0$ for the derived new solution.

4. EXTENSION TO THE REGIONS BEFORE THE COLLISION

The line element (2.19) describes a colliding wave metric structure if one can match this spacetime with the corresponding ones of the incoming waves. Since the solution (2.19) was obtained via a SU(2, 1) symmetry transformation, by Refs. [8] and [9], it is guaranteed that this solution represents a colliding plane gravitational wave situation. The extension to the regions before the collision can be performed from the metric written in null coordinates (u, v) as in (3.3). The region I of interaction is constrained to $0 \le u < 1$, $0 \le v < 1$. One extends the metric (3.3) to region II, $u < 0$, $v > 0$; region III, $u > 0$, $v < 0$ and region IV, $u < 0$, $v < 0$, by subjecting u and v to the so-called Penrose extension [13]:

$$u \to uH(u), \qquad v \to vH(v), \tag{4.1}$$

where H is the Heaviside unit step function. Singularities or discontinuities may arise from the substitution (4.1) along the null boundaries $u=0$ and $v=0$ separating the different regions. To determine the behaviour of the $C^{(i)}$'s on the null boundaries we use the relations given by Chandrasekhar and Xanthopoulos [14]. They define $\tilde{f}(x) = f(xH(x))$ and by using a Taylor expansion, it can be obtained that

$$\lim_{x\to 0} \tilde{f}'(x) = f'(0)H(x), \quad \lim_{x\to 0} \tilde{f}''(x) = f''(0)H(x) + f'(0)\delta(x), \tag{4.2}$$

where $\delta(x)$ is the Dirac delta function. For short we shall use δ, H and c for "δ function singularity", "Heaviside function discontinuity" and "continuous function", respectively. If, for instance, a quantity L exhibits an H and δ behaviour we shall write simply $L(H+\delta)$. In accordance with this rules, when crossing from region I to region II on $u=0$ we have

$$C^{(1)}(H+\delta), \quad C^{(5)}(c), \quad C^{(3)}(H), \quad C_{33}(H), \quad C_{11}(H), \quad C_{44}(c). \tag{4.3}$$

On the boundary $v=0$ separating regions I, III we have

$$C^{(5)}(H+\delta), \quad C^{(3)}(H), \quad C^{(1)}(c), \quad C_{33}(c), \quad C_{11}(H), \quad C_{44}(H), \tag{4.4}$$

while when passing from II to IV on $v=0$ one has

$$C_{44}(H), \quad C^{(5)}(\delta+H), \tag{4.5}$$

and finally when crossing from III to IV on $u = 0$ one has,

$$C_{33}(H), \quad C^{(1)}(\delta + H), \tag{4.6}$$

To interpret the parameters appearing in the metric (2.19) defined for I∪II∪III∪IV we follow Ref. [7], §21.5, where for plane waves (regions II and III) with $C^{+}_{abcd} = 2\psi_4 V_{ab} V_{cd}$, $\psi_4 = |\psi_4| e^{i\theta}$; there it is proposed to call $|\psi_4|$ the amplitude and associate θ with the polarization in a similar manner to the interpretation given to $\Phi_2 = Fe^{i\varphi}$ of a null electromagnetic field $F^{+}_{ab} = 2\Phi_2 V_{ab}$. In these expressions $V_{\mu\nu} = 2k_{[\mu} m_{\nu]}$.

For the vector basis (3.4) one identifies in region III (see Ref. [15])

$$k_\mu = e^3_\mu, \quad m_\mu = e^1_\mu, \quad 2\psi_4 = -C^{(1)}, \quad \Phi_2 = F_{31}, \quad C_{33} = -4F_{13}F_{23}, \tag{4.7}$$

and in region II

$$k_\mu = e^4_\mu, \quad 2\psi_0 = -C^{(5)}, \quad \Phi_1 = F_{41}, \quad C_{44} = -4F_{14}F_{24}, \tag{4.8}$$

In order to analize the polarization of the incoming waves, which is directly related with $\tilde{W}$ in (2.19), we calculate the Weyl tensor $C^{(a)}$, $a = 1, \ldots, 5$ in region III, $v < 0$, $u > 0$. For region II the situation is symmetric. We pass from (2.19) to region III with the transformations

$$p \to \frac{C}{\epsilon} u + \frac{n}{\epsilon}, \qquad q \to \frac{A}{\epsilon} u + \frac{m}{\epsilon}. \tag{4.9}$$

In region III all functions depend just on u and we have besides (3.6), $C^{(5)} = 0 = C^{(3)}$. Then only $C^{(1)} \neq 0$, and it is given by expression (3.5.a), with

$$\begin{aligned} C^{(1)}_s = {} & u\left(\frac{D_u}{D} - \frac{\Delta_u}{\Delta} + \frac{2u}{1-u^2}\right) - 2 + \frac{U^2(DA_u - AD_u)^2}{(\alpha^2+\beta^2)^2 D^2 \Delta^2 PQ} \\ & + U^2\left\{\frac{\Delta_{uu}}{\Delta} - \frac{2\Delta_u^2}{\Delta} + \frac{D_u}{D}\left(\frac{\Delta_u}{\Delta} + \frac{2u}{1-u^2}\right) + \frac{D_u^2 - D_{uu}}{D^2}\right\} \\ & + i\frac{f}{\mathcal{K}}\left\{-W_u\left(u + 3U^2\frac{\Delta_u}{\Delta} - U^2\right) - \frac{U^2(A_{uu}D - AD_{uu})}{D^2(\alpha^2+\beta^2)}\right\}, \end{aligned} \tag{4.10}$$

where $A = P(\alpha - \beta q^2)(\beta + \alpha q^2) + Q(\alpha + \beta p^2)(\beta - \alpha p^2)$.

The imaginary part of $C^{(1)}$ in principle defines the gravitational polarization in region III.

To determine the spacetime polarization plane $V^{(1)}$ one proceeds as follows: The bivector $V_{\mu\nu}$, which determines the bivector basis in region III ($C^{+} = 2\psi_4 VV$) is related with the "spacetime propagation direction", $k_\mu = e^3_\mu$, according to $V_{\mu\nu} = 2e^3_{[\mu} e^1_{\nu]}$; both, k_μ and $V_{\mu\nu}$ occur to be covariantly constant ($k_{\mu;\nu} = 0 = V_{\mu\nu;\lambda}$). This bivector can be decomposed in two parts, $V = V^{(1)} + V^{(2)}$, where $V^{(1)}$ is just the spacetime polarization plane.

Introducing the orthonormal tetrad $e^{a'}$,

$$e^1 = (e^2)^* = \tfrac{1}{\sqrt{2}}(e^{1'} + ie^{2'}), \quad (e^3) = \tfrac{1}{\sqrt{2}}(e^{3'} + e^{4'}), \quad (e^4) = \tfrac{1}{\sqrt{2}}(e^{3'} - e^{4'}), \tag{4.11}$$

one obtains

$$V^{(1)}_{\mu\nu} = \sqrt{2}\, e^3_{[\mu} e^{1'}_{\nu]}, \quad V^{(2)}_{\mu\nu} = \sqrt{2}\, e^3_{[\mu} e^{2'}_{\nu]}. \tag{4.12}$$

Labeling (x^0, x^1, x^2, x^3) as $(u, v, \varphi', \sigma')$, the only nonzero components are

$$V^{(1)}_{0\nu} = \frac{2\mathcal{K} e^\gamma}{U\sqrt{f}}\, |\psi|^2 \delta^{\varphi'}_\nu, \quad V^{(2)}_{0\nu} = 2\sqrt{2}\, e^\gamma \frac{\sqrt{f}}{U}(\delta^{\sigma'}_\nu - \tilde{W}\delta^{\varphi'}_\nu). \tag{4.13}$$

On the other hand, with respect to an observer moving along the direction $e^{4'}$ one defines the electric intensity as $E_\mu = -F^\nu_\mu e^{4'}_\nu$ which explicitly amounts to

$$E_\mu = -\tfrac{1}{2}\left[(F_{31} + F^*_{31})e^{1'}_\mu + i(F_{31} - F^*_{31})e^{2'}_\mu\right], \tag{4.14}$$

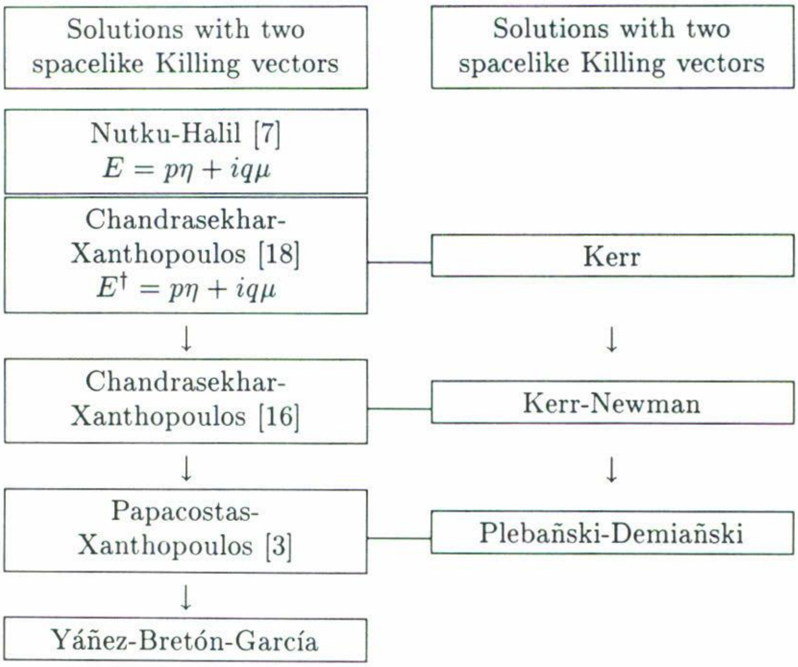
this vector E_μ determines the direction of polarization of the electromagnetic field. One can represent $F_{31} = Fe^{i\phi}$, and consider the direction of the vector $e^{1'}$ as the direction of polarization and ϕ as the polarization angle. The polarization direction $e^{1'}$ is also the polarization direction of the gravitational wave.

5. Concluding remarks

Applying a Harrison transformation we have obtained a generalization of the Papacostas-Xanthopoulos [3] solution which describes the collision of two plane gravitational waves supporting an electromagnetic field. No new singularities arise due to the electric and magnetic fields incorporated, as shows the analysis of the behaviour of the Weyl coefficients. The singularities that appear are the same ones as those of the seed metric. We conclude then that for a particular range of the free parameters, no curvature singularities occur in the generalized Papacostas-Xanthopoulos solution.

The solution derived here is a natural generalization of the Chandrasekhar-Xanthopoulos solution of Ref. [16] in the following sense: The algebraically general solution determined in this paper is an electromagnetic generalization of the Papacostas-Xanthopoulos [3] type D solution; the Papacostas-Xanthopoulos solution corresponds, for spacetimes with one spacelike and one timelike Killing vectors, to the Plebański-Demiański [6] solution. The Plebański-Demiański solution is in turn a generalization of the Kerr-Newman solution. The Chandrasekhar-Xanthopoulos solution [16] is a type D solution derived from the same potentials, fulfilling the Ernst equation, that are used to derive the Kerr-Newman field. In fact these two spacetimes are isometric in the region where the Kerr-Newman has two spacelike Killing vectors (interior to its ergosphere).

It is noteworthy to point out the following fact: Taking the Ernst potential given by $E = p\eta + iq\mu$, $p^2 + q^2 = 1$, when χ and ω are considered as the metric functions, one arrives at the Nutku-Halil [17] solution. While when χ and ω are taken as the potentials, with $E^{\dagger} = p\eta + iq\mu$, $p^2 + q^2 = 1$, it is obtained a type D solution [18] which is the *strict* analogue of the Kerr solution and which is locally isometric to the Kerr spacetime in the region interior to its ergosphere, where Kerr spacetime has two spacelike Killing vectors. It is in this sense that the solution derived in this paper can be considered as a generalization of all type D solutions for colliding plane waves obtained from a single Ernst potential $E^{\dagger}$ being common to the Kerr, Kerr-Newman and Plebański-Demiański solution. These statements can be put in a more transparent form with the following diagram:



In this diagram it is shown the relationship between the type D solutions and the solution derived in this paper. All of the solutions are of the Petrov type D except the Natku-Halil and Yáñez-Bretón-García ones. On the left hand side are shown the colliding plane wave solutions and on the right hand side are the stationary axisymmetric solutions. The lines between the blocks join the solutions corresponding to the same Ernst potentials. Each arrow starts at one solution and points to its corresponding generalization.

ACKNOWLEDGMENTS

We appreciate the referee's comments on various respects. This work was partially supported by CONACyT. One of the authors (G.Y.) acknowledges to CINVESTAV (México) for financial support.

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