Simulation of random field distribution with given structure function

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Recibido el 9 de diciembre de 1994; aceptado el 18 de abril de 1995

ABSTRACT. The distortions of the wavefront in the Earth's atmosphere are discussed. The approximation of the distortions of the wave localized in a thin layer is considered. A model to construct random realizations of the field with a given structure function is suggested. In the first approximation the role of the thickness of the atmosphere is described by a linear diffraction equation which gives an estimation of the dislocations of the wavefront of the initial plane wave. It is shown that for this case the density is less than 1 dislocation per 100 square meters, at least for good seeing conditions. This indicates the possibility of correcting wavefronts with a flexible mirror.

RESUMEN. Se discuten las distorsiones del frente de onda en la atmósfera de la Tierra. Se considera la aproximación de la distorsión de la onda localizada en un estrato delgado. Se sugiere el modelo para construir al azar la realización del campo con la función de estructura. La ecuación lineal de difracción describe la importancia del espesor de la atmósfera. Para las condiciones de buena calidad de la imagen del objeto, la estimación de la densidad de dislocaciones es menor a 1 en 100 metros cuadrados. Es posible la corrección del frente de onda con un espejo flexible.

PACS: 95.75.Qr; 94.20.Bb; 94.10.Lf

1. INTRODUCTION

Atmospheric turbulence is the main factor which limits the resolution of optical telescopes on the Earth [1–7]. The possibility of improving the resolution by partial correction of the wavefront using adaptive optics has been demonstrated in Refs. [3–11]. One fundamental limit of this method may be connected with the points in which the amplitude of the field is zero and the phase is indefinite. Such points are not so important, because the position of the mirror surface at such points has no influence on the quality of the corrected image. But, unfortunately, usually the phase in the vicinity of the zero-point is also indefinite. (It changes by 2π on the contour around the zero-point.) In 3–dimensional space such zero-points define lines. In analogy with crystallography such lines may be called "dislocations" of the wavefront. In the surface of the adaptive mirror of the telescope these lines could define points of the dislocations. An example of a field with such dislocations is presented in Fig. 1. The dependence of the phase has a 2π "jump". The position of the jump may change because only the relative phase is important, and the region for the relative phase may be defined in any way *e.g.* $-\pi : \pi$ or $0 : 2\pi$. This location tends to minimize the

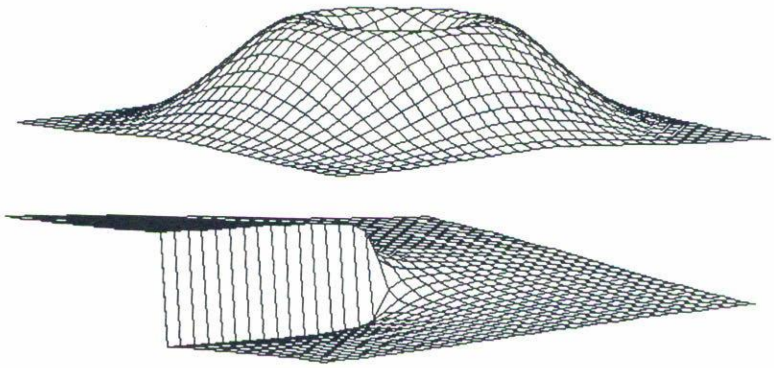

FIGURE 1. Distribution of the amplitude and phase of the smooth field: $E(x, y) = (x + iy)e^{-x^2-y^2}$.

physical significance or importance of the discontinuities, a significant consideration for deformable-mirror adaptive optics, for which there is an unavoidable correction error in the vicinity of the branch cut [15]. In any case, a wavefront with such a jump cannot be corrected completely by a mirror with a smooth profile, so, the investigation of the possibility of the realization of such structures on real wavefronts is important. It has been indicated [10] that dislocations limit the possibility of correcting wavefronts with flexible mirrors. For a field of Gaussian statistics, the density of dislocations was estimated in Refs. [9,10], but the surface density of dislocations due to the atmosphere was not estimated.

The aim of this work is to propose an algorithm to generate quasi-random realizations of the field with a given structure function, which describes the distortion of the field phase by the atmosphere and to estimate the surface density of dislocations, *i.e.*, the mean number of dislocations per unit of the Earth's surface.

2. THE MODEL

In the simplest model the initial distortions of the plane wave in the atmosphere are phase ones. They can be described [4,12,16,17] by the empirical formula for the structure function:

$$D(r) = \left\langle \Big(\Phi(x) - \Phi(x+r) \Big)^2 \right\rangle_x = 6.88 \left(\frac{|r|}{r_0} \right)^{5/3}, \tag{1}$$

where $r = (r_1, r_2)$, $x = (x_1, x_2)$, Φ is the phase of the field. Formula (1) describes the fluctuations of the phase at distance r, while r_0 is the "coherence scale" parameter. For visible light $r_0 \approx 10$ cm [12–19]. If all the turbulence is localized near the Earth's surface, there is no fluctuation of the field amplitude. But it has been shown [19] that this layer gives a relatively small contribution to the distortion of the light, at least for places with good *seeing*. For estimations we may assume that all phase distortions are localized in a thin layer at a distance z above the Earth's surface, and then the propagation of the wave is described by the free paraxial diffraction equation.

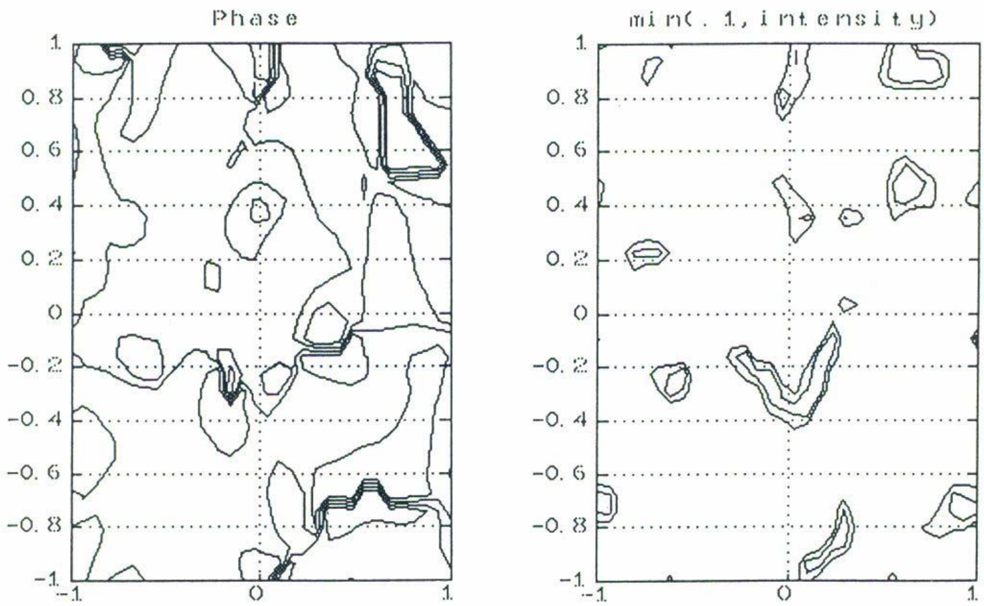

FIGURE 2. The distribution of the phase (a) and of the field and zeros of the amplitude (b) after propagation at the distance $z = 60$ km, ($h = 1.2$). The field was calculated by formula (7).

So, we should provide the algorithm to generate realizations of the phase which reproduces the structure function (1). Then we should construct a field with a uniform amplitude and a given phase. Without loss of generality the amplitude of the field may be set to unity. The primary distorted field can be defined by the formula $E(x, 0) = \exp(i\Phi(x))$. Then linear propagation in a homogeneous linear medium should be described by the diffraction equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2i\kappa_0}\left(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} + \frac{\mathrm{d}^2}{\mathrm{d}x_2^2}\right)\right) E(x, z) = 0, \tag{2}$$

where κ_0 is the wavenumber.

The function $f(x) = i\log(E(z, x))$ describes the phase of the field. If this function is regular, there are no dislocations. If there are cuts at which the function jumps by 2π it is not enough to say that the field has dislocations. May be, these "cuts" can be removed by the transformation

$$f(x) \to \tilde{f}(x) = f(x) + 2\pi n(x), \tag{3}$$

where $n(x)$ is a function of the transverse coordinates which takes only integer values. But if any "cuts" cannot be removed by the transformation (3), then the wavefront has dislocations. To avoid the use of a complicated algorithm to count them, we prefer to see the map $f(x)$ to count dislocations with our fingers. An example of a map of lines of equal phase is shown in Fig. 2a. The region where there are many equilines at a distance of one step of the grid indicates the "cut"; the ends of the "cut" show points of the dislocation. The calculations are done in Sect. 4.

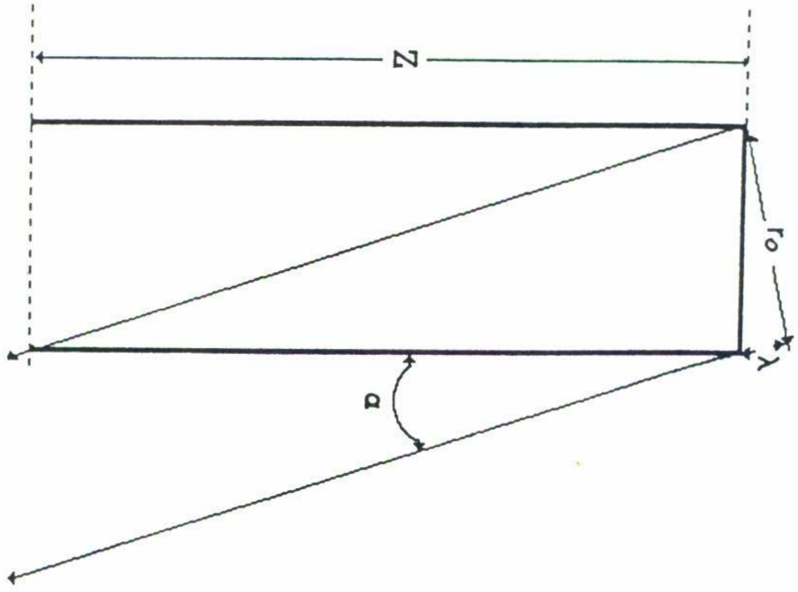

FIGURE 3. The estimation of the distance z of the propagation which causes a complete amplitude modulation from the initial phase modulation.

3. THE FRESNEL ESTIMATION

Before numerical calculations it is useful to compare orders of magnitudes and estimate, what should be expected from numerical simulations. We trying to estimate the distance of propagation at which a phase modulation of 2π per $r_0 \approx 10$ cm should cause complete amplitude modulation. At complete amplitude modulation there are some points at which the amplitude of the field is zero; so, the phase may have some peculiarities, *i.e.*, dislocations.

Consider a column of thickness r_0 and length z . If the phase of the field at one side of the column differs by 2π from the value at the other side of the column, it changes the direction of the beam by the angle $\alpha \approx \lambda/r_0$, Fig. 3. From Fig. 3 we see that a beam of the diameter of the column should leave this column at the distance of propagation $z \approx r_0/\alpha \approx r_0^2/\lambda$. For $\lambda \approx 10^{-4}$ cm this gives the estimation $z \approx 10$ km. Note that this is the scale of the thickness of the atmosphere of the Earth. So, this estimation does not prohibit dislocations of the wavefront by the atmosphere, and more careful estimations are necessary.

The statistical average of the number of zeros per unit area of wavefront is found for a speckle-nonuniform field with Gaussian statistics [2,10,11]. Other accurate considerations may be found in [20-23], but the surface density of dislocations due to the atmosphere is not estimated there.

4. THE CALCULATION ALGORITHM

Consider the discrete Fourier transform

$$\Phi_k = \frac{1}{L^2} \int e^{\frac{2\pi ikx}{L}} \Phi(x)\, \mathrm{d}^2 x, \tag{4}$$

where $k = k_1, k_2$, $k_{12} = 0, \pm 1, \pm 2, \ldots$ The function $\Phi(x)$ is assumed to be periodic; the period L should be much greater than the turbulence scale r_0. The phase of the field is real, so, $\Phi_{-k} = \Phi_k^*$. The calculus of the square of the modulus of the phase comes next

$$|\Phi_k|^2 = \frac{1}{L^4} \int e^{\frac{-2\pi ikx}{L}} \Phi(x)\, \mathrm{d}^2 x \int e^{\frac{2\pi iky}{L}} \Phi(y)\, \mathrm{d}^2 y, \tag{5}$$

or

$$|\Phi_k|^2 = \frac{1}{L^4} \iint e^{\frac{2\pi ik(y-x)}{L}} \Phi(x)\Phi(y)\, \mathrm{d}^2 x\, \mathrm{d}^2 y. \tag{6}$$

If $y = u + x$, then the last expression can be written as

$$|\Phi_k|^2 = \frac{1}{L^4} \iint e^{\frac{2\pi iku}{L}} \Phi(x)\Phi(x+u)\, \mathrm{d}^2 x\, \mathrm{d}^2 y, \tag{7}$$

$$= \frac{1}{L^2} \int e^{\frac{2\pi iku}{L}} \left(\frac{1}{L^2} \int \Phi(x)\Phi(x+u)\, \mathrm{d}^2 x \right) \mathrm{d}^2 u. \tag{8}$$

Finally the square of the modulus of the phase is

$$|\Phi_k|^2 = \frac{1}{L^2} \int e^{\frac{2\pi ikx}{L}} C(r)\, \mathrm{d}^2 r, \tag{9}$$

where

$$C(r) = \frac{1}{L^2} \int \Phi(x)\Phi(x+r)\, \mathrm{d}^2 x, \tag{10}$$

Following Refs. [1,12] we can relate the phase correlation C with the structure function D (In the approximation of large period L): from formula (1)

$$D(r) = \frac{1}{L^2} \int (\Phi(x) - \Phi(x+r))^2\, \mathrm{d}^2 x \tag{11}$$

$$= \frac{1}{L^2} \int \Phi(x)^2\, \mathrm{d}^2 x + \frac{1}{L^2} \int \Phi(x+r)^2\, \mathrm{d}^2 x$$

$$- \frac{2}{L^2} \int \Phi(x)\Phi(x+r)\, \mathrm{d}^2 x = 2C(0) - 2C(r). \tag{12}$$

So, the phase correlation $C(r) = C(0) - D(r)/2$. But the structure function D increases infinitely at $|r| \to \infty$, while the phase correlation should be finite and preferably positive.

So, we should restrict the structure function. The necessity of the restriction with some parameter A may be grounded from some physical speculation: the difference in optical paths for two points at the Earth's surface never is greater than the thickness of the atmosphere. (Practically many orders of magnitude less). Let us introduce the parameter $A > 1$ and define the restricted structure function

$$\tilde{D}(r) = \begin{cases} 6.88 \left|\dfrac{r}{r_0}\right|^{5/3}, & \text{for } \left|\dfrac{r}{r_0}\right| \le A; \\ 6.88\, A^{5/3}, & \text{for } \left|\dfrac{r}{r_0}\right| \ge A. \end{cases} \tag{13}$$

This function coincides with the empirical formula (1) at $|r| \le A r_0$ and is equal to the constant $6.88\,A$ at large values of $|r|$. It gives the finite non-negative correlation

$$\tilde{C}(r) = \tilde{C}(0) - \frac{\tilde{D}(r)}{2} = \begin{cases} 3.44 \left(A - \left|\dfrac{r}{r_0}\right|^{5/3}\right), & \text{for } |r| \le \frac{A}{r_0}; \\ 0, & \text{for } |r| \ge \frac{A}{r_0}. \end{cases} \tag{14}$$

We can generate quasirandom realizations of the phase with given correlation $C(r)$; at $|r| \le A^{5/3} r_0$ they reproduce the empirical formula (1) for the structure function asymptotically at large values of A. Consider the Fourier-transform of $C(r)$. It should be positive; so, it is necessary to restrict negative values, *i.e.*, set them to zero. Then the amplitude of the Fourier-components of the phase is defined. Note that phases of these Fourier-components have no influence on the sharpness of the structure function (1), so, they should be random values between $-\pi, \pi$. The field is defined by $\exp(i\Phi(x))$. The solution of the diffraction equation has the form

$$E(x, z) = \sum_k \exp\left(-\frac{2\pi i k x}{L}\right) \exp\left(\frac{-\pi \lambda z k^2}{L^2}\right) F(k), \tag{15}$$

where

$$F(k) = \frac{1}{L^2} \int \exp\left(\frac{2\pi i k x}{L}\right) E(x, 0)\, \mathrm{d}^2 x. \tag{16}$$

For technical reasons the grid was taken to be 32×32 points. For the step of the grid $\delta x = 3.125$ cm, the period of the grid $L = 1$ m. In this case $r_0 \approx 3.2\,\delta x$. For visible light $\lambda \approx 0.6 \times 10^{-6}$ m. Define the dimensionless parameter h which corresponds to the length of the propagation:

$$\left(\frac{\delta x}{r_0}\right)^2 h = \frac{\pi \lambda}{L^2} z. \tag{17}$$

For values mentioned above $h \approx \frac{z}{50\ \text{km}}$; giving the realistic value $h \approx 0.2$. Thus, we have a dimensionless approximation for numerical simulations of the distribution of the field. The empirical formula (1) is not adequate at $z \gg 10$ km, but it gives an upper limit to the path of the propagation without dislocations.

TABLE I. The program of calculations, written in matlab.

```
i=sqrt(-1); pi=3.1415925636; N=32; A=4^(5/3); h=.2; eps=.2;                %1;
x=-N/2:N/2-1; r=ones(x')*x+i*x'*ones(x); r=abs(r)/3.2; y=-1:2/31:1;        %2;
F=3.44*max(0,A-r.^(5/3)); C=fafo(F); mesh(C); C=max(0,real(C));           %3;
f=ifafo(C); pause; clg; subplot(211); mesh(F); mesh(f); pause; clg;       %4;
Fk=exp(2*pi*i*rand(N)).*sqrt(C); subplot(111); mesh(Fk); pause;           %5;
Fx=2*real(fafo(Fk)); mesh(Fx); pause; clg;                                %6;
E=exp(i*Fx); subplot(211); mesh(E); mesh(i*E); pause;                     %7;
C=fafo(E); mesh(C); mesh(i*C); pause; clg; subplot(121);                  %8;
F=exp((i*h-eps)*r.*r).*C; E=ifafo(F);                                     %9;
contour(i*log(E),5,y,y); title('Phase'); grid;                            %10;
contour(min(.1,abs(E).^2),3,y,y); title('min(.1,intensity)'); grid;       %11;
```

5. CALCULATIONS

The program to realize the algorithm mentioned above is so short (11 lines, written in matlab), that we reproduce it here (Table I). The meaning of every line of the program is simple:

1. Defines i, π, A, h. While $N = 32$, the calculations should be made over the grid of 32×32 points. Parameter "eps" is used later only to smooth the visible picture to make it easier to count dislocations.

2. Defines the radius-vector of each point of the grid.

3. Defines the correlation C by formula (14). (It is plotted by the next line.) The operator "mesh" plots the distribution of the real part of its Fourier-transform. Unfortunately, it also has negative values, so, it is truncated to zero at these points.

4. Shows the distortion of the correlation by this truncation. (Fig. 4) This result is not used in the following calculations.

5. Defines the random phase of the Fourier-components of the field and plots them.

6. Calculates the field as a function of the transverse coordinate and plots it.

7. Calculates the field and plots its real and imaginary part.

8. Calculates the Fourier-transform of the field [Eq. (16)] and plots its real and imaginary part.

9. Calculates the field distribution at the distance of propagation z. The value of z is defined by the parameter h.

10. Plots the lines of countours of the phase of the field (Fig. 2a).

11. Shows zeros of the field amplitude. (Fig. 2b). Points with intensities .033 and .067 are connected by lines.

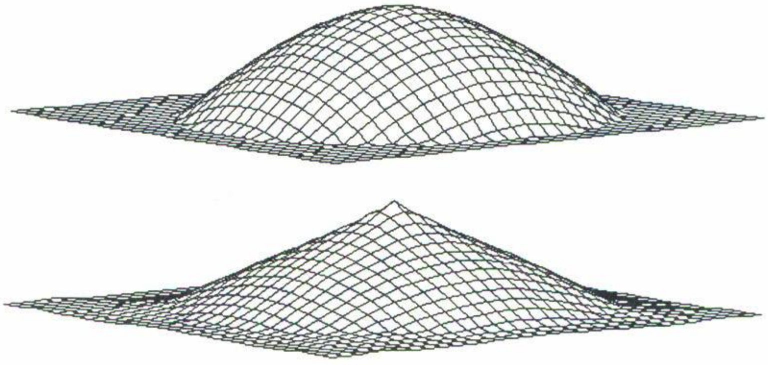

FIGURE 4. (a) Plot of the restricted correlation of phase by formula (9). (b) Actual phase correlation generated by the program.

6. Results and discussion

We considered some hundreds of pictures of the phase distribution at $z \approx 10$ km. No dislocations were detected. Among more than 100 realizations of the field they never appear at $h < 1$. In Fig. 2 one such dislocation is marked by the ends of the line of the cut of the phase $\arg(E)$. (For one realization.) Fig. 2a represents the map of zeros of the field. (Equilines with the intensity 0.033 and 0.067 are marked.)

For some realizations of the initial distribution of the phase of the field dislocations appear at $h \approx 1$, which corresponds to $z \approx 50$ km. Observation of the set of realizations enable us to formulate the estimation: The surface density of dislocations of the wavefront due to the atmosphere at a distance of propagation in the atmosphere $z \approx 10$ km is less than $0.01/\mathrm{m}^2$. So, dislocations should not remove the possibility of correcting wavefronts by flexible mirrors at least while its size is not greater than 10 m.

To make more accurate estimations it is necessary to consider a model for the 3–dimensional distribution of the fluctuations of the refractive index.

Acknowledgments

This work was stimulated by A. Chelli, P. Loza, and M. Kallistratova. D. Kouznetsov was partially supported by *Sistema Nacional de Investigadores.* We thank Neil Bruce for help with the text.

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