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Forces between two uniformly charged cylinders versus forces between two conducting cylinders

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ABSTRACT. The forces between two bodies with fixed electrical charge distributions and the forces between two conducting bodies with the same shape and geometrical arrangement are compared, emphasizing the qualitative difference in the respective electrostatic situations. The choice of parallel circular cylinders lends itself to a quantitative illustration of the differences, which persist even in the limit of vanishing radii.

RESUMEN. Se comparan las fuerzas entre dos cuerpos con distribuciones de carga eléctrica fijas y las fuerzas entre dos cuerpos conductores con las mismas formas y arreglo geométrico, destacando la diferencia cualitativa en las situaciones electrostáticas respectivas. La selección de cilindros circulares paralelos se presta para una ilustración cuantitativa de las diferencias, las cuales persisten aun en el límite cuando los radios de los cilindros tienden a cero.

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1. INTRODUCTION

Coulomb's law describes the forces of interaction between two point charges [1–3]. In practice, it is the interaction of finite size bodies that can be studied experimentally. Then the question arises, "Under what conditions can the forces between two finite size bodies be described by Coulomb's law?". Newton studied the gravitational counterpart, establishing that the forces between two spherical symmetric distributions of mass obey his universal law [4]. Correspondingly, the answer for the electrical question is that the bodies must have spherically symmetric distributions of charge. However, in the electrical practice the use of conducting bodies is the common rule, which leads to the question: "Under what conditions can the forces between two conducting spheres be approximated by Coulomb's law?". It must be recognized that in electrostatics there is a qualitative difference between the two situations of given charge distributions versus given equipotential surfaces, leading

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to different forces for each case and leaving the last question still open. The force between two conducting spheres forming a bispherical capacitor can be evaluated on the basis of a recent work [5], but this requires some elaborate mathematics. The same physics can be quantitatively illustrated considering a cylindrical geometry. In fact, it is shown in this paper that the force between two conducting electrodes in a bicylindrical capacitor is definitely different from the force between the uniformly charged cylinders, even in the limit of vanishing radii. The difference for any geometry is due to the mutual and self induction effects in the conductors.

In this paper we present a comparative study of the interaction between two uniformly charged cylinders and the interaction between two conducting cylinders in the same geometrical arrangement, which may be appropriate for the junior-senior level. Section 2 contains the evaluation of the electric intensity field and the mutual forces of the two uniformly charged cylinders, using polar coordinates. In Sect. 3, the electrostatic field, charge distributions and forces associated with the electrodes of a bicylindrical capacitor are studied, using bipolar coordinates. Sect. 4 consists of a comparison of the forces and sources involved in both situations, recognizing their qualitative and quantitative differences. In Sect. 5 we discuss the specific characteristics of the cylindrical geometry and the general results for any geometry.

2. FORCES BETWEEN TWO UNIFORMLY CHARGED CIRCULAR CYLINDERS

The electrostatic field of a uniformly charged straight line with linear charge density λ is studied and evaluated in the introductory courses on electromagnetism using Gauss' law [1–3], the electric intensity field being radial and inversely proportional to the distance from the line:

$$\vec{E}(\vec{\rho}) = \frac{2\lambda\hat{\rho}}{\rho}. \tag{1}$$

This field is also valid for the region outside any cylinder coaxial with the original straight line with the same charge distributed uniformly on its surface; the field inside the cylinder is null.

Let us consider two such cylinders with axes in the positions $\vec{\rho}_1$, $\vec{\rho}_2$, radii R_1, R_2 and linear charge densities λ_1, λ_2, respectively. By using the superposition principle the total electric intensity at any point defined by the position vector $\vec{\rho}$, is written

$$\vec{E}(\vec{\rho}) = \frac{2\lambda_1(\vec{\rho}-\vec{\rho}_1)}{|\vec{\rho}-\vec{\rho}_1|^2} + \frac{2\lambda_2(\vec{\rho}-\vec{\rho}_2)}{|\vec{\rho}-\vec{\rho}_2|^2}. \tag{2}$$

The force exerted by cylinder 1 on cylinder 2 is evaluated through integration of the product of the electric intensity and the charge over the surface of the latter:

$$\vec{F}_{1\to 2} = \oint_S \vec{E}(\vec{\rho})\sigma(\vec{\rho})\, da, \tag{3}$$

where $\vec{E}(\vec{\rho})$ is the total field in Eq. (2), but the integrated contribution of the second term, *i.e.* the self-force, vanishes; $\sigma(\vec{\rho}) = \sigma(\vec{\rho}_2 + \vec{R}_2) = \frac{\lambda_2}{2\pi R_2}$ is the surface charge density; and $da = R_2\, d\varphi_2\, dz$ is the elementary area element on the surface of the cylinder. For purposes of the integration and without any loss of generality, the axis of cylinder 1 can be taken at $\vec{\rho}_1 = 0$, then

$$\vec{F}_{1\to 2} = \frac{2\lambda_1\lambda_2}{2\pi R_2} \int_0^{2\pi} \int_0^h \frac{(\vec{\rho}_2 + \vec{R}_2) R_2\, d\varphi_2\, dz}{\rho_2^2 + R_2^2 + 2\rho_2 R_2 \cos\varphi_2}. \tag{4}$$

By taking cartesian coordinates with the x–axis along the line joining the axes of cylinder 1 and 2, the vector in the integrand of Eq. (4) has the components

$$\vec{\rho}_2 + \vec{R}_2 = \hat{\imath}(\rho_2 + R_2\cos\varphi_2) + \hat{\jmath} R_2 \sin\varphi_2. \tag{5}$$

Upon integration the $\hat{\jmath}$–component of the force in Eq. (4) vanishes, since the contributions from $0 < \varphi_2 < \pi$ cancel their counterparts from $\pi < \varphi < 2\pi$. The integration of the $\hat{\imath}$–component follows after rewriting the corresponding integrand and evaluating each integral:

$$\begin{aligned}\int_0^{2\pi} \frac{(\rho_2 + R_2\cos\varphi_2)\, d\varphi_2}{\rho_2^2 + R_2^2 + 2\rho_2 R_2\cos\varphi_2} &= \frac{1}{2\rho_2}\int_0^{2\pi}\left[1 + \frac{\rho_2^2 - R_2^2}{\rho_2^2 + R_2^2 + 2\rho_2 R_2 \cos\varphi_2}\right] d\varphi_2 \\ &= \frac{2\pi}{2\rho_2}\left[1 + \frac{\rho_2^2 - R_2^2}{\sqrt{(\rho_2^2 + R_2^2)^2 - 4\rho_2^2 R_2^2}}\right].\end{aligned} \tag{6}$$

At this point, it is important to distinguish between the two situations $\rho_2 > R_2$, *i.e.*, axis 1 is external to cylinder 2, and $\rho_2 < R_2$, *i.e.*, axis 1 is inside cylinder 2. In the first situation the fraction in Eq. (6) is +1 and the force of Eq. (4) becomes

$$\vec{F}_{1\to 2} = \frac{2\lambda_1\lambda_2 h\hat{\imath}}{\rho_2}. \tag{7}$$

In the second situation the fraction in Eq. (6) is −1 and the force vanishes.

Comparison of Eqs. (1) and (7) indicates that the force between the uniformly charged cylinders, when they are external to each other, is the same as the force between two charged lines at their respective axes. On the other hand when one cylinder is inside the other one, the force vanishes, since the field of the external cylinder is null in its interior as pointed out in the paragraph after Eq. (1).

3. Bicylindrical capacitor and forces between its electrodes

The geometrical configuration of the two circular cylinders is naturally described in bipolar coordinates [6]

$$x = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}, \qquad y = \frac{a\sin\xi}{\cosh - \cos\xi}, \qquad z = z, \tag{8}$$

where the surfaces with fixed values of $0 \leq \xi < \pi$ correspond to circular arc cylinders with radius $a|\csc\xi|$ and axis at $(x = 0, y = a\cot\xi, z)$ meeting at $(x = \pm a, y = 0, z)$, with $\xi = 0, 2\pi$ corresponding to the outer portions of the xz plane with $y = 0_+$ and $y = 0_-$, respectively, $\xi = \frac{\pi}{2}, \frac{3\pi}{2}$ are half circle cylinders of radius a, and $\xi = \pi$ is the inner portion of the yz plane; and the surfaces with fixed values of $-\infty < \eta < \infty$ correspond to nested circular cylinders with radius $a|\text{csch}\eta|$ and axis at $(x = a\coth\eta, y = 0, z)$, with $\eta = 0$ corresponding to the yz plane, and $\eta = \pm\infty$ to the lines $(x = \pm a, y = 0, z)$.

The unit vectors and scale factors associated with these coordinates follow from

$$d\vec{r} = \hat{\imath}\, dx + \hat{\jmath}\, dy + \hat{k}\, dz = \hat{\xi} h_\xi\, d\xi + \hat{\eta} h_\eta\, d\eta + \hat{k}\, dz, \tag{9a}$$

$$\hat{\xi} = \frac{-\hat{\imath}\sinh\eta\sin\xi + \hat{\jmath}(\cosh\eta\cos\xi - 1)}{\cosh\eta - \cos\xi}, \quad \hat{\eta} = \frac{-\hat{\imath}(\cosh\eta\cos\xi - 1) - \hat{\jmath}\sinh\eta\sin\xi}{\cosh\eta - \cos\xi}, \tag{9b}$$

$$h_\xi = h_\eta = \frac{a}{\cosh\eta - \cos\xi}. \tag{9c}$$

The unit vectors $\hat{\xi}$, $\hat{\eta}$, $\hat{k}$ form a right handed orthonormal set reflecting the orthonormality of the coordinates themselves.

The bipolar coordinates also give a natural description of the electrostatic field of capacitors with circular cylindrical electrodes. In fact, the Laplace equation

$$\left[\frac{1}{h_\xi h_\eta}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \frac{\partial^2}{\partial z^2}\right]\phi(\xi, \eta, z) = 0 \tag{10}$$

is separable in such coordinates, allowing the fulfillment of the boundary conditions

$$\phi(\xi, \eta = \eta_1, z) = V_1, \qquad \phi(\xi, \eta = \eta_2, z) = V_2 \tag{11}$$

in an immediate way. For infinite cylinders the z-dependence in Eqs. (10)-(11) disappears, and the most general solution of the Laplace equation in the bipolar coordinates is

$$\phi(\xi, \eta) = \sum_{m=0}^{\infty}(A_m \sin m\xi + B_m \cos m\xi)(C_m \sinh m\eta + D_m \cosh m\eta). \tag{12}$$

The boundary conditions of Eqs. (11) select the $m = 0$ term in the summation and determine the respective coefficients leading to the electrostatic potential function linear in the coordinate η:

$$\phi(\xi, \eta) = \frac{V_2(\eta - \eta_1) + V_1(\eta_2 - \eta)}{\eta_2 - \eta_1}. \tag{13}$$

The electric intensity field is the negative gradient of this potential:

$$\vec{E}(\xi, \eta) = -\left(\frac{\hat{\xi}}{h_\xi}\frac{\partial}{\partial\xi} + \frac{\hat{\eta}}{h_\eta}\frac{\partial}{\partial\eta}\right)\phi(\xi, \eta) = -\frac{\hat{\eta}}{h_\eta}\frac{V_2 - V_1}{\eta_2 - \eta_1}. \tag{14}$$

Notice that according to Eq. (13) the equipotential surfaces are the nested circular cylinders for each value of η between the electrodes, and the electric field lines according to Eq. (14) are normal to these cylinders.

The charge distributions on the electrodes are obtained by application of Gauss' law in terms of the surface charge densities:

$$\sigma(\xi,\eta=\eta_1)=\frac{\hat{\eta}\cdot\vec{E}}{4\pi}=-\frac{1}{4\pi a}\frac{V_2-V_1}{\eta_2-\eta_1}(\cosh\eta_1-\cos\xi), \tag{15a}$$

$$\sigma(\xi,\eta=\eta_2)=-\frac{\hat{\eta}\cdot\vec{E}}{4\pi}=\frac{1}{4\pi a}\frac{V_2-V_1}{\eta_2-\eta_1}(\cosh\eta_2-\cos\xi), \tag{15b}$$

The total charge on each electrode is evaluated by integrating the surface charge density over the surface of the cylinder:

$$Q_1=\int_0^{2\pi}\int_0^h \sigma h_\xi\,d\xi\,dz=\frac{h}{2a}\frac{V_1-V_2}{\eta_2-\eta_1}=-Q_2. \tag{16}$$

As expected, both charges have the same magnitude and opposite signs, and correspond to linear charge densities

$$\lambda_1=\frac{Q_1}{h}=-\lambda_2=\frac{V_1-V_2}{\eta_2-\eta_1}. \tag{17}$$

The force exerted by cylinder 1 on cylinder 2 is evaluated through Eq. (3) using the appropriate electric intensity Eq. (14) and surface charge density Eq. (15b):

$$\begin{aligned}\vec{F}_{1\to 2}&=-\frac{1}{4\pi}\left(\frac{V_2-V_1}{\eta_2-\eta_1}\right)^2\int_0^{2\pi}\int_0^h\frac{1}{h_\eta^2}\hat{\eta}h_\xi\,d\xi\,dz\Bigg|_{\eta=\eta_2}\\&=-\frac{h}{4\pi a}\left(\frac{V_2-V_1}{\eta_2-\eta_1}\right)^2\int_0^{2\pi}\left[-\hat{\imath}(\cosh\eta_2\cos\xi-1)-\hat{\jmath}\sinh\eta_2\sin\xi\right]d\xi\\&=-\frac{\hat{\imath}h}{4\pi a}\left(\frac{V_2-V_1}{\eta_2-\eta_1}\right)^2 2\pi=\frac{2\lambda_1\lambda_2 h\hat{\imath}}{a}.\end{aligned} \tag{18}$$

In the second line the explicit form of the unit vector $\hat{\eta}$ [Eq. (9b)] is used, and in the final form the force is expressed in terms of the linear charge densities of each cylinder. Notice that this force is attractive and the same for any two cylinders η_1 and η_2 with same charges, for a given value of the distance a. This distance can be expressed in terms of the radius of each cylinder, $R_1=a\,\mathrm{csch}\,\eta_1$ and $R_2=a\,\mathrm{csch}\,\eta_2$, and the distance between their axes, $d=a\coth\eta_2-a\coth\eta_1$, as

$$a=\frac{\sqrt{(d+R_1+R_2)(d+R_1-R_2)(d-R_1-R_2)(d-R_1+R_2)}}{2d}. \tag{19}$$

4. COMPARISON OF FORCES AND SOURCES

The comparison of the forces of the electrostatic situations studied in Sects. 2 and 3, respectively, can be made right away through Eq. (7) with $\rho_2 = d$, and Eq. (18) with the value of a given by Eq. (19). Their obviously common characteristics are their radial direction, their direct proportionality with the charge per unit length λ_1 of cylinder 1 and the charge $\lambda_2 h$ of the chosen portion of cylinder 2. Their space dependences are definitely different, even though they have the same geometrical arrangement. Indeed, the force between the uniformly charged cylinders external to each other depends only on the distance between their axes and not at all on their radii, Eq. (7); if one cylinder is inside the other, the force vanishes. The force between the electrodes of the bicylindrical capacitor, on the other hand, has a fixed value for any two cylinders with the same value of a, Eq. (18); it changes with the radii of the cylinders and distance between their axes according to Eq. (19). It must be emphasized that the difference persists even in the limit of cylinders with vanishing radii, $R_1 \to 0$, $R_2 \to 0$, for which Eq. (19) gives $a = d/2$, and the force between the two conductors is double the force between the two uniformly charged cylinders. The conclusion is that there are not common geometrical situations for which the force between the conducting cylinders, Eq. (18), may approximate the force between the uniformly charged cylinders, Eq. (7).

It is instructive to compare also the charge distributions in the cylinders for the respective situations of Sects. 2 and 3, since they determine the nature of the corresponding forces. In the first situation, each cylinder has the lowest circular harmonic, *i.e.*, uniform, source distribution [7]; correspondingly, the force reduces to the bidimensional "monoline-monoline"contribution, Eq. (7). The electrostatic potential function, Eq. (1), electric intensity field, Eq. (14), and surface charge distributions, Eqs. (15), correspond also to the lowest harmonic, but in bipolar coordinates. The source distributions of Eqs. (15) can be rewritten in terms of their respective circular harmonics with the help of Eqs. (8); for the points on cylinder 2,

$$x_2 = \frac{a \sinh \eta_2}{\cosh \eta_2 - \cos \xi} = a \coth \eta_2 + a \operatorname{csch} \eta_2 + a \operatorname{csch} \eta_2 \cos \varphi_2, \tag{20}$$

and Eq. (15b) becomes

$$\begin{aligned}
\sigma(\varphi_2, \eta = \eta_2) &= \frac{1}{4\pi a} \frac{V_2 - V_1}{\eta_2 - \eta_1} \frac{\sinh^2 \eta_2}{\cosh \eta_2 + \cos \varphi_2} \\
&= \frac{1}{4\pi a} \frac{V_2 - V_1}{\eta_2 - \eta_1} \frac{\sinh^2 \eta_2}{\cosh \eta_2} \sum_{s=0}^{\infty} (-)^s \left(\frac{\cos \varphi_2}{\cosh \eta_2} \right)^s \\
&= \frac{1}{4\pi a} \frac{V_2 - V_1}{\eta_2 - \eta_1} \frac{\sinh^2 \eta_2}{\cosh \eta_2} \\
&\quad \times \sum_{m=0}^{\infty} \left[\frac{(-)^m}{(2\cosh \eta_2)^m} \sum_{k=0}^{\infty} \binom{m+2k}{k} \frac{1}{(2\cosh \eta_2)^{2k}} \right] \epsilon_m \cos m\varphi_2 .
\end{aligned} \tag{21}$$

In going from the first to the second line the inverse of the binomial is expanded through its geometric series, and the final form is obtained by expressing the powers of the cosine function as the superposition of cosines of the integer multiples of the azimuthal angle φ_2, where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$.

Equation (21) shows that the lowest bipolar harmonic corresponds to an infinite superposition of circular harmonics; there is also the counterpart for cylinder 1. The lowest circular harmonic in each cylinder is associated with the uniformly distributed charge, and the higher harmonics arise from the mutual and self electrostatic induction in both cylinders. Correspondingly, the force in Eq. (18) contains not only the "monoline-monoline" contribution, but also the contributions from the higher harmonic interactions, which due to their inductive nature make an attractive contribution to the force.

5. Discussion

We have presented a comparative study of the forces between two uniformly charged cylinders and between two conducting cylinders in the same geometrical arrangement. The emphasis has been on the differences arising from the different electrostatic situations involved; fixed charge distributions in one and fixed electrostatic potentials in the other. The cylindrical geometry allows simple and direct quantitative treatments of both cases, as well as the comparison of the forces and sources.

Other geometries may not allow a quantitative treatment so simple as the one in this work, but the physical elements are the same. The case of two uniformly charged spheres is mathematically the same problem studied by Newton for the gravitational interaction of two spheres with uniform mass distributions [4]; the case of two equipotential spheres was studied by Maxwell using the method of images and using spherical harmonics centered in each sphere [8], and also by other authors using bispherical coordinates [5,9]. In general, the forces in both situations of bodies with fixed charge distributions and conducting bodies with the same geometrical arrangement, are different since their electrostatic fields and sources have different harmonic compositions.

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