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Solution of the inhomogeneous equations of equilibrium for an isotropic elastic medium

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Abstract. It is shown that the method of adjoint operators allows one to solve certain inhomogeneous systems of linear partial differential equations. As an example, this method is applied to the equations of equilibrium for an isotropic elastic medium.

Resumen. Se muestra que el método de operadores adjuntos permite resolver ciertos sistemas inhomogéneos de ecuaciones diferenciales parciales lineales. Como ejemplo, este método se aplica a las ecuaciones de equilibrio para un medio elástico isótropo.

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1. Introduction

The complete solution of some homogeneous systems of linear partial differential equations can be obtained by using the method of adjoint operators [1,2], which allows one to express the solution of such systems of equations in terms of one or several scalar potentials. Among the systems of differential equations of mathematical physics that can be solved by this method are the source-free Maxwell equations (in flat space-time [2,3] or in an algebraically special space-time [1]), the Einstein vacuum field equations linearized about the Minkowski metric [4] or about an algebraically special vacuum space-time [1], and the equations of equilibrium for an isotropic elastic medium in the absence of body forces [5].

The method of adjoint operators can be applied directly to systems of differential equations that can be written in the form

$$\mathcal{E}(f) = 0, \tag{1}$$

where $\mathcal{E}$ is a self-adjoint linear operator and f represents the unknown variables, if there exists a scalar made out of the field components and their derivatives $\chi = \mathcal{T}(f)$, where $\mathcal{T}$ is a linear operator, that obeys a decoupled equation

$$\mathcal{O}(\chi) = 0, \tag{2}$$

where $\mathcal{O}$ is another linear operator. The fact that Eq. (2) is a consequence of Eq. (1) is equivalent to the existence of a linear operator $\mathcal{S}$ such that

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}. \tag{3}$$

Then, $f = \mathcal{S}^\dagger(\psi)$ satisfies Eq. (1), provided that

$$\mathcal{O}^\dagger(\psi) = 0, \tag{4}$$

which follows from the adjoint of Eq. (3),

$$\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger, \tag{5}$$

using the fact that $\mathcal{E}$ is self-adjoint. (For details see Refs. [1,2].) In some cases it is necessary to find several decoupled equations of the form (2) in order to get the general solution of Eq. (1).

The method of adjoint operators also allows one to solve the inhomogeneous system of differential equations

$$\mathcal{E}(f) = g, \tag{6}$$

associated to Eq. (1), in the following manner. At those points where the source term g vanishes, we have $f = \mathcal{S}^\dagger(\psi)$, where ψ is a solution of Eq. (4) (for the sake of simplicity, we are assuming here that a single scalar potential generates the general solution of Eq. (1); in the example given in Sect. 2, three scalar potentials are necessary) then, from Eqs. (3) and (6), one gets the equality

$$\mathcal{O}\mathcal{T}\mathcal{S}^\dagger(\psi) = \mathcal{S}(g), \tag{7}$$

which determines the solution of Eq. (4) that represents the effect of the source g.

In this paper we solve the equations of equilibrium for an infinite isotropic elastic medium using the method outlined above. In Sect. 2 the solution of these equations is written in cartesian and circular cylindrical coordinates and in Sect. 3 we conclude with some remarks.

2. Solution of the inhomogeneous equations for isotropic elastic media

2.1. *Basic equations*

As shown in Ref. [5], the equations of equilibrium for an isotropic elastic medium in the absence of body forces,

$$(1 - 2\sigma)\nabla^2\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}) = 0, \tag{8}$$

where σ denotes Poisson's ratio and $\mathbf{u}$ is the displacement vector [6], are of the form (1) with $\mathcal{E}$ being self-adjoint. From Eqs. (8) one can obtain the decoupled equations

$$\begin{aligned} \nabla^2(\hat{e}_z \cdot \nabla \times \mathbf{u}) &= 0, \\ \nabla^2(\nabla \cdot \mathbf{u}) &= 0, \\ \nabla^2[z\nabla \cdot \mathbf{u} + 2(1-2\sigma)\hat{e}_z \cdot \mathbf{u}] &= 0, \end{aligned} \tag{9}$$

whose existence imply that of three scalar potentials, ψ_1, ψ_2, ψ_3, such that

$$\mathbf{u} = \nabla \times (\psi_1 \hat{e}_z) - \nabla(\psi_2 + z\psi_3) + 4(1-\sigma)\psi_3 \hat{e}_z \tag{10}$$

satisfies Eqs. (8) provided that

$$\nabla^2\psi_1 = \nabla^2\psi_2 = \nabla^2\psi_3 = 0, \tag{11}$$

(for details see Ref. [5]).

An expression analogous to Eq. (10) for the solution of Eqs. (8) in terms of *four* scalar potentials was obtained by Papkovich (1932) and Neuber (1934) (see, *e.g.*, Refs. [7–9]). The Papkovich-Neuber solution can be written in the form

$$\mathbf{u} = -\nabla(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) + 4(1-\sigma)(\phi_1\hat{e}_x + \phi_2\hat{e}_y + \phi_3\hat{e}_z), \tag{12}$$

where the four functions ϕ_0, ϕ_1, ϕ_2, ϕ_3 are harmonic. Attempts have been made to prove that only three of these potentials are necessary and this point has been a subject of much discussion (see, *e.g.*, Refs. [7–9] and the references cited therein). In Ref. [8], it is shown that, if $4\sigma \neq 1$, the function ϕ_0 can be omitted. However, one can show that ϕ_0 can be omitted in all cases, provided that the potentials are allowed to have singularities (the displacement $\mathbf{u}$ given by Eq. (12) may be well-behaved even if the potentials have singularities, see, *e.g.*, Ref. [5]). In fact, it is easy to see that the right-hand side of Eq. (12) is unchanged if one sets ϕ_0 equal to zero and replaces ϕ_i by $\phi_i + \partial f/\partial x_i$ $(i = 1, 2, 3)$, where

$$f = r^{4(1-\sigma)} \int \phi_0 r^{4\sigma-5}\, dr,$$

the x_i are cartesian coordinates and r is the usual radial coordinate. (Note that $\nabla^2\phi_0 = 0$ implies that $\nabla^2 f = 0$ and, hence, $\nabla^2(\partial f/\partial x_i)$ also vanishes.)

Equations (8) can be solved by separation of variables, making use of spin-weighted functions [5], and it turns out that their most general solution can be expressed in the form (10) (see also Sect. 3). Comparison of Eqs. (10) and (12) shows that ϕ_0 corresponds to ψ_2, while ϕ_3 corresponds to ψ_3. The Papkovich-Neuber potentials ϕ_1, ϕ_2, ϕ_3 can be derived from the decoupled equations

$$\begin{aligned} \nabla^2[x\nabla \cdot \mathbf{u} + 2(1-2\sigma)\hat{e}_x \cdot \mathbf{u}] &= 0, \\ \nabla^2[y\nabla \cdot \mathbf{u} + 2(1-2\sigma)\hat{e}_y \cdot \mathbf{u}] &= 0, \\ \nabla^2[z\nabla \cdot \mathbf{u} + 2(1-2\sigma)\hat{e}_z \cdot \mathbf{u}] &= 0, \end{aligned}$$

which follow from Eqs. (8).

Now, we want to solve the inhomogeneous system of partial differential equations

$$(1-2\sigma)\nabla^2\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}) = \mathbf{K}, \tag{13}$$

where $\mathbf{K}$ is a given vector field. (The vector field $\mathbf{K}$ is related to the body force $\mathbf{F}$ through $\mathbf{K} = -2(1-2\sigma)(1+\sigma)\,\mathbf{F}/E$, where E is the Young modulus [6].) Then, from Eq. (13) one finds that [5]

$$\begin{aligned}(1-2\sigma)\nabla^2(\hat{e}_z\cdot\nabla\times\mathbf{u}) &= \hat{e}_z\cdot\nabla\times\mathbf{K} \equiv \mathcal{S}_1(\mathbf{K}),\\ 2(1-\sigma)\nabla^2(\nabla\cdot\mathbf{u}) &= \nabla\cdot\mathbf{K} \equiv \mathcal{S}_2(\mathbf{K}),\\ 2(1-\sigma)\nabla^2[z\nabla\cdot\mathbf{u} + 2(1-2\sigma)\hat{e}_z\cdot\mathbf{u}] &= z\nabla\cdot\mathbf{K} + 4(1-\sigma)\hat{e}_z\cdot\mathbf{K} \equiv \mathcal{S}_3(\mathbf{K}).\end{aligned} \tag{14}$$

(Note that when $\mathbf{K}=0$, Eqs. (13) and (14) reduce to Eqs. (8) and (9), respectively.)

The scalar inhomogeneous equations (14) can be solved using the Green's functions for the Laplace operator appropriate to the boundary conditions; for instance, in the case of free space the displacement (and its derivatives) must vanish at infinity, therefore

$$\begin{aligned}(1-2\sigma)\hat{e}_z\cdot\nabla\times\mathbf{u}(\mathbf{r}) &= -\frac{1}{4\pi}\int\frac{[\mathcal{S}_1(\mathbf{K})](\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv',\\ 2(1-\sigma)\nabla\cdot\mathbf{u}(\mathbf{r}) &= -\frac{1}{4\pi}\int\frac{[\mathcal{S}_2(\mathbf{K})](\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv',\\ 2(1-\sigma)[z\nabla\cdot\mathbf{u}(\mathbf{r}) + 2(1-2\sigma)\hat{e}_z\cdot\mathbf{u}(\mathbf{r})] &= -\frac{1}{4\pi}\int\frac{[\mathcal{S}_3(\mathbf{K})](\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv'.\end{aligned} \tag{15}$$

Making use of the definition of the adjoint operator [1,5] (which amounts to integrating by parts), we can write

$$\int\frac{[\mathcal{S}_i(\mathbf{K})](\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' = \int\mathbf{K}(\mathbf{r}')\cdot\mathcal{S}_i'^{\dagger}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv', \qquad i=1,2,3, \tag{16}$$

where the prime on $\mathcal{S}_i'^{\dagger}$ means that this operator acts on the primed variables. From the definitions given by Eqs. (14), one finds that [5]

$$\begin{aligned}\mathcal{S}_1^{\dagger}(\psi) &= \nabla\times(\psi\hat{e}_z) = -\hat{e}_z\times\nabla\psi,\\ \mathcal{S}_2^{\dagger}(\psi) &= -\nabla\psi,\\ \mathcal{S}_3^{\dagger}(\psi) &= -\nabla(z\psi) + 4(1-\sigma)\psi\hat{e}_z.\end{aligned} \tag{17}$$

(Note that Eq. (10) amounts to $\mathbf{u} = \mathcal{S}_1^{\dagger}(\psi_1) + \mathcal{S}_2^{\dagger}(\psi_2) + \mathcal{S}_3^{\dagger}(\psi_3)$.)

We shall assume that $\mathbf{K}$ vanishes outside some bounded region Ω. Then, for points outside Ω, $\mathbf{u}$ is given by Eq. (10) which, when substituted into Eqs. (15), yields

$$\begin{aligned}
(1-2\sigma)\frac{\partial^2\psi_1}{\partial z^2} &= -\frac{1}{4\pi}\int \mathbf{K}(\mathbf{r}')\cdot \mathcal{S}_1'^{\dagger}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dv', \\
4(1-\sigma)(1-2\sigma)\frac{\partial\psi_3}{\partial z} &= -\frac{1}{4\pi}\int \mathbf{K}(\mathbf{r}')\cdot \mathcal{S}_2'^{\dagger}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dv', \qquad (18)\\
4(1-\sigma)(1-2\sigma)\left[-\frac{\partial\psi_2}{\partial z}+(3-4\sigma)\psi_3\right] &= -\frac{1}{4\pi}\int \mathbf{K}(\mathbf{r}')\cdot \mathcal{S}_3'^{\dagger}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) dv',
\end{aligned}$$

where we have used Eq. (16).

Thus, the scalar potentials ψ_i, that determine the displacement vector outside Ω have to satisfy Eqs. (11) and (18). Since Eqs. (10), (17) and (18) are adapted to cylindrical coordinates, in the following subsections we express the solutions of these equations in cartesian coordinates and in circular cylindrical coordinates.

2.2. Cartesian coordinate expansion

We shall consider the upper half-space $z > 0$ (the case $z < 0$ is analogous), then, in order for $\mathbf{u}$ to vanish at infinity, the solutions of Eqs. (11) must be of the form

$$\begin{aligned}
\psi_1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\alpha,\beta)\, e^{i(\alpha x+\beta y)-\sqrt{\alpha^2+\beta^2}\, z}\, d\alpha\, d\beta, \\
\psi_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(\alpha,\beta)\, e^{i(\alpha x+\beta y)-\sqrt{\alpha^2+\beta^2}\, z}\, d\alpha\, d\beta, \qquad (19)\\
\psi_3 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\alpha,\beta)\, e^{i(\alpha x+\beta y)-\sqrt{\alpha^2+\beta^2}\, z}\, d\alpha\, d\beta,
\end{aligned}$$

where a, b, and c are complex-valued functions of two variables. Then, making use of the expansion

$$\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i\alpha(x-x')+i\beta(y-y')-\sqrt{\alpha^2+\beta^2}\,|z-z'|}}{\sqrt{\alpha^2+\beta^2}}\, d\alpha\, d\beta, \qquad (20)$$

which follows from the well-known relation

$$\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2}\, d^3k,$$

assuming that $z > z'$, from Eqs. (18–20) one finds that the coefficients $a(\alpha,\beta)$, $b(\alpha,\beta)$, and $c(\alpha,\beta)$ are given by

$$a(\alpha,\beta) = -\frac{1}{8\pi^2(1-2\sigma)}\frac{1}{(\alpha^2+\beta^2)^{3/2}}\int \mathbf{K}(\mathbf{r}')\cdot \mathbf{w}_{1\alpha\beta}(\mathbf{r}')\, dv',$$

$$b(\alpha,\beta) = -\frac{1}{8\pi^2(1-2\sigma)}\frac{1}{4(1-\sigma)}\left[\frac{1}{\alpha^2+\beta^2}\int \mathbf{K}(\mathbf{r}')\cdot\mathbf{w}_{3\alpha\beta}(\mathbf{r}')\,dv' + \frac{3-4\sigma}{(\alpha^2+\beta^2)^{3/2}}\int \mathbf{K}(\mathbf{r}')\cdot\mathbf{w}_{2\alpha\beta}(\mathbf{r}')\,dv'\right], \tag{21}$$

$$c(\alpha,\beta) = \frac{1}{8\pi^2(1-2\sigma)}\frac{1}{4(1-\sigma)}\frac{1}{\alpha^2+\beta^2}\int \mathbf{K}(\mathbf{r}')\cdot\mathbf{w}_{2\alpha\beta}(\mathbf{r}')\,dv',$$

where we have introduced the vector fields

$$\mathbf{w}_{i\alpha\beta}(\mathbf{r}) \equiv \mathcal{S}_i^\dagger\left(e^{-i(\alpha x+\beta y)+\sqrt{\alpha^2+\beta^2}\,z}\right), \qquad i=1,2,3, \tag{22}$$

which, according to Eqs. (10–11) and (17), satisfy the homogeneous equations (8).

On the other hand, substituting Eqs. (19) into Eq. (10) one finds that, outside Ω,

$$\mathbf{u} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{a(\alpha,\beta)\,\mathbf{u}_{1\alpha\beta} + b(\alpha,\beta)\,\mathbf{u}_{2\alpha\beta} + c(\alpha,\beta)\,\mathbf{u}_{3\alpha\beta}\right\}d\alpha\,d\beta, \tag{23}$$

where

$$\mathbf{u}_{i\alpha\beta}(\mathbf{r}) \equiv \mathcal{S}_i^\dagger\left(e^{i(\alpha x+\beta y)-\sqrt{\alpha^2+\beta^2}\,z}\right), \qquad i=1,2,3, \tag{24}$$

which are also solutions of the homogeneous equations (8). Note that as $z\to\infty$, the vector fields $\mathbf{u}_{i\alpha\beta}$ vanish, while the vector fields $\mathbf{w}_{i\alpha\beta}$ diverge. Thus, the displacement vector can be expanded in terms of the elementary solutions $\mathbf{u}_{i\alpha\beta}$ of Eqs. (8), and the corresponding coefficients are related to the inner products of $\mathbf{K}$ with $\mathbf{w}_{i\alpha\beta}$ (*cf.* Ref. [10]); this is a consequence of the fact that the integrand in the expansion of the Green's function (20) is the product of a function of $\mathbf{r}$ and a function of $\mathbf{r}'$.

Substituting now Eqs. (21) into Eq. (23), using Eq. (20) and the fact that

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= -\frac{1}{\pi^2}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^4}\,d^3k \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{|z-z'|}{\alpha^2+\beta^2} + \frac{1}{(\alpha^2+\beta^2)^{3/2}}\right] e^{i\alpha(x-x')+i\beta(y-y')-\sqrt{\alpha^2+\beta^2}\,|z-z'|}\,d\alpha\,d\beta\end{aligned}$$

one finds that the cartesian components of $\mathbf{u}$ are given by

$$u_i(\mathbf{r}) = \int G_{ij}(\mathbf{r}-\mathbf{r}')K_j(\mathbf{r}')\,dv', \tag{25}$$

where we are using the summation convention and

$$\begin{aligned}G_{ij}(\mathbf{r}-\mathbf{r}') &= -\frac{1}{1-2\sigma}\frac{1}{4\pi}\left[\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4(1-\sigma)}\frac{\partial^2|\mathbf{r}-\mathbf{r}'|}{\partial x_i\partial x_j}\right] \\ &= -\frac{1}{1-2\sigma}\frac{1}{16\pi(1-\sigma)}\left[(3-4\sigma)\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|} + \frac{(x_i-x_i')(x_j-x_j')}{|\mathbf{r}-\mathbf{r}'|^3}\right].\end{aligned} \tag{26}$$

An alternative derivation of Eq. (26) is given in Ref. [6] (see also Ref. [8], Eq. (91.2)).

2.3. Circular cylindrical coordinate expansion

Considering again the half-space $z > 0$, we seek solutions of Eqs. (11) of the form

$$\begin{aligned} \psi_1 &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} a_m(k)\, J_m(k\rho) e^{im\varphi - kz}\, dk, \\ \psi_2 &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} b_m(k)\, J_m(k\rho) e^{im\varphi - kz}\, dk, \\ \psi_3 &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} c_m(k)\, J_m(k\rho) e^{im\varphi - kz}\, dk, \end{aligned} \tag{27}$$

where J_m is a Bessel function. The coefficients $a_m(k)$, $b_m(k)$ and $c_m(k)$ are evaluated by substituting Eqs. (17) and (27) and the expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} J_m(k\rho) J_m(k\rho') e^{im(\varphi-\varphi') - k|z-z'|}\, dk$$

into Eqs. (18). One finds

$$\begin{aligned} a_m(k) &= -\frac{1}{4\pi(1-2\sigma)} \frac{1}{k^2} \int \mathbf{K}(\mathbf{r}') \cdot \mathbf{w}_{1mk}(\mathbf{r}')\, dv', \\ b_m(k) &= -\frac{1}{4\pi(1-2\sigma)} \frac{1}{4(1-\sigma)} \left[\frac{1}{k} \int \mathbf{K}(\mathbf{r}') \cdot \mathbf{w}_{3mk}(\mathbf{r}')\, dv' \right. \\ &\qquad \left. + \frac{3-4\sigma}{k^2} \int \mathbf{K}(\mathbf{r}') \cdot \mathbf{w}_{2mk}(\mathbf{r}')\, dv' \right], \\ c_m(k) &= \frac{1}{4\pi(1-2\sigma)} \frac{1}{4(1-\sigma)} \frac{1}{k} \int \mathbf{K}(\mathbf{r}') \cdot \mathbf{w}_{2mk}(\mathbf{r}')\, dv', \end{aligned} \tag{28}$$

where, now,

$$\mathbf{w}_{imk}(\mathbf{r}) \equiv \mathcal{S}_i^{\dagger} \Big(J_m(k\rho) e^{-im\varphi + kz} \Big). \tag{29}$$

From Eqs. (10) and (27) it follows that the displacement vector is given by

$$\mathbf{u} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} \Big\{ a_m(k)\, \mathbf{u}_{1mk} + b_m(k)\, \mathbf{u}_{2mk} + c_m(k)\, \mathbf{u}_{3mk} \Big\} dk, \tag{30}$$

where the $\mathbf{u}_{imk}$ are the solutions of the homogeneous equations (8) defined by

$$\mathbf{u}_{imk}(\mathbf{r}) \equiv \mathcal{S}_i^{\dagger} \Big(J_m(k\rho) e^{im\varphi - kz} \Big). \tag{31}$$

Note that, also in this case, the coefficients in the expansion (30) are related to the inner products of **K** with the vector fields $\mathbf{w}_{imk}$, which are solutions of the homogeneous equations (8) that diverge as $z \to \infty$.

3. DISCUSSION

One can verify by a direct computation that Eq. (25) is, indeed, a solution of Eq. (13); this result provides another proof of the fact that the most general solution of Eqs. (8) can be expressed in the form (10), in terms of three scalar potentials only. If the body forces vanish but there are forces applied to the boundary of the elastic medium, the potentials ψ_i are determined by substituting Eq. (10) into the boundary conditions coming from the definition of the stress tensor (see Ref. [6], Eq. (2.8)). In fact, in Ref. [6] the case where the boundary is an infinite plane is treated in this manner, making use of an expression analogous to Eq. (10), involving four harmonic functions instead of the three potentials ψ_i, and of some unproven assumptions.

The procedure followed here to find the coefficients appearing in Eqs. (23) and (30) is similar to that employed in the multipole expansion of the electromagnetic field, where the relationship between the multipole coefficients and the sources is obtained making use of the decoupled equations satisfied by $\mathbf{r}\cdot\mathbf{E}$ and $\mathbf{r}\cdot\mathbf{B}$ (see, *e.g.*, Ref. [11]). However, in the derivation given in Sect. 2, the decoupled equations (14) that allow us to find the coefficients (21) and (28) also lead to the general solution (10) of the homogeneous equations (8) (*cf.* also Refs. [4,10]).

Equations (21–24) and (28–31) signify that the Green's function for Eq. (13), in cartesian and circular cylindrical coordinates, admits a sort of factorization (*cf.* also Ref. [10]).

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