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Perturbations of Carter's A solution

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ABSTRACT. Considering the equations for the spin-3/2 perturbations of the Carter A solution, it is shown that the maximal spin-weight components, which obey decoupled equations, satisfy differential identities that are equivalent to the Teukolsky-Starobinsky identities fulfilled by the separated functions (in terms of which are given the maximal spin-weight components). Furthermore, assuming that the electromagnetic field is absent, we obtain analogous relations for the maximal spin-weight components of the Weyl spinor perturbations.

RESUMEN. Considerando las ecuaciones para las perturbaciones por campos de espín 3/2 de la solución A de Carter, se muestra que las componentes con peso de espín maximal, las cuales obedecen ecuaciones desacopladas, satisfacen identidades diferenciales que son equivalentes a las identidades de Teukolsky-Starobinsky satisfechas por las funciones separadas. Además, asumiendo que el campo electromagnético no está presente, se obtienen relaciones análogas para las componentes con peso de espín maximal de las perturbaciones del espinor de Weyl.

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1. INTRODUCTION

In general relativity there are theorems proving that the exterior space-time of an uncharged, rotating black hole is described by the Kerr solution [1]. The equilibrium configuration of such a black hole is completely characterized by its mass and angular momentum, whereas a black hole with mass, angular momentum and electric charge has to be described by the Kerr-Newman metric. Because of their astrophysical applications, these solutions have received considerable interest in the last decades. Thus, among other things, their perturbations by different test fields have been studied with very interesting results. One of these, is that from the massless free field equations for spin 1/2, 1, 3/2 and 2, one can obtain decoupled equations for the maximal spin-weight components of the perturbations, which admit separable solutions [2–6]. For the Dirac field and when the background space-time is the Kerr metric, Chandrasekhar [7] showed that this equation can be solved by separation of variables too. Furthermore, it has been found that the maximal spin-weight components satisfy certain differential identities, that are equivalent to the so-called Teukolsky-Starobinsky identities. The success of these investigations has motivated the generalization of some of these results. For example, Torres del Castillo [8,9,10] found that the solution of the Rarita-Schwinger equation can be put in terms of one complex-valued function (Debye potential), when the background space-time is an algebraically special solution of the Einstein vacuum field equations with or without cosmological constant or of the Einstein-Maxwell field equations without cosmological term, these works (in a certain sense) generalize the works of Güven [6] and Aichelburg and Güven [11]. Furthermore, he has shown that when the space-time is of type D, the equations governing the maximal spin-weight components admit separable or R-separable solutions.

The aim of the present paper is to give the differential identities satisfied by the maximal spin-weight components of the spin-3/2 perturbations of the Carter A solution, when the cosmological constant is equal to zero, as well as for the maximal spin-weight components of the Weyl spinor perturbations, when the electromagnetic field is absent. On the other hand, for the spin-3/2 perturbations the real and imaginary parts of the Starobinsky constants are determined. The spinor formalism and the Newman-Penrose notation are used throughout.

2. THE SPIN-3/2 PERTURBATIONS OF THE CARTER A SOLUTION

When the back reaction of the spin–3/2 fields on the geometry is neglected, the extended supergravity field equations reduce to the Rarita-Schwinger equation on a fixed background space-time, which satisfies the Einstein-Maxwell field equations without cosmological constant [12,13]. The Rarita-Schwinger equation, in spinor notation is given by

$$\nabla_{A\dot{B}}\psi^{jA}{}_{C\dot{D}} + i\sqrt{2}\epsilon^{jk}\varphi^{A}{}_{C}\psi^{k}{}_{\dot{D}\dot{B}A} = \nabla_{C\dot{D}}\psi^{jA}{}_{A\dot{B}}, \tag{1}$$

where $j,\ \underline{k=1,}\ 2$, φ_{AB} is the electromagnetic spinor, ϵ^{jk} is the Levi-Civita symbol and $\psi^{j}{}_{\dot{A}\dot{B}C} \equiv \psi^{j}{}_{AB\dot{C}}$. Equation (1) is equivalent to [10]

$$H^{j}{}_{ABC} = H^{j}{}_{(ABC)}, \qquad H^{j}{}_{A\dot{B}\dot{C}} = 0, \tag{2}$$

where

$$\begin{aligned} H^{j}{}_{ABC} &\equiv \nabla^{\dot{R}}{}_{(B}\psi^{j}{}_{|A|C)\dot{R}} - i\sqrt{2}\,\epsilon^{jk}\,\varphi_{A(B}\,\psi^{k\dot{R}}{}_{|\dot{R}|C)}, \\ H^{jA}{}_{\dot{B}\dot{C}} &\equiv \nabla^{D}{}_{(\dot{B}}\psi^{jA}{}_{|D|\dot{C})} - i\sqrt{2}\,\epsilon^{jk}\,\varphi^{AR}\,\psi^{k}{}_{(\dot{B}\dot{C})R}. \end{aligned} \tag{3}$$

(The round brackets denote symmetrization on the indices enclosed and the indices between bars are excluded from the symmetrization.)

As in this linearized theory the back reaction of the spin-3/2 fields on the background solution is neglected, then the supersymmetry transformations affect only the spin-3/2 fields and are given by

$$\psi^j{}_{A\dot{B}\dot{C}} \to \psi^j{}_{A\dot{B}\dot{C}} + \nabla_{\dot{B}\dot{C}}\, \alpha^j{}_A - i\sqrt{2}\,\epsilon^{jk}\,\varphi_{AB}\,\alpha^k{}_{\dot{C}}, \tag{4}$$

where $\alpha^j{}_A$ is a pair of arbitrary spinor fields. These transformations induce the following changes on the fields $H^j{}_{ABC}$ and $H^j{}_{A\dot{B}\dot{C}}$:

$$\begin{aligned} H^j{}_{ABC} &\to H^j{}_{ABC} + \Psi^D{}_{ABC}\,\alpha^j{}_D + i\sqrt{2}\,\epsilon^{jk}\,\alpha^{k\dot{S}}\,\nabla_{(B|\dot{S}|}\,\varphi_{C)A}, \\ H^j{}_{A\dot{B}\dot{C}} &\to H^j{}_{A\dot{B}\dot{C}}. \end{aligned} \tag{5}$$

From Eqs. (1) and (3) one obtains that

$$\nabla^{A\dot{R}} H^j{}_{ABC} = \Psi_{ABC}{}^D\,\psi^{jA}{}_D{}^{\dot{R}} + i\sqrt{2}\,\epsilon^{jk}\,\psi^{k\dot{S}\dot{R}A}\,\nabla_{B\dot{S}}\,\varphi_{AC}, \tag{6}$$

where Ψ_{ABCD} is the Weyl spinor.

In the case where the background space-time is a type-D solution of the Einstein-Maxwell field equations with an algebraically general electromagnetic field, and their principal directions are aligned with the principal directions of the Weyl spinor, one can choose a spin frame o^A, ι^A such that the nonvanishing components of Ψ_{ABCD} and φ_{AB} are Ψ_2 and φ_1, respectively. Then the scalars $H^j{}_0 \equiv H^j{}_{ABC}\, o^A\, o^B\, o^C$ and $H^j{}_3 \equiv H^j{}_{ABC}\, \iota^A\, \iota^B\, \iota^C$, which are invariant under the transformations (4), satisfy the following decoupled equations [9]:

$$\begin{aligned} &\Big[(D - 2\epsilon + \bar{\epsilon} - 3\rho - \bar{\rho})(\Delta - 3\gamma + \mu) \\ &\qquad - (\delta - 2\beta - \bar{\alpha} - 3\tau + \bar{\pi})(\bar{\delta} - 3\alpha + \pi) - \Psi_2\Big] H^j{}_0 = 0, \\ &\Big[(\Delta + 2\gamma - \bar{\gamma} + 3\mu + \bar{\mu})(D + 3\epsilon - \rho) \\ &\qquad - (\bar{\delta} + 2\alpha + \bar{\beta} + 3\pi - \bar{\tau})(\delta + 3\beta - \tau) - \Psi_2\Big] H^j{}_3 = 0. \end{aligned} \tag{7}$$

The scalars $H^j{}_1 \equiv H^j{}_{ABC}\, \iota^A\, o^B\, o^C$ and $H^j{}_2 \equiv H^j{}_{ABC}\, \iota^A\, \iota^B\, o^C$, are not invariant under the supersymmetry transformations and, by constrast to the case where the electromagnetic field is absent, they cannot be always simultaneously eliminated by means of a supersymmetry transformation. It turns out that in the case where the background space-time is the Carter A solution, these scalars cannot be eliminated if

$$(m^2 + n^2) - \varepsilon_0(e^2 + g^2) = 0. \tag{8}$$

For the Kerr-Newman black hole, Eq. (8) reduces to $(m^2 - e^2) = 0$. When Eq. (8) is fulfilled, apart from ${H^j}_0$ and ${H^j}_3$, we have an additional invariant given by [11]

$$ {H^j}_{\mathrm{I}} = \overline{\Psi_2}\,{H^j}_1 + 2i\sqrt{2}\,\varphi_1\,\epsilon^{jk}\left[\tau\overline{{H^k}_1} + \rho\overline{{H^k}_2}\right]. \tag{9} $$

In this paper we study only the scalars ${H^j}_0$ and ${H^j}_3$.

The Carter A solution [14], which contains the Kerr-Newman metric as a particular case and, of course, the Kerr solution, is given by

$$ ds^2 = \frac{\mathcal{Q}}{p^2+q^2}\,(du - p^2\,dv)^2 - \frac{p^2+q^2}{\mathcal{Q}}\,dq^2 - \frac{\mathcal{P}}{p^2+q^2}\,(du + q^2\,dv)^2 - \frac{p^2+q^2}{\mathcal{P}}\,dp^2, \tag{10} $$

where $\{p, q, u, v\}$ is a real coordinate system such that ∂_u and ∂_v are Killing vectors; and $\mathcal{P} = \mathcal{P}(p)$ and $\mathcal{Q} = \mathcal{Q}(q)$, are polynomial functions given by

$$ \begin{aligned} \mathcal{P} &= b - g^2 + 2np - \epsilon_0 p^2 - (\lambda_0/3)p^4, \\ \mathcal{Q} &= b + e^2 - 2mq + \epsilon_0 q^2 - (\lambda_0/3)q^4. \end{aligned} \tag{11} $$

The parameters m, n, e, g, and λ_0 correspond to mass, NUT parameter, electric and magnetic charge, and cosmological constant, respectively; while ϵ_0 and b are two additional parameters. The Kerr-Newman metric is obtained if one takes $b = a^2$, $g = 0 = n$ and $\epsilon_0 = 1$. In terms of the Boyer-Linquist coordinates $q = r$, $p = -a\cos\theta$, $u = -t + a\varphi$ and $v = \varphi/a$.

The tangent vectors

$$ \begin{aligned} D &= \partial_q + \frac{1}{\mathcal{Q}}(\partial_v - q^2\partial_u), \\ \Delta &= \tfrac{1}{2}\phi\overline{\phi}\mathcal{Q}\left(-\partial_q + \frac{1}{\mathcal{Q}}(\partial_v - q^2\partial_u)\right), \\ \delta &= \left(\frac{\mathcal{P}}{2}\right)^{1/2}\overline{\phi}\left(\partial_p + \frac{i}{\mathcal{P}}(\partial_v + p^2\partial_u)\right), \\ \overline{\delta} &= \left(\frac{\mathcal{P}}{2}\right)^{1/2}\phi\left(\partial_p - \frac{i}{\mathcal{P}}(\partial_v + p^2\partial_u)\right), \end{aligned} \tag{12} $$

where

$$ \phi \equiv \frac{1}{q+ip}, \tag{13} $$

form a null tetrad, such that D and Δ are double principal null directions of the conformal curvature. The spin-coefficients are given by

$$ \begin{aligned} &\kappa = \sigma = \lambda = \nu = 0, \qquad \epsilon = 0, \qquad \beta = \delta\ln\mathcal{P}^{1/4}, \\ &\alpha = -\overline{\delta}\ln\frac{\mathcal{P}^{1/4}\mathcal{Q}^{1/2}}{q+ip}, \qquad \gamma = -\Delta\ln\frac{\mathcal{P}^{1/4}\mathcal{Q}^{1/2}}{q+ip}, \\ &\rho = D\ln\phi, \qquad \tau = \delta\ln\phi, \qquad \pi = -\overline{\delta}\ln\phi, \qquad \mu = -\Delta\ln\phi, \end{aligned} \tag{14} $$

and the only curvature components different from zero are

$$\Psi_2 = \Big\{-(m+in) + (e^2+g^2)\overline{\phi}\Big\}\phi^3, \qquad \Lambda = \frac{\lambda_0}{6}. \tag{15}$$

Whereas the electromagnetic field is determined by

$$\varphi_1 = \tfrac{1}{2}(e+ig)\phi^2. \tag{16}$$

Assuming that the components ${H^j}_0$ and ${H^j}_3$ have a dependence in the ignorable coordinates u and v of the form $e^{i(ku+lv)}$, where k and l are constants, and using Eqs. (11)–(15) we have that the solutions of Eqs. (7) are given by [10]

$$\begin{aligned} {H^j}_0 &= a^j e^{i(ku+lv)}\, R_{+3/2}(q)\, S_{+3/2}(p), \\ {H^j}_3 &= -\tfrac{1}{2\sqrt{2}}\, b^j \phi^3 e^{i(ku+lv)}\, R_{-3/2}(q)\, S_{-3/2}(p), \end{aligned} \tag{17}$$

where the a^j and b^j are constants, and the functions $R_{+3/2}$, $R_{-3/2}$, $S_{+3/2}$ and $S_{-3/2}$ satisfy the following equations:

$$\begin{aligned} \Big[\mathcal{Q}\mathcal{D}_{-1/2}\mathcal{D}_0^\dagger + 4ikq\Big]\mathcal{Q}^{3/2}\, R_{+3/2} &= A\mathcal{Q}^{3/2}\, R_{+3/2}, \\ \Big[\mathcal{Q}\mathcal{D}^\dagger_{-1/2}\mathcal{D}_0 - 4ikq\Big]R_{-3/2} &= AR_{-3/2}, \\ \Big[\mathcal{L}^\dagger_{-1/2}\mathcal{L}_{3/2} + 4kp\Big]S_{+3/2} &= -AS_{+3/2}, \\ \Big[\mathcal{L}_{-1/2}\mathcal{L}^\dagger_{3/2} - 4kp\Big]S_{-3/2} &= -AS_{-3/2}, \end{aligned} \tag{18}$$

where A is a separation constant, whereas

$$\begin{aligned} \mathcal{D}_n &\equiv \partial_q + \frac{i}{\mathcal{Q}}(l-kq^2) + n\frac{\dot{\mathcal{Q}}}{\mathcal{Q}}, \\ \mathcal{D}_n^\dagger &\equiv \partial_q - \frac{i}{\mathcal{Q}}(l-kq^2) + n\frac{\dot{\mathcal{Q}}}{\mathcal{Q}}, \\ \mathcal{L}_n &\equiv \sqrt{\mathcal{P}}\left(\partial_p + \frac{1}{\mathcal{P}}(l+kp^2) + \frac{n}{2}\frac{\dot{\mathcal{P}}}{\mathcal{P}}\right), \\ \mathcal{L}_n^\dagger &\equiv \sqrt{\mathcal{P}}\left(\partial_p - \frac{1}{\mathcal{P}}(l+kp^2) + \frac{n}{2}\frac{\dot{\mathcal{P}}}{\mathcal{P}}\right). \end{aligned} \tag{19}$$

A direct computation shows that the function $Q^{3/2}\big(\mathcal{D}_0^\dagger\big)^3 Q^{3/2}\, R_{+3/2}$ has to be a multiple of $R_{-3/2}$, as well as $Q^{3/2}\big(\mathcal{D}_0\big)^3 R_{-3/2}$ must be a multiple of $Q^{3/2}R_{+3/2}$. Furthermore, by normalizing appropriately, the functions $Q^{3/2}R_{+3/2}$ and $R_{-3/2}$ satisfy

$$\begin{aligned} Q^{3/2}\left(\mathcal{D}_0^\dagger\right)^3 Q^{3/2}\, R_{+3/2} &= C_1 R_{-3/2}, \\ Q^{3/2}\left(\mathcal{D}_0\right)^3 R_{-3/2} &= C_2\, Q^{3/2}\, R_{+3/2}, \end{aligned} \tag{20}$$

where C_1 and C_2 are constants such that for k and l real $\overline{C_2} = C_1$. Analogously, one can normalize the functions $S_{+3/2}$ and $S_{-3/2}$ in such a way that

$$\begin{aligned} \mathcal{L}_{-1/2}\,\mathcal{L}_{1/2}\,\mathcal{L}_{3/2}\, S_{+3/2} &= -B_1\, S_{-3/2}, \\ \mathcal{L}^\dagger_{-1/2}\,\mathcal{L}^\dagger_{1/2}\,\mathcal{L}^\dagger_{3/2}\, S_{-3/2} &= B_2\, S_{+3/2}, \end{aligned} \tag{21}$$

where B_1 and B_2 are real constants for k and l real. The constants C_1, C_2, B_1 and B_2 are such that [10]

$$C_1 C_2 = B_1 B_2 + 16\,(e^2 + g^2)(l^2 + k^2). \tag{22}$$

On the other hand, when the background space-time is a type-D solution of the Einstein-Maxwell field equations without cosmological constant, the solution of Eq. (1) is given by [9,15]

$$\begin{aligned} \psi^j{}_{\text{i}\text{i}1} &= -2\sqrt{2}\,\phi^{-3}(\delta + 2\beta + \overline{\alpha} - 2\tau)(\delta + 3\beta - 3\tau)\, T^j{}_3 \\ &\quad + 4i\epsilon^{jk}\overline{\varphi_1\phi^{-3}}(\overline{\delta} + 3\overline{\beta} - 3\overline{\tau})\,\overline{T^j{}_3}, \\ \psi^j{}_{\text{i}\dot{0}1} &= -2\sqrt{2}\,\phi^{-3}\Big[(D + 2\epsilon + \overline{\epsilon} - 2\rho)(\delta + 3\beta - 3\tau) - \overline{\pi}(D + 3\epsilon - 3\rho)\Big] T^j{}_3, \\ \psi^j{}_{\dot{0}\text{i}1} &= -2\sqrt{2}\,\phi^{-3}\Big[(\delta + 2\beta - \overline{\alpha} - 2\tau)(D + 3\epsilon - 3\rho) + \overline{\rho}(\delta + 3\beta - 3\tau)\Big] T^j{}_3, \\ \psi^j{}_{\dot{0}\dot{0}1} &= -2\sqrt{2}\,\phi^{-3}(D + 2\epsilon - \overline{\epsilon} - 2\rho)(D + 3\epsilon - 3\rho)\, T^j{}_3, \\ \psi^j{}_{\text{i}\text{i}0} &= +4i\epsilon^{jk}\,\overline{\varphi_1\phi^{-3}}\,(D + 3\overline{\epsilon} - 3\overline{\rho})\,\overline{T^k{}_3}, \\ \psi^j{}_{\text{i}\dot{0}0} &= \psi^j{}_{\dot{0}\text{i}0} = \psi^j{}_{\dot{0}\dot{0}0} = 0, \end{aligned} \tag{23}$$

where $T^j{}_3$ is a solution of the second equation of (7), up to the gauge transformations (4). From Eqs. (3) and (23) one finds that

$$\begin{aligned} \overline{H^j{}_0} &= -2\sqrt{2}\,\phi^{-3}(D - 2\overline{\varepsilon} + \varepsilon - 4\rho)(D - \overline{\varepsilon} + 2\varepsilon - 2\rho)(D + 3\varepsilon - 3\rho)\, T^j{}_3, \\ \overline{H^j{}_3} &= -2\sqrt{2}\,\phi^{-3}(\delta + 2\overline{\alpha} + \beta - 4\tau)(\delta + \overline{\alpha} + 2\beta - 2\tau)(\delta + 3\beta - 3\tau)\, T^j{}_3 \\ &\quad - 4i\,\overline{\varphi_1\phi^{-3}}\,\epsilon^{jk}\Big[(\Delta + 2\overline{\gamma} - \gamma + \mu + \overline{\mu})(D + 3\overline{\varepsilon} - 3\overline{\rho}) \\ &\quad - (\delta + 2\overline{\alpha} + \beta - \tau + \overline{\pi})(\overline{\delta} + 3\overline{\beta} - 3\overline{\tau})\Big]\overline{T^k{}_3}. \end{aligned} \tag{24}$$

Furthermore, using the second equation of (7) we have

$$\begin{aligned}\overline{H^j{}_3} &= -2\sqrt{2}\,\phi^{-3}(\delta+2\overline{\alpha}+\beta-4\tau)(\delta+\overline{\alpha}+2\beta-2\tau)(\delta+3\beta-3\tau)\,T^j{}_3\\ &\quad - 8i\,\overline{\varphi_1\phi^{-3}}\,\epsilon^{jk}\Big[\overline{\pi}(\overline{\delta}+3\overline{\beta})-\overline{\mu}(D+3\overline{\varepsilon})+\overline{\tau}(\delta+3\overline{\alpha})\\ &\quad -\overline{\rho}(\Delta+3\overline{\gamma})+\tfrac{3}{2}\overline{\Psi_2}\Big]\overline{T^k{}_3}. \end{aligned} \tag{25}$$

Therefore, substituting Eq. (17) into Eq. (24) one arrives at

$$\begin{aligned}\overline{H^j{}_0} &= b^j C_2\, e^{i(ku+lv)}\, R_{+3/2}\, S_{-3/2},\\ \overline{H^j{}_3} &= \tfrac{1}{2\sqrt{2}}\,\overline{\phi}^3\Big\{b^j B_2\, e^{i(ku+lv)}\, R_{-3/2}\, S_{+3/2}\\ &\qquad - 4i(e-ig)\epsilon^{jk}b^k\,\overline{\Big[(l+ik)e^{i(ku+lv)}\,R_{-3/2}\,S_{-3/2}\Big]}\Big\}. \end{aligned} \tag{26}$$

Looking for differential identities, analogous to those found by Torres del Castillo [8,9,16] for the spin-3/2 perturbations of a type-D solution of the Einstein vacuum field equations with or without cosmological constant, we have that, when the background space-time is the Carter A solution, the perturbations (26) satisfy

$$\begin{aligned}&(D+\epsilon-2\overline{\epsilon}-5\rho)(D+2\epsilon-\overline{\epsilon}-3\rho)(D+3\epsilon-\rho)H^j{}_3 =\\ &\quad(\overline{\delta}-\alpha-2\overline{\beta}+5\pi)(\overline{\delta}-2\alpha-\overline{\beta}+3\pi)(\overline{\delta}-3\alpha+\pi)H^j{}_0\\ &\quad+2\sqrt{2}i\,\varphi_1\,\epsilon^{jk}\Big\{\rho(\Delta-3\overline{\gamma})+\mu(D-3\overline{\varepsilon})-\tau(\overline{\delta}-3\overline{\beta})\\ &\quad-\pi(\delta-3\overline{\alpha})+\tfrac{3}{2}\phi\,\overline{\phi^{-1}\Psi_2}\Big\}\overline{H^j{}_0},\\ &(\Delta-\gamma+2\overline{\gamma}+5\mu)(\Delta-2\gamma+\overline{\gamma}+3\mu)(\Delta-3\gamma+\mu)H^j{}_0 =\\ &\quad(\delta+\beta+2\overline{\alpha}-5\tau)(\delta+2\beta+\overline{\alpha}-3\tau)(\delta+3\beta-\tau)H^j{}_3\\ &\quad-2\sqrt{2}\,i\varphi_1\,\epsilon^{jk}\Big\{\rho(\Delta+3\overline{\gamma})+\mu(D+3\overline{\varepsilon})-\tau(\overline{\delta}+3\overline{\beta})\\ &\quad-\pi(\delta+3\overline{\alpha})-\tfrac{3}{2}\phi\,\overline{\phi^{-1}\Psi_2}\Big\}\overline{H^j{}_3}. \end{aligned} \tag{27}$$

According to the previously expounded, we observe that Eqs. (27) restrict the solutions of the Rarita-Schwinger equation, in such a way that the separable solutions (17) cannot belong to the same solution.

Substituting the symmetric solutions of the decoupled equations (7) given by

$$\begin{aligned}H^j{}_0 &= a^j e^{i(ku+lv)}\, R_{+3/2}\, S_{+3/2} + \varepsilon^{jk}a^k e^{-i(ku+lv)}\, Q^{-3/2}R_{-3/2}\, S_{-3/2},\\ H^j{}_3 &= -\tfrac{1}{2\sqrt{2}}\phi^3\Big[a^j e^{i(ku+lv)}\, R_{-3/2}\, S_{-3/2} - \varepsilon^{jk}a^k e^{-i(ku+lv)}\, Q^{3/2}R_{+3/2}\, S_{+3/2}\Big], \end{aligned} \tag{28}$$

into Eqs. (27), and making use of Eqs. (20) and (21) we find that

$$C_2 = B_1 + 4i(e+ig)(l+ik),$$
$$C_1 = B_2 - 4i(e+ig)(l+ik), \tag{29}$$

from these equations one finds that,

$$C_1C_2 = B_1B_2 + 16(e+ig)^2(l+ik)^2 + 4i(e+ig)(l+ig)(B_2 - B_1). \tag{30}$$

Comparing with Eq. (22) one arives at

$$B_1 - B_2 = 8(ek+gl). \tag{31}$$

3. Differential identities for the perturbations of the Weyl spinor

If we denote by ϕ_{ABCD} the perturbations of the Weyl spinor and assuming that the electromagnetic field of the space-time is absent, then [17]

$$\nabla^{A\dot{A}}\phi_{ABCD} = \tfrac{1}{2}h^{RS\dot{A}\dot{B}}\nabla_{B\dot{B}}\Psi_{RSCD} - \Psi_{RS(BC}\nabla_{D)}{}^{\dot{B}}\,h^{RS\dot{A}}{}_{\dot{B}}$$
$$- \tfrac{1}{2}\Psi_{RS(BC}\nabla^{R\dot{B}}\,h_{D)}{}^{S\dot{A}}{}_{\dot{B}}, \tag{32}$$

where $h_{AB\dot{C}\dot{D}}$ denote the metric perturbations. When the space-time is type-D the components $\phi_0 \equiv \phi_{ABCD}\, o^A\, o^B\, o^C\, o^D$ and $\phi_4 \equiv \phi_{ABCD}\, \iota^A\, \iota^B\, \iota^C\, \iota^D$ are invariant under the transformations

$$\phi_{ABCD} \mapsto \phi_{ABCD} - \zeta^{E\dot{E}}\,\nabla_{\dot{E}(A}\Psi_{BCD)E} - 2\Psi_{E(ABC}\nabla_{D)\dot{E}}\zeta^{E\dot{E}}, \tag{33}$$

which are induced by the gauge transformations $h_{ab} \mapsto h_{ab} - 2\nabla_{(a}\zeta_{b)}$, where ζ_a is an arbitrary vector field. Furthermore, these components are such that [2]

$$\Big[(D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})(\Delta - 4\gamma + \mu)$$
$$- (\delta - 3\beta - \bar{\alpha} - 4\tau + \bar{\pi})(\bar{\delta} - 4\alpha + \pi) - 3\Psi_2\Big]\phi_0 = 0,$$
$$\Big[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho)$$
$$- (\bar{\delta} + 3\alpha + \bar{\beta} + 4\pi - \bar{\tau})(\delta + 4\beta - \tau) - 3\Psi_2]\phi_4 = 0. \tag{34}$$

For our case, these equations admit R-separable solutions given by

$$\phi_0 = e^{i(ku+lv)}\,R_{+2}(q)\,S_{+2}(p),$$
$$\phi_4 = \tfrac{1}{4}\phi^4 e^{i(ku+lv)}\,R_{-2}(q)\,S_{-2}(p), \tag{35}$$

where k and l are separation constants and the functions $R_{\pm 2}$, and $S_{\pm 2}$ obey the differential equations

$$
\begin{aligned}
(\mathcal{Q}\mathcal{D}_{-1}\mathcal{D}_0^{\dagger} + 6ikq)\mathcal{Q}^2 R_{+2} &= A\mathcal{Q}^2 R_{+2}, \\
(\mathcal{Q}\mathcal{D}^{\dagger}_{-1}\mathcal{D}_0 - 6ikq)R_{-2} &= AR_{-2}, \\
(\mathcal{L}^{\dagger}_{-1}\mathcal{L}_2 + 6kp)S_{+2} &= -AS_{+2}, \\
(\mathcal{L}_{-1}\mathcal{L}_2^{\dagger} - 6kp)S_{-2} &= -AS_{-2}.
\end{aligned} \tag{36}
$$

In these equations A is another separation constant. As in the previous case, the functions $\mathcal{Q}^2 R_{+2}$, R_{-2} and $S_{\pm 2}$ satisfy the so-called Teukolsky-Starobinsky identities given by [18]

$$
\begin{aligned}
\mathcal{Q}^2(\mathcal{D}_0^{\dagger})^4 \mathcal{Q}^2 R_{+2} &= C_1 R_{-2}, \\
\mathcal{Q}^2(\mathcal{D}_0)^4 R_{-2} &= C_2 \mathcal{Q}^2 R_{+2}, \\
\mathcal{L}_{-1}\mathcal{L}_0\mathcal{L}_1\mathcal{L}_2 S_{+2} &= B_1 S_{-2}, \\
\mathcal{L}^{\dagger}_{-1}\mathcal{L}_0^{\dagger}\mathcal{L}_1^{\dagger}\mathcal{L}_2^{\dagger} S_{-2} &= B_2 S_{+2},
\end{aligned} \tag{37}
$$

where C_1, C_2, B_1 and B_2 satisfy [18]

$$
C_1 C_2 = B_1 B_2 + 144(m^2 + n^2)(k^2 + l^2). \tag{38}
$$

On the other hand, we have that when the space-time is type-D, the maximal spin-weight components of the Weyl spinor can be written in the following form [19–21]:

$$
\begin{aligned}
\phi_0 &= 4\overline{\phi^{-4}}(D - 3\epsilon + \overline{\epsilon} - 5\overline{\rho})(D - 2\epsilon + 2\overline{\epsilon} - 5\overline{\rho})(D - \epsilon + 3\overline{\epsilon} - 5\overline{\rho})(D + 4\overline{\epsilon} - \overline{\rho})\,\overline{T_4}, \\
\phi_4 &= 4\overline{\phi^{-4}}(\overline{\delta} + 3\alpha + \overline{\beta} - 5\overline{\tau})(\overline{\delta} + 2\alpha + 2\overline{\beta} - 5\overline{\tau})(\overline{\delta} + \alpha + 3\overline{\beta} - 5\overline{\tau})(\overline{\delta} + 4\overline{\beta} - \overline{\tau})\,\overline{T_4} \\
&\quad - 12\phi^{-4}\Psi_2\{\rho(\Delta + 4\gamma) + \mu(D + 4\epsilon) - \tau(\overline{\delta} + 4\alpha) \\
&\quad - \pi(\delta + 4\beta) - 2\Psi_2 - 4\Lambda\}T_4,
\end{aligned} \tag{39}
$$

where T_4 is a solution of second equation (34). Therefore, making use of Eqs. (11)–(15), (35) and (39) we have

$$
\begin{aligned}
\phi_0 &= C_1 e^{-i(ku+lv)}\,\mathcal{Q}^{-2} R_{-2}\,S_{-2}, \\
\phi_4 &= \tfrac{1}{4}\phi^4\Big\{B_2 e^{-i(ku+lv)}\,\mathcal{Q}^2 R_{+2}\,S_{+2} \\
&\qquad + 12(m + in)(l + ik)e^{i(ku+lv)}\,R_{-2}\,S_{-2}\Big\},
\end{aligned} \tag{40}
$$

where we have assumed that $\overline{R_{+2}} = \mathcal{Q}^2 R_{+2}$ and that $S_{\pm 2}$ are real for k and l real.

Looking for differential identities analogous to (27), we have

$$
\begin{aligned}
&(D+\epsilon-3\bar{\epsilon}-5\rho)(D+2\epsilon-2\bar{\epsilon}-5\rho)(D+3\epsilon-\bar{\epsilon}-5\rho)(D+4\epsilon-\rho)\phi_4 = \\
&\qquad (\overline{\delta}-\alpha-3\overline{\beta}+5\pi)(\overline{\delta}-2\alpha-2\overline{\beta}+5\pi)(\overline{\delta}-3\alpha-\overline{\beta}+5\pi)(\overline{\delta}-4\alpha+\pi)\phi_0 \\
&\qquad - 3\Psi_2\{\rho(\Delta-4\overline{\gamma})+\mu(D-4\bar{\epsilon}) \\
&\qquad - \tau(\overline{\delta}-4\overline{\beta})-\pi(\delta-4\overline{\alpha})+2\phi\,\overline{\phi^{-1}\Psi_2}\}\overline{\phi_0}, \\
&(\Delta-\gamma+3\overline{\gamma}+5\mu)(\Delta-2\gamma+2\overline{\gamma}+5\mu)(\Delta-3\gamma+\overline{\gamma}+5\mu)(\Delta-4\gamma+\mu)\phi_0 = \\
&\qquad (\delta+\beta+3\overline{\alpha}-5\tau)(\delta+2\beta+2\overline{\alpha}-5\tau)(\delta+3\beta+\overline{\alpha}-5\tau)(\delta+4\beta-\tau)\phi_4 \\
&\qquad + 3\Psi_2\Big\{\rho(\Delta+4\overline{\gamma})+\mu(D+4\bar{\epsilon})-\tau(\overline{\delta}+4\overline{\beta}) \\
&\qquad - \pi(\delta+4\overline{\alpha})-2\phi\overline{\phi^{-1}\Psi_2}\Big\}\overline{\phi_4}.
\end{aligned} \tag{41}
$$

These differential identities were found by Torres del Castillo [18] when the background is any type-D solution of the Einstein vacuum field equations with cosmological constant; however, in the form given en Ref. [18], they do not apply in all null tetrads such that D and Δ are principal directions of the Weyl spinor.

4. Conclusions

Since all terms that appear in the differential identities (27) and (41) have a well-defined type (in the sense of Geroch-Held-Penrose [7,22]), these identities are invariant under the transformations given by

$$
o^A \to z o^A, \qquad \iota^A \to z^{-1}\iota^A,
$$

where z is an arbitrary (nowhere vanishing) complex scalar field. Therefore, (27) and (41) are valid in all null tetrads such that D and Δ are double principal null directions of the conformal curvature.

On the other hand, in accordance to the results obtained by Aichelburg and Güven [11] and Eq. (8), it should be possible to search for a solution of the field equations of O(2) extended supergravity theory, which should contain the solution obtained by Aichelburg and Güven [23] as a particular case.

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