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Dynamics of relativistic membranes with boundaries

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ABSTRACT. A covariant formalism is presented to describe the dynamics of an arbitrary membrane with smooth physical boundaries. We exploit the fact that the boundary worldsheet is an embedded surface of codimension one in the worldsheet spanned by the membrane in spacetime. The formalism applies to any effective theory for the dynamics of the membrane and of its boundary. The case of extremal membranes with extremal boundaries is discussed in detail.

RESUMEN. Presentamos una formulación covariante para describir la dinámica de una membrana arbitraria con fronteras físicas suaves. Usamos el hecho de que la superficie de mundo de la frontera es una superficie de co-dimensión uno en la superficie de mundo de la membrana en el espacio-tiempo. Esta formulación se aplica a cualquier teoría efectiva para la dinámica de una membrana y de sus fronteras. El caso de membranas extremales con fronteras extremales está tratado en detalle.

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The dynamics of many physical systems can be modeled by the dynamics of relativistic membranes of an appropriate dimension, in a fixed background spacetime (For examples, see Refs. [1-4]). The phenomenological action determining the dynamics of the membrane is constructed using a linear combination of the geometrical scalars of its worldsheet. At lowest order, this action is proportional to the volume of the worldsheet, and it is known as the Dirac-Nambu-Goto [DNG] action.

Here, we consider the case of membranes with smooth physical boundaries. An elementary example is given by an open string with monopoles at its ends. Another is a domain wall bounded by a string. In cosmology, objects of this type can be generated by a hierarchy of phase transitions in the early universe [1]. In the limit that the mass of the boundary tends to zero we recover the null boundary dynamics familiar in the theory of open membranes [2]. The main point of this paper is to provide a geometrical treatment of the dynamics of relativistic membranes with boundaries. Our key idea is to consider the boundary worldsheet itself as an embedded surface in the worldsheet of the relativistic membrane. The boundary is thus treated as a membrane itself, of one dimension less than the original membrane.

In order to simplify our presentation, we consider the case of an extremal membrane, described by the DNG action, with a single extremal boundary, described by a DNG action of one lower dimension. We derive the equations of motion for this system, and we discuss briefly their structure. This analysis can be easily generalized to any dynamics, both for a different choice of action for the membrane and for the boundary. It can also be generalized in a straightforward way to the case in which the boundary has several disconnected components.

To begin with, consider an oriented *timelike* worldsheet m of dimension D, which corresponds to the trajectory of a relativistic membrane in an N–dimensional spacetime $\{M, g_{\mu\nu}\}$. The worldsheet m is described by the embedding

$$x^\mu = X^\mu(\xi^a), \tag{1}$$

where x^μ are coordinates on M, and ξ^a coordinates on m ($\mu, \nu, \ldots = 0, \ldots, N-1$, and $a, b, \ldots = 0, \ldots, D-1$). The D vectors,

$$e_a := X^\mu_{,a}\, \partial_\mu, \tag{2}$$

form a basis of tangent vectors to m, at each point of m. The metric induced on the worldsheet is then given by

$$\gamma_{ab} = X^\mu_{,a} X^\nu_{,b}\, g_{\mu\nu} = g(e_a, e_b). \tag{3}$$

Let the spacetime vectors n^i denote the i^{th} unit normal to the worldsheet ($i, j, \ldots = 1, \ldots, N-D$), defined, up to a local $O(N-D)$ rotation, with

$$g(n^i, n^j) = \delta^{ij}, \qquad g(e_a, n^i) = 0. \tag{4}$$

Normal vielbein indices are raised and lowered with δ^{ij} and δ_{ij}, respectively, whereas tangential indices are raised and lowered with γ^{ab} and γ_{ab}, respectively.

The vectors $\{e_a, n^i\}$ form a basis for spacetime vectors adapted to the situation of interest here.

The worldsheet projection of the spacetime covariant derivatives is defined with $D_a := e^\mu_a D_\mu$, where D_μ is the (torsionless) covariant derivative compatible with $g_{\mu\nu}$. The classical Gauss equation (see Ref. [5]) is given by

$$D_a e_b = \gamma_{ab}{}^c e_c - K_{ab}{}^i n_i. \tag{5}$$

The $\gamma_{ab}{}^c = \gamma_{ba}{}^c$ are the connection coefficients compatible with the worldsheet metric γ_{ab}. The quantity $K_{ab}{}^i$ is the i^{th} extrinsic curvature of the worldsheet, defined by

$$K_{ab}{}^i = -g(D_a e_b, n^i) = K_{ba}{}^i. \tag{6}$$

The extrinsic geometry of m is determined by $K_{ab}{}^{i}$, and by the extrinsic twist potential, associated with the covariance under normal frame rotations, which we will not need here (see, *e.g.*, Ref. [6]).

As is well known, not every specification of the intrinsic and of the extrinsic geometry is consistent with some embedding. There are integrability conditions, the Gauss-Codazzi, Codazzi-Mainardi, and Ricci equations, which must be satisfied by the intrinsic and extrinsic geometry, for an embedding to exist. We will not need their explicit form in this paper.

We turn now to the definition of the intrinsic and extrinsic geometry of the worldsheet boundary ∂m. The point is to see ∂m as a timelike surface of dimension $D-1$, described by the embedding in the worldsheet m:

$$\xi^a = \chi^a(\mu^A), \tag{7}$$

where $A, B, \ldots = 0, 1, \ldots, D-2$.

The definition of the extrinsic and intrinsic geometry of the worldsheet boundary is a special case of the discussion given above for an arbitrary worldsheet. In order to establish our notation, we repeat it, specializing to the case of co-dimension one. The $D-2$ vectors $\epsilon_A := \chi^a{}_{,A}\,\partial_a$ are tangent to the boundary worldsheet ∂m. The metric induced on ∂m is then

$$h_{AB} = \gamma_{ab}\chi^a{}_{,A}\,\chi^b{}_{,B} = \gamma(\epsilon_A, \epsilon_B). \tag{8}$$

The normal to ∂m is defined by

$$\gamma(\eta, \epsilon_A) = 0, \qquad \gamma(\eta, \eta) = 1. \tag{9}$$

The Gauss equation, for a co-dimension one embedding, takes the form

$$\nabla_A \epsilon_B = \gamma_{AB}{}^{C}\epsilon_C - k_{AB}\eta, \tag{10}$$

where $\nabla_A = \epsilon^a{}_A \nabla_a$ is the gradient along the tangential basis $\{\epsilon^A\}$. $\gamma_{AB}{}^{C}$ are the connection coefficients compatible with the boundary worldsheet metric h_{AB}, and $k_{AB} = k_{BA}$ is the boundary worldsheet extrinsic curvature. For a co-dimension one embedding, the extrinsic geometry is determined completely by the extrinsic curvature, and the Ricci integrability conditions are vacuus.

The dynamics of the membrane is specified by the choice of an appropriate phenomenological action, constructed with scalars built with the quantities that characterize the intrinsic and extrinsic geometry of the membrane worldsheet. In the presence of boundaries, one needs also to specify some dynamical rule for the boundaries themselves. For the sake of concreteness, in the following we choose the DNG action for the membrane, and the same action for its single boundary. Our analysis can be easily generalized to any dynamics, and to the case of a boundary with many disconnected components.

The action we consider is

$$S = S_0 + S_b, \tag{11}$$

where

$$S_0[X] = -\sigma_0 \int_m d^D\xi \sqrt{-\gamma}, \tag{12a}$$

$$S_b[\chi(X)] = -\sigma_b \int_{\partial m} d^{(D-1)}\mu \sqrt{-h}, \tag{12b}$$

σ_0 is the membrane tension, σ_b is the tension of the boundary membrane, γ the determinant of the membrane worldsheet metric γ_{ab}, and h is the determinant of the boundary worldsheet metric h_{AB}. This action is a functional of the embedding X^μ of m in M, and of the embedding χ^a of ∂m in m.

To derive the equations of motion of the action (11), consider first a variation of the embedding of m, $X^\mu \to X^\mu + \delta X^\mu$. The displacement is assumed to vanish on two spacelike hypersurfaces of m, which play the role of initial and final times (see *e.g.*, Refs. [6,7]).

The variation of the membrane action S_0 gives

$$\delta S_0 = -\sigma_0 \int_m d^D\xi\, \partial_b \left[\sqrt{-\gamma}\gamma^{ab} e^\alpha{}_a\, \delta X_\alpha\right] + \sigma_0 \int_m d^D\xi\, \partial_b \left[\sqrt{-\gamma}\gamma^{ab} e^\alpha{}_a\right] \delta X_\alpha. \tag{13}$$

We decompose the displacement with respect to the spacetime basis $\{e^a, n^i\}$, as

$$\delta X = \Phi_a e^a + \Phi_i n^i. \tag{14}$$

Note also that the contracted Gauss equation, [Eq. (5)] can be expressed in the form

$$\partial_b \left[\sqrt{-\gamma}\, \gamma^{ab} e^\alpha{}_a\right] = \sqrt{-\gamma}\, \gamma^{ab} \nabla_b\, e^\alpha{}_a = \sqrt{-\gamma}\, K^i n^\alpha{}_i, \tag{15}$$

where $K^i = \gamma^{ab} K_{ab}{}^i$ is the mean extrinsic curvature. Inserting this expression in the variation of the action, one obtains

$$\delta S_0 = -\sigma_0 \int_{\partial m} d^{D-1}\mu\, \eta_b \sqrt{-h}\, \Phi^b + \sigma_0 \int_m d^D\xi \sqrt{-\gamma}\, K^i \Phi_i. \tag{16}$$

We find then that only the normal projection of the variation contributes to the equations of motion of the membrane. The tangential variation gives a boundary term that will contribute to the equations of motion of the boundary.

An interesting special case worth mentioning is the string, *i.e.*, $D = 2$. For an open string without physical boundaries, one imposes the boundary condition that the worldsheet spatial derivative of the embedding function vanish at the ends. This cancels the boundary term in Eq. (16). Physically, this is equivalent to assuming that no momentum flows off the ends which now travel at the speed of light [2]. (For a geometric treatment of this type of boundary contributions, see Ref. [8].)

Let us consider now the boundary action. The displacement of the boundary worldsheet induced by the displacement of the worldsheet m can be written as

$$\delta\xi^a = \Phi^a = \Phi\eta + \Phi_A\,\epsilon^A. \tag{17}$$

Then the variation of the boundary action gives

$$\delta S_b = \sigma_b \int_{\partial m} d^{D-1}\mu\,\sqrt{-h}\,h^{ab}k_{ab}\Phi. \tag{18}$$

We have used part of the previous derivation here, together with the well known fact that the boundary of a boundary is zero. It is at this level that the assumption of dealing with a *smooth* boundary comes in.

The variation of the total action [Eq. (11)], gives the equations of motion for the membrane

$$K^i = 0, \tag{19}$$

i.e., the mean extrinsic curvature vanishes. We find that the form of the equations of motion of the original membrane is not affected by the addition of a boundary.

The equations of motion for the boundary take the form

$$\sigma_b k = \sigma_0, \tag{20}$$

where $k = h^{AB}k_{AB}$ is the mean extrinsic curvature of the boundary worldsheet.

These equations of motion constitute a highly non-linear system of coupled hyperbolic partial differential equations. While the system is well defined in principle, one can hope to find explicit solutions only for highly symmetrical configurations. However, even in this case, the interplay between the dynamics of the original membrane and the dynamics of the physical boundaries can make the solution very difficult. In practice, what one needs to do is to treat the reaction of the boundary dynamics on the membrane in an iterative way.

We conclude with a brief remark about the stability analysis of extended objects with boundary. In Ref. [6], a generally covariant treatment of the kinematics of infinitesimal deformations of membranes without boundaries was given. It can be easily generalized to the case in which smooth boundaries are present using the geometric point of view presented in this paper.

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