Revista Mexicana de Física **41**, *No. 6 (1995) 791–795*

From vacuum field equations on principal bundles to Einstein equations with fluids

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Recibido el 24 de marzo de 1995; aceptado el 29 de agosto de 1995

ABSTRACT. In the present work we show that the Einstein equations on M without cosmological constant and with perfect fluid as source, can be obtained from the field equations for vacuum with cosmological constant on the principal fibre bundle $P\left(\frac{1}{I}M, \mathrm{U}(1)\right)$, M being the space-time and I the radius of the internal space $\mathrm{U}(1)$.

RESUMEN. Mostramos que las ecuaciones de Einstein sobre M sin constante cosmológica y con fluido perfecto como fuente, pueden obtenerse a partir de las ecuaciones de campo para vacío con constante cosmológica sobre el haz fibrado principal $P\left(\frac{1}{I}M, \mathrm{U}(1)\right)$, donde M es el espacio-tiempo e I el radio del espacio interno $\mathrm{U}(1)$.

PACS: 11.10.Kk; 98.80.Dr

1. INTRODUCTION

In a recent work [1] it has been shown that vacuum solutions in scalar-tensor theories are equivalent to solutions of general relativity with imperfect fluid as source. The above models have the defect that the scalar fields do not arise from a natural framework of unification, but they are put by hand as in the inflationary models [2] and are therefore artificial fields in the theory. On the other hand, we know that the geometric formalism of principal fibre bundles [3, 4] is a natural scheme to unify the general relativity theory with gauge field theories (Abelian and Non-Abelian). If the principal fibre bundle $P(\tilde{M}, \mathrm{U}(1))$ is endowed with a metric "dimensionally reducible" to $\tilde{M}$ by means of the reduction theorem [5], *i.e.*, if the metric can be built out from quantities defined only on M, then the scalar fields arise in a natural way. Therefore, it is important to study the above model in the context of [1] for the particular principal fibre bundle $P(\frac{1}{I}M, \mathrm{U}(1))$, M being the space-time and I the scalar field. This paper is organized as follows: in the next section we review the geometric formalism of principal fibre bundles while in Sect. 3 we deduce the Einstein equations without cosmological constant and perfect fluid as source from the field equations on $P(\frac{1}{I}M, \mathrm{U}(1))$ for vacuum and cosmological constant. We give an example in Sect. 4 when $\tilde{M}$ is conformally FRW. Finally we summarize the results in Sect. 5.

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2. THE GEOMETRY

The actual version of the Kaluza-Klein theories is based on the mathematical stucture of principal fibre bundles [5, 6]. In this scheme, the unification of the general relativity theory with the gauge theories is a natural fact. Moreover, the reduction theorem provides a metric on (right) principal fibre bundles $P(\tilde{M}, G)$ which is right-invariant under the action of the structure group G on the whole space P. In the trivialization of the bundle this metric reads [5, 6]

$$\hat{g} = \tilde{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + \xi_{mn}\,(\omega^m + A^m_\alpha\, dx^\alpha) \otimes (\omega^n + A^n_\beta\, dx^\beta), \tag{1}$$

where the metric of the base space $\tilde{M}$ (generally identified with the space-time of general relativity) is $\tilde{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ while the metric on the fibre ($x^\alpha = \text{const.}$) is $\xi_{mn}\, \omega^m \otimes \omega^n$ and $\{\omega^m\}$ is a basis of right-invariant 1-forms on G. The quantities $\tilde{g}_{\alpha\beta}$, ξ_{mn} and A^n_α depend only on the coordinates on $\tilde{M}$ and the A^n_α correspond to Yang-Mills potentials in the gauge theory while the ξ_{mn} are the scalar fields.

In particular, the principal fibre bundle $P(\tilde{M}, \mathrm{U}(1))$ has the metric

$$\hat{g} = \tilde{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + I^2\,(d\psi + A_\alpha\, dx^\alpha) \otimes (d\psi + A_\beta\, dx^\beta), \tag{2}$$

where the scalar field I correspond to the radius of the internal space U(1) and ψ is the coordinate on U(1) too. However, the magnitude of the internal radius I depends on the particular cases; cosmological or astrophysical models (for details on units and magnitude on the scalar field I see Refs. [6, 7]). For vanishing electromagnetic potential, $A_\alpha = 0$, we obtain the unification of $\tilde{g}_{\alpha\beta}$ with the scalar field I:

$$\hat{g} = \tilde{g}_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + I^2\, d\psi^2. \tag{3}$$

By using Eq. (3) we compute the Ricci tensor

$$\hat{R}_{\alpha\beta} = \tilde{R}_{\alpha\beta} - I^{-1}\, I_{;\alpha\beta}, \tag{4}$$

$$\hat{R}_{\alpha 4} = 0, \tag{5}$$

$$\hat{R}_{44} = -I\, \Box\, I, \tag{6}$$

where greek indices run on $0, 1, 2, 3$ and the label "4" corresponds to the fifth dimension.

Usually the base space $\tilde{M}$ of $P(\tilde{M}, G)$ is identified as the space-time; in this paper we adopt the version where the base space $\tilde{M}$ of $P(\tilde{M}, \mathrm{U}(1))$ is conformally the space-time M of general relativity, *i.e.*, $\tilde{M} = \frac{1}{I} M$. That is to say, we start with the metric (compare Ref. [8]):

$$\hat{g} = \frac{1}{I} g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + I^2\, d\psi^2, \tag{7}$$

where $g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ is the space-time metric. Then by using Eq. (7) we obtain the Ricci tensor

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} + \tfrac{1}{2}\left(I^{-1}\,\Box I - I_{;\lambda}\, I^{;\lambda}\right) g_{\alpha\beta} - \tfrac{3}{2} I^{-2} I_{;\alpha}\, I_{;\beta}, \tag{8}$$

$$\hat{R}_{\alpha 4} = 0, \tag{9}$$

$$\hat{R}_{44} = -I^2\,\Box I + I_{;\lambda}\, I^{;\lambda}. \tag{10}$$

In what follows, we use the signature $(-,+,+,+)$ for the space-time metric on M.

3. Perfect fluid structure

The field equations on $P(\frac{1}{I}M, \mathrm{U}(1))$ in vacuum with cosmological constant Λ are given by $\hat{R}_{AB} - \frac{\hat{R}}{2}\,\hat{g}_{AB} = \Lambda\,\hat{g}_{AB}$ or in equivalent form

$$\hat{R}_{AB} = -\tfrac{2}{3}\Lambda\,\hat{g}_{AB}, \tag{11}$$

where A, B run on greek indices α and 4.

By using Eqs. (8)–(10) and (11) we obtain

$$R_{\alpha\beta} = I^{-2}\left(\tfrac{1}{2} I_{;\lambda} I^{;\lambda}\, g_{\alpha\beta} + \tfrac{3}{2} I_{;\alpha} I_{;\beta}\right) - I^{-1}\left(\tfrac{1}{2}\Box I + \tfrac{2}{3}\Lambda\right) g_{\alpha\beta}, \tag{12}$$

$$\Box I = \frac{1}{I} I_{;\lambda} I^{;\lambda} + \tfrac{2}{3}\Lambda. \tag{13}$$

By substituting the field equation for I [Eq. (13)] into the Ricci tensor [Eq. (12)] we obtain the equivalent system of equations

$$R_{\alpha\beta} = \tfrac{3}{2} I^{-2}\, I_{;\alpha}\, I_{;\beta} - I^{-1}\Lambda\, g_{\alpha\beta}, \tag{14}$$

$$\Box I = \frac{1}{I} I_{;\lambda} I^{;\lambda} + \tfrac{2}{3}\Lambda. \tag{15}$$

On the other hand, by using the Einstein equations without cosmological constant,

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{T}{2} g_{\alpha\beta}, \tag{16}$$

and Eq. (14), we can define the energy-momentum tensor associated with the scalar field I:

$$T_{\alpha\beta} = \tfrac{3}{2} I^{-2}\, I_{;\alpha}\, I_{;\beta} + \left(-\tfrac{3}{4} I^{-2} I_{;\lambda} I^{;\lambda} + I^{-1}\Lambda\right) g_{\alpha\beta}. \tag{17}$$

This energy-momentum tensor is covariantly conserved, ${T^{\alpha\beta}}_{;\beta} = 0$, as follows from the field equation for I. Finally, by comparing the above energy-momentum tensor associated with the scalar field I with that of an imperfect fluid:

$$T_{\alpha\beta} = \rho U_\alpha U_\beta + 2q_{(\alpha} U_{\beta)} + p\, h_{\alpha\beta} + \pi_{\alpha\beta}, \tag{18}$$

where ρ is the energy density of fluid, U_α the velocity, q_α the heat flux vector, p the pressure, $\pi_{\alpha\beta}$ the anisotropic stress tensor and

$$h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta \tag{19}$$

is the projection orthogonal to the velocity, we conclude [1]

$$q_\alpha = 0, \tag{20}$$

$$\pi_{\alpha\beta} = 0, \tag{21}$$

$$\rho = -\tfrac{3}{4} I^{-2} I_{;\lambda}\, I^{;\lambda} - \frac{\Lambda}{I}, \tag{22}$$

$$p = -\tfrac{3}{4} I^{-2} I_{;\lambda}\, I^{;\lambda} + \frac{\Lambda}{I}, \tag{23}$$

where the velocity has been choosen in the form [1]

$$U_\alpha \equiv \frac{I_{;\alpha}}{\sqrt{-I_{;\lambda}\, I^{;\lambda}}}. \tag{24}$$

That is to say, Eqs. (20)–(23) imply that Eq. (17) has the structure corresponding to a perfect fluid. Moreover, if $\Lambda = 0$ then Eqs. (17) and (20)–(23) correspond to the so called "Zeldovich ultrastiff matter" fluid, $p = \rho$ (see Ref. [1]).

4. EXAMPLE: THE P.F.B. $P(\frac{1}{I}\mathrm{FRW}, \mathrm{U}(1))$

We start from the metric

$$\hat{g} = \frac{1}{I(t)} \left[-dt^2 + R^2(t) \left(\frac{dr^2}{1 - k\, r^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right) \right] + I^2(t)\, d\psi^2, \tag{25}$$

where $I = I(t)$ on account of the isotropy and homogeneity of the FRW metric. In this case the Eqs. (14), (15) read

$$-3\left(\frac{\ddot{R}}{R}\right) = \frac{3}{2}\left(\frac{\dot{I}}{I}\right)^2 + \left(\frac{1}{I}\right)\Lambda, \tag{26}$$

$$2\left(\frac{k}{R^2}\right) + 2\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{\ddot{R}}{R}\right) = -\left(\frac{1}{I}\right)\Lambda, \tag{27}$$

$$3\left(\frac{\dot{R}}{R}\right)\left(\frac{\dot{I}}{I}\right) - \left(\frac{\dot{I}}{I}\right)^2 + \left(\frac{\ddot{I}}{I}\right) = -\frac{2}{3}\left(\frac{1}{I}\right)\Lambda, \tag{28}$$

where dot means derivation with respect to the cosmological time t. These equations are equivalent to the Einstein equations for FRW with perfect fluid as source, provided that

$$\rho = \frac{3}{4}\left(\frac{\dot{I}}{I}\right)^2 - \left(\frac{1}{I}\right)\Lambda, \tag{29}$$

$$p = \frac{3}{4}\left(\frac{\dot{I}}{I}\right)^2 + \left(\frac{1}{I}\right)\Lambda. \tag{30}$$

By the way, the field equation for I [Eq. (28)] is the covariant conservation of $T_{\alpha\beta}$, ${T^{\alpha\beta}}_{;\beta} = 0$

$$\dot{\rho} + 3\left(\frac{\dot{R}}{R}\right)(\rho + p) = 0. \tag{31}$$

5. Conclusion

We have shown that the field equations with cosmological constant Λ on the principal fibre bundle $P(\frac{1}{I}M, \mathrm{U}(1))$ are equivalent to the Einstein equations without cosmological constant on M and with perfect fluid as source. In order to show it, we start from the field equations on $P(\frac{1}{I}M, \mathrm{U}(1))$, $\hat{R}_{AB} = -\frac{2}{3}\Lambda\, \hat{g}_{AB}$, and separate them in their 4–dimensional and fifth dimension parts. We have found that from the 4–dimensional part of these equations it is possible to define an effective energy-momentum tensor $T_{\alpha\beta}$ and that it is covariantly conserved, being ${T^{\alpha\beta}}_{;\beta} = 0$ equivalent to the field equation for I. Finally, we applied the above result to the particular bundle $P(\frac{1}{I}\mathrm{FRW}, \mathrm{U}(1))$.

Acknowledgments

The authors are grateful to CONACyT for financial support.

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