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On the relativistic hydrogen atom

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ABSTRACT. In the present article we revisit the problem of a relativistic Dirac electron. Using a second order formalism which reduces the problem of finding the energy spectrum to solving the Whittaker equation, we show that the only physical solution is obtained by truncating the hypergeometric series. Therefore the energy spectrum does not depend on any free parameters for $119 < Z < 137$.

RESUMEN. En el presente artículo se estudia el problema de un electrón relativista de Dirac. Haciendo uso de un formalismo de segundo orden que reduce el problema de encontrar el espectro de energía a resolver la ecuación de Whittaker, se muestra que la única solución físicamente aceptable se obtiene truncando la serie hipergeométrica. Por consiguiente, el espectro de energía no depende de ningún parámetro libre para $119 < Z < 137$.

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1. INTRODUCTION

The Dirac equation is a system of four coupled partial differential equations which describes the relativistic electron and other spin 1/2 particles. Despite the considerable effort made during the last decades in order to find exact solutions describing the relativistic electron in the presence of external fields, the number of solvable problems is relatively few [1] Perhaps the most remarkable success of the Dirac theory of the electron is the prediction of the hydrogen atom energy spectrum [2, 3]. This problem has been extensively discussed in the literature [4–7], and different techniques have been suggested for its solution. Essentially, all the approaches, after separating the angular variables from the radial and time dependence, reduce the problem to solving a system of coupled first order differential equations, whose solution can be expressed, in almost all the standard representations of the Dirac matrices, as a combination of confluent hypergeometric functions. Motivated by the idea of obtaining a Dirac spinor solution similar to the one obtained in studying the free-electron problem in spherical coordinates, Biedenharn [8] has solved the Dirac-Coulomb problem in a representation which diagonalizes the Dirac $\hat{K}$ operator and gives as a result spinor solutions with a radial dependence having the same form as for the nonrelativistic Coulomb problem. Using the Biedenharn approach, the solutions for bound states of the Dirac-Coulomb equation have been computed by Wong and Yeh [9].

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Here the equations governing the radial dependence of the spinor are reduced to two decoupled second order Whittaker equations. Recently the Dirac-Coulomb problem has been revisited [10]. After some algebra, the authors transform the coupled system of differential equations for the radial dependence of the wave function into two second order Whittaker differential equations. They choose as solution of this system the functions $W_{\lambda,\mu}(z)$ [11], obtaining in this way, not only the standard hydrogen energy spectrum, but some physical implications for $119 < Z < 137$.

The purpose of the present article is twofold: first, we show that the reduction of the system of equations governing the radial Dirac wave solution to a system of Whittaker equations can be obtained with the help of a similarity transformation which gives as a result a solution equivalent to the one obtained by Wong [9]. Second, we show that the solutions reported in Ref. [10] are unphysical and therefore no anomalous behavior should be expected for $119 < Z < 137$.

2. Separation of variables in the Dirac equation

In this section we separate variables in the Dirac equation

$$\left\{\tilde{\gamma}^{\mu}(\partial_{\mu} - \Gamma_{\mu} - iA_{\mu}) + m\right\}\Psi = 0, \tag{1}$$

where for the Dirac-Coulomb problem the only nonzero component of the vector potential A_{μ} is $A_0 = -Ze^2/r$, the curved gamma matrices satisfy the commutation relations $\{\tilde{\gamma}^{\mu}, \tilde{\gamma}^{\nu}\}_{+} = 2g^{\mu\nu}$ and Γ_{μ} are the spin connections [12]. If we choose to work in the fixed Cartesian gauge, the spinor connections are zero and the $\tilde{\gamma}$ matrices take the form

$$\begin{aligned}
\tilde{\gamma}^0 &= \gamma^0 = \bar{\gamma}^0, \\
\tilde{\gamma}^1 &= \left[(\gamma^1 \cos\varphi + \gamma^2 \sin\varphi)\sin\vartheta + \gamma^3 \cos\vartheta\right] = \bar{\gamma}^1, \\
\tilde{\gamma}^2 &= \frac{1}{r}\left[(\gamma^1 \cos\varphi + \gamma^2 \sin\varphi)\cos\vartheta - \gamma^3 \sin\vartheta\right] = \frac{\bar{\gamma}^2}{r}, \\
\tilde{\gamma}^3 &= \frac{1}{r\sin\vartheta}(-\gamma^1 \sin\varphi + \gamma^2 \cos\varphi) = \frac{\bar{\gamma}^3}{r\sin\vartheta},
\end{aligned} \tag{2}$$

and the Dirac equation in the fixed tetrad frame (2) becomes

$$\left\{\bar{\gamma}^0(\partial_t + iV(r) + \bar{\gamma}^1\partial_r + \frac{\bar{\gamma}^2}{r}\partial_{\vartheta} + \frac{\bar{\gamma}^3}{r\sin\vartheta}\partial_{\varphi} + m\right\}\tilde{\Psi} = 0. \tag{3}$$

In order to separate variables in the Dirac equation, we are going to work in the diagonal tetrad gauge where the gamma matrices $\tilde{\gamma}_{\rm d}$ are

$$\tilde{\gamma}^0_{\rm d} = \gamma^0, \quad \tilde{\gamma}^1_{\rm d} = \gamma^1, \quad \tilde{\gamma}^2_{\rm d} = \frac{1}{r}\gamma^2, \quad \tilde{\gamma}^3_{\rm d} = \frac{1}{r\sin\vartheta}\gamma^3. \tag{4}$$

Since the curvilinear matrices $\tilde{\gamma}^{\mu}$ and $\tilde{\gamma}_{\mathrm{d}}$ satisfy the same anticommutation relations, they are related by a similarity transformation, unique up to a factor. In the present case we choose this factor in order to eliminate the spin connections in the resulting Dirac equation. The transformation S can be written as [13]

$$S = \frac{1}{r(\sin\vartheta)^{1/2}} \exp\left(-\frac{\varphi}{2}\gamma^1\gamma^2\right) \exp\left(-\frac{\vartheta}{2}\gamma^3\gamma^1\right) \mathsf{a} = \mathsf{S}_0\, \mathsf{a}, \tag{5}$$

where a is the constant non singular matrix given by $\mathsf{a} = \frac{1}{2}(\gamma^1\gamma^2 - \gamma^1\gamma^3 + \gamma^2\gamma^3 + I)$, which applied on the gammas's acts as follows:

$$\mathsf{a}\,\gamma^1\,\mathsf{a}^{-1} = \gamma^3, \quad \mathsf{a}\,\gamma^2\,\mathsf{a}^{-1} = \gamma^1, \quad \mathsf{a}\,\gamma^3\,\mathsf{a}^{-1} = \gamma^2, \tag{6}$$

the transformation S acts on the curvilinear $\tilde{\gamma}$ matrices, reducing them to the rotating diagonal gauge as follows:

$$S^{-1}\tilde{\gamma}^{\mu} S = g^{\mu\mu}\gamma^{\mu} = \tilde{\gamma}^{\mu}_{\mathrm{d}} \quad \text{(no summation)}, \tag{7}$$

then, the Dirac equation in spherical coordinates, with the radial potential $V(r)$, in a locally rotating frame reads

$$\left\{\gamma^0\partial_t + \gamma^1\partial_r + \frac{\gamma^2}{r}\partial_\vartheta + \frac{\gamma^3}{r\sin\vartheta}\partial_\varphi + m + i\gamma^0 V(r)\right\}\Psi = 0, \tag{8}$$

where we have introduced the spinor Ψ, related to $\tilde{\Psi}$ by the expression $\tilde{\Psi} = S\Psi = S_0\mathsf{a}\Psi$, and γ^{μ} are the standard Dirac flat matrices.

Applying the algebraic method of separation of variables [14, 15], it is possible to write Eq. (8) as a sum of two first order linear differential operators $\hat{K}_1$, $\hat{K}_2$ satisfying the relation

$$\left[\hat{K}_1, \hat{K}_2\right] = 0, \qquad \left\{\hat{K}_1 + \hat{K}_2\right\}\Phi = 0, \tag{9}$$

$$\hat{K}_1\Phi = \lambda\Phi = -\hat{K}_2\Phi, \tag{10}$$

then, if we separate the time and radial dependence from the angular one, we obtain

$$\hat{K}_2\Phi = \left[\gamma^2\partial_\vartheta + \frac{\gamma^3}{\sin\vartheta}\partial_\varphi\right]\gamma^0\gamma^1\Phi = -i\kappa\Phi, \tag{11}$$

$$\hat{K}_1\Phi = r\left[\gamma^0\partial_t + \gamma^1\partial_r + m + i\gamma^0 V\right]\gamma^0\gamma^1\Phi = i\kappa\Phi, \tag{12}$$

with $\Psi = \gamma^0\gamma^1\Phi$, where we have made the identification $i\kappa = \lambda$. Notice that (11) is the angular momentum $\hat{K}$ obtained by Brill and Wheeler [12].

Here it is necessary to remark that the operator $\hat{K}_2$ appearing in (11) is not single valued and does not satisfy the properties of a "good" angular momentum operator. The true angular operator should be obtained from $\hat{K}_2$ with the help of the transformation S (5)

$$S\left[\gamma^2\partial_\vartheta + \frac{\gamma^3}{\sin\vartheta}\partial_\varphi\right]\gamma^0\gamma^1 S^{-1}S\Phi = -i\kappa S\Phi; \tag{13}$$

since $S\gamma^\mu S^{-1} = \bar{\gamma}^\mu$, we obtain

$$\left[\bar{\gamma}^2\bar{\gamma}^0\bar{\gamma}^1\partial_\vartheta + \frac{\bar{\gamma}^3\bar{\gamma}^0\bar{\gamma}^1}{\sin\vartheta}\partial_\varphi + \bar{\gamma}^2\bar{\gamma}^0\bar{\gamma}^1 S\partial_\vartheta S^{-1} + \frac{\bar{\gamma}^3\bar{\gamma}^0\bar{\gamma}^1}{\sin\vartheta}S\partial_\varphi S^{-1}\right]\tilde{\Phi} = -i\kappa\tilde{\Phi}, \tag{14}$$

where $S\Phi = \tilde{\Phi}$.

Using the explicit form of S given by (5) we have

$$S\partial_\vartheta S^{-1} = \tfrac{1}{2}\cot\vartheta + \tfrac{1}{2}(\cos\varphi - \gamma^1\gamma^2\sin\varphi)\gamma^3\gamma^1 \quad S\partial_\varphi S^{-1} = \tfrac{1}{2}\gamma^1\gamma^2; \tag{15}$$

substituting (15) into (14)

$$\left[\bar{\gamma}^0\bar{\gamma}^2\bar{\gamma}^1\partial_\vartheta + \frac{\bar{\gamma}^0\bar{\gamma}^3\bar{\gamma}^1}{\sin\vartheta}\partial_\varphi + \bar{\gamma}^0\right]\tilde{\Phi} = i\kappa\tilde{\Phi}, \tag{16}$$

expression that can be written as follows:

$$\hat{K} = \gamma^0\left[(\sigma\hat{\mathbf{L}}) + 1\right]. \tag{17}$$

The operator $\hat{K}$ satisfies the relation

$$\hat{K}^2 = \hat{J}^2 + \tfrac{1}{4}, \quad \hat{J}^2 = \left(L + \tfrac{1}{2}\boldsymbol{\sigma}\right)^2, \tag{18}$$

and the eigenvalue k is related to j as follows:

$$k^2 = j(j+1) + \tfrac{1}{4} = \left(j + \tfrac{1}{2}\right)^2. \tag{19}$$

3. Solution of the radial equation

In order to reduce Eq. (12) to a system of ordinary differential equations, we choose to work in the following representation of the gamma matrices [16]:

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{20}$$

Then, substituting (20) into (12) we obtain

$$\left(d_r - \frac{k}{r}\right)\Phi_1 + \sigma_1(E - V - m)\Phi_2 = 0, \tag{21}$$

$$\left(d_r + \frac{k}{r}\right)\Phi_2 - \sigma_1(E - V + m)\Phi_1 = 0, \tag{22}$$

where the spinor Φ has the form

$$\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \tag{23}$$

Taking into account the structure of the system of Eqs. (21)–(22), we can gather the first and forth components of Φ, $\Phi_1 = \xi$, $\Phi_4 = \zeta$ as follows:

$$\left(d_r - \frac{k}{r}\right)\xi + (E - V - m)\zeta = 0, \tag{24}$$

$$\left(d_r + \frac{k}{r}\right)\zeta - (E - V + m)\xi = 0; \tag{25}$$

the system of Eqs. (24)–(25), after substituting $V = -Z\alpha/r$, is just the standard one governing the radial dependence of the wave equation in Dirac-Coulomb problem. After introducing the auxiliary spinor Θ defined by

$$\Theta = \begin{pmatrix}\xi\\ \zeta\end{pmatrix}, \tag{26}$$

the system (24)–(25) reduces to

$$\left(d_r + m\sigma_1 + i(E - V)\sigma_2 - \frac{k}{r}\sigma_3\right)\Theta = 0. \tag{27}$$

Here, in order to solve (27), we do not proceed in the standard way [17, 5], but with the help a similarity transformation T, we try to reduce (27) to an equivalent system of Whittaker equations. Here the idea is to present a solution, as in the nonrelativistic case, where the components of the spinor solution are not a sum of two special functions. The transformation T can be written as follows [10]:

$$\begin{aligned} T &= \begin{pmatrix}\sqrt{m+E} & -\sqrt{m-E}\\ \sqrt{m+E} & \sqrt{m-E}\end{pmatrix} \\ &= \frac{1}{2}\left[\left(\sqrt{m+E} + \sqrt{m-E}\right)(1 - i\sigma_2) + \left(\sqrt{m+E} - \sqrt{m-E}\right)(\sigma_3 + \sigma_1)\right]. \end{aligned} \tag{28}$$

The matrix transformation T (28) acts on the Pauli matrices σ_i as follows:

$$\begin{aligned}
T\sigma_3 T^{-1} &= \sigma_1, \\
T\sigma_2 T^{-1} &= \frac{1}{2}\left[\sigma_2\left(\frac{\sqrt{m+E}}{\sqrt{m-E}}+\frac{\sqrt{m-E}}{\sqrt{m+E}}\right)+i\sigma_3\left(\frac{\sqrt{m+E}}{\sqrt{m-E}}-\frac{\sqrt{m-E}}{\sqrt{m+E}}\right)\right], \\
T\sigma_1 T^{-1} &= \frac{1}{2}\left[i\sigma_2\left(\frac{\sqrt{m+E}}{\sqrt{m-E}}-\frac{\sqrt{m-E}}{\sqrt{m+E}}\right)-\sigma_3\left(\frac{\sqrt{m+E}}{\sqrt{m-E}}+\frac{\sqrt{m-E}}{\sqrt{m+E}}\right)\right],
\end{aligned} \tag{29}$$

then, after substituting (29) into (27) and introducing the variable $\rho = 2\sqrt{m^2-E^2}r$, we arrive at

$$\left(\rho\frac{d}{d\rho}-\frac{Z\alpha E}{\sqrt{m^2-E^2}}+\frac{\rho}{2}\right)\Phi_+ + \left(\frac{mZ\alpha}{\sqrt{m^2-E^2}}-k\right)\Phi_- = 0, \tag{30}$$

$$\left(\rho\frac{d}{d\rho}+\frac{Z\alpha E}{\sqrt{m^2-E^2}}-\frac{\rho}{2}\right)\Phi_- - \left(\frac{mZ\alpha}{\sqrt{m^2-E^2}}+k\right)\Phi_- = 0, \tag{31}$$

where

$$T\Theta = \begin{pmatrix}\Phi_+ \\ \Phi_-\end{pmatrix}. \tag{32}$$

Substituting (30) into (31) and vice-versa, we obtain the following Whittaker equation:

$$\left(\frac{d^2}{d\rho^2}-\frac{1}{4}+\left(\frac{Z\alpha E}{\sqrt{m^2-E^2}}\pm\frac{1}{2}\right)\frac{1}{\rho}+\frac{Z^2\alpha^2-k^2+\frac{1}{4}}{\rho^2}\right)\begin{pmatrix}\hat{\Phi}_+ \\ \hat{\Phi}_-\end{pmatrix} = 0, \tag{33}$$

where we have made the substitution $\hat{\Phi}_\pm = \rho^{1/2}\Phi_\pm$. The solutions of (33) are given by

$$\hat{\Phi}_\pm = c_\pm \rho^{\mu+1/2}\, e^{-\rho/2}\, M\left(\left(\mu-\left(\frac{Z\alpha E}{\sqrt{m^2-E^2}}\right)\pm\frac{1}{2}\right)+\frac{1}{2},\, 2\mu+1,\, \rho\right), \tag{34}$$

with $\mu = \sqrt{k^2-Z^2\alpha^2}$, where $M(a,c,z)$ is the confluent hypergeometric function which is regular at the origin and $c_\pm$ are constant coefficients. Substituting (34) into (30) we find

$$c_+ = \frac{k-\dfrac{mZ\alpha}{\sqrt{m^2-E^2}}}{\mu-\dfrac{Z\alpha E}{\sqrt{m^2-E^2}}}\,c_-. \tag{35}$$

Since the hypergeometric series diverges as $\rho\to\infty$, we have that

$$\mu-\frac{Z\alpha E}{\sqrt{m^2-E^2}} = -n, \tag{36}$$

where n is a nonnegative integer. Then we obtain

$$E = m\left(1 + \frac{Z^2\alpha^2}{(\mu+n)^2}\right)^{-1/2} \tag{37}$$

Here it is worth mentioning that the confluent hypergeometric $M(a,c,z)$ functions reduce to Laguerre polynomials when condition (36) is satisfied. In this case we obtain a orthonormal set of eigenfunctions. If we try to use the second family of functions $U(a,c,z)$ [10] in order to solve (33) we will not be able to find well behaved solutions in the asymptotic regions. In fact, considering the functions $U(a,c,z)$, the solution of the system of equations (30)–(31) takes the form

$$\Phi_\pm = d_\pm \rho^\mu e^{-\rho/2} U\left(\left(\mu - \left(\frac{Z\alpha E}{\sqrt{m^2-E^2}}\right) \pm \frac{1}{2}\right) + \frac{1}{2},\, 2\mu+1,\, \rho\right), \tag{38}$$

where $d_\pm$ are constants to be fixed from (30) and (31). Using the recurrence relation

$$a(c-a-1)U(a+1,c,z) + aU(a,c,z) + z\frac{dU(a,c,z)}{dz} = 0, \tag{39}$$

we obtain

$$d_+ = \frac{k - \dfrac{mZ\alpha}{\sqrt{m^2-E^2}}}{\mu^2 - \dfrac{Z^2\alpha^2E^2}{E^2-m^2}}\, d_-. \tag{40}$$

Since the transformation T does not depend on coordinates, we have that the components $\Phi_\pm$ of the spinor (38) should satisfy

$$\int_0^\infty (\Phi_\pm)^2\, dr < \infty. \tag{41}$$

Expression (41) is a trivial consequence of the normalization condition on the spinor wave function. The absence of a weight function in the product is the result of working in the locally rotating frame. Since the functions $U(a,c,z)$ have the following asymptotic behavior for small z and $\mathrm{Re}\, c > 0$:

$$U(a,c,z) \to z^{1-c}\frac{\Gamma(c-1)}{\Gamma(a)}, \tag{42}$$

then, in the vicinity of zero, Φ_+ must take the form

$$\Phi_+ \to d_\pm \rho^{-\mu} e^{-\rho/2} \frac{\Gamma(2\mu)}{\Gamma\left(\mu - \dfrac{Z\alpha E}{\sqrt{m^2-E^2}}\right)}; \tag{43}$$

a solution that makes the integral diverge (41) at zero unless $\mu < 1/2$ or $a = -n$. Taking into account that $\mu > 0$, we obtain $0 < k^2 - Z^2\alpha^2 < 1/4$, which implies

$$\frac{k}{\alpha} > Z > \frac{\sqrt{k^2 - \frac{1}{4}}}{\alpha}. \tag{44}$$

Since $k = \pm 1, \pm 2, \pm 3, \ldots$, we have that for $k = 1$, no polynomials regular solutions of the form (38) for the Whittaker equation are possible if

$$118.6 < Z < 137. \tag{45}$$

Here it is necessary to point out that the Wronskian of the solutions gives

$$W\{M(a,c,z), U(a,c,z)\} = -\frac{\Gamma(c)}{\Gamma(a)}\, z^{-c}\, e^{z}, \tag{46}$$

and the relation for c$\neq 0, \pm 1, \pm 2,$

$$U(a,c,z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)} M(a,c,z) + \frac{\Gamma(c-1)}{\Gamma(a)}\, z^{1-c}\, M(1+a-c, 2-c, z). \tag{47}$$

$M(a,c,z)$, and $U(a,c,z)$ are linearly dependent for nonnegative integer values of $-a$. Therefore, non polynomial regular solutions of the Whittaker equation (44) are only possible in terms of $U(a,c,z)$ if the condition (44) is fulfilled.

Now we proceed to analyze some properties of the anomalous solution found. This solution has been reported in the literature by Armstrong [18], and more recently has been quoted by de Lange and Raab [19], but it is also the solution presented in Ref. [10]. The expression (38) renders their solutions physically unsatisfactory. The first problem is found when we try to verify the orthogonality property. In fact, with the help of the definite integral [20]

$$\int_0^\infty x^{-1}\, W_{k,\mu}(x)\, W_{\lambda,\mu}(x)\, dx = \frac{1}{(k-\lambda)\sin(2\pi\mu)}$$
$$\times \left[\frac{1}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-\lambda-\mu)} - \frac{1}{\Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}-\lambda+\mu)}\right], \quad |\mathrm{Re}\,\mu| < \tfrac{1}{2}, \tag{48}$$

where $W_{k,\mu}(x)$ is given by the expression $W_{k,\mu}(x) = x^{\mu+1/2} e^{-x/2} U(\frac{1}{2} - k + \mu, 2\mu + 1, x)$, then putting $k = \mp\frac{1}{2} + \frac{Z\alpha E}{\sqrt{m^2 - E^2}}$, $x = \rho$ we find that the functions $\Phi_\pm$ given by (38) and related to the solution of the Whittaker equation (33) by $\hat{\Phi}_\pm = \rho^{1/2}\Phi_\pm$ do not satisfy the orthogonality relation

$$\int_0^\infty \Phi_\pm(k,\mu,\rho)\, \Phi_\pm(\lambda,\mu,\rho)\, d\rho = 0, \text{ for } k \neq \lambda. \tag{49}$$

This result means that two solutions of the Dirac equation associated with two different values of the energy, with a radial dependence given by (38), and with $\mu < 1/2$, are not orthogonal unless the condition (36) is satisfied. Also we have that the expectation value of the Coulomb potential in the basis given by (38)is not finite. In fact, the integral

$$2Z\alpha\sqrt{m^2 - E^2}\int_0^\infty \frac{Z\alpha}{\rho}\,\Phi_\pm(k,\mu,\rho)\,\Phi_\pm(\lambda,\mu,\rho)\,d\rho \tag{50}$$

is divergent because the argument has the following behavior as $\rho \to 0$:

$$c\rho^{-2\mu-1}\,e^{-\rho}; \tag{51}$$

the same situation occurs when we compute the kinetic energy of the Dirac particle. Therefore, the solutions of the Dirac equation given in terms of $U(a,c,z)$ with $\mu < 1/2$ should be regarded as non physical, consequently, no anomalous behavior should be expected for $119 < Z < 137$.

In order to conclude this section we have to say that the result obtained by Wong [9], for the wave spinor of the Dirac Coulomb-Problem, can be obtained from (32) after applying the transformation Q:

$$Q = \frac{1}{2}\left[\left(m - \frac{Ek}{\lambda}\right)^{1/2} + \left(m + \frac{Ek}{\lambda}\right)^{1/2}\frac{c_-}{c_+}\right] + \frac{1}{2}\left[\left(m - \frac{Ek}{\lambda}\right)^{1/2} - \left(m + \frac{Ek}{\lambda}\right)^{1/2}\frac{c_-}{c_+}\right]\sigma_3. \tag{52}$$

The following matrix transformation allows us to work with the two component spinor Θ (32) instead of Φ, we have

$$\mathcal{L} = \tfrac{1}{2}(1 + \gamma^3 - i\gamma^1\gamma^2 - i\gamma^1\gamma^2\gamma^3), \tag{53}$$

this transforms the spinor Φ (23) as follows:

$$\mathcal{L}\Phi = \mathcal{L}\begin{pmatrix}\xi\\ \chi\\ \eta\\ \zeta\end{pmatrix} = \begin{pmatrix}\xi\\ \zeta\\ \eta\\ \chi\end{pmatrix}, \tag{54}$$

and therefore the upper two components of (54) satisfy the coupled system of equations (30)–(31). Notice that $\mathcal{L}^{-1} = \mathcal{L}$ and therefore it is straightforward to go back to the original representation.

4. CONCLUSIONS

In this article we have rederived the energy spectrum of the relativistic Hydrogen atom. In order to separate variables we have worked in the diagonal (rotating) tetrad gauge. After separating variables, we have reduced the radial equations to a system of Whittaker equations without introducing the Lippmann and Johnson [21] operator. Also we have shown that the energy spectrum obtained in this way is that already reported in the literature. The result presented by Cohen and Kuharetz [10] on the dependence of the energy spectrum on a free parameter for $119 < Z < 137$ is shown to be unphysical. Some comments regarding the self adjointness of the Dirac Hamiltonian are in order. In fact, as it was pointed out by different authors [22,23] the Dirac operator is essentially self adjoint for $Z < 119$. Any self adjoint extension for $119 < Z < 137$ requires a clear physical meaning. A good criterion for that selection is that expectation value of each component of the Hamiltonian be finite in the basis selected. In particular the wave functions should possess finite kinetic energy [24]. It is straigthforward to verify that solution (38) fails to satisfy the finiteness condition of the kinetic energy, and the expectation value of the Coulomb potential is also divergent (50). A self adjoint extension in the basis of the hypergeometric functions given by (34) is physically acceptable for $Z < 137$.

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