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A quasi-statistical approach to digital binary image representation

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Abstract. A new quasi-statistical description of digital binary patterns is presented here. The method is based on mapping the configuration space of a primary image onto the smaller configuration space of an auxiliary system called here local scanning basis. A theorem is proved which establishes the relationship between the new representation and the correlation moments of an image-signal. The method's application to description, classification and possible encoding of patterns, especially to ones having texture and/or local symmetry, is discussed here on a basic level.

Resumen. Presentamos una nueva descripción cuasi-estadística de los patrones digitales binarios. El método se basa en el mapeo del espacio de configuración de una imagen primaria en un espacio de configuración menor de un sistema auxiliar que llamamos base local de análisis (*local scanning basis*). Demostramos aquí un teorema que establece la relación entre la nueva representación y los momentos de correlación de una imagen-señal. La aplicación del método a la descripción, clasificación y codificación posible de patrones, especialmente aquellos que poseen textura y/o simetría local, es indicada aquí a un nivel básico.

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1. Introduction

In order to be digitally processed, any real image-signal has to be prepared in appropriate manner and, for that reason, a signal passes through the following stages: sampling of a signal and quantization of samples to any of K discrete levels.

Let us consider a possible way for preparing a full-color picture to be digitally processed. In order to simplify the consideration we can first perform spectral filtering of a picture and, thus, to get the latter as a monochrome picture image. A continuous monochrome intensity distribution of the image can be sampled to convert the image into a set of continuous-valued functions on a grid, which has been chosen for sampling. The next step is to divide the range of the intensity variation into K discrete levels, and then to prescribe the intensity on each element of the grid to one of K discrete levels to have the intensity distribution quantized. This is a typical way of picture digitalization. Being prepared in this way, an image-signal is kept in matrix or vector space representation, which are the basic representations [1–6].

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In general, a picture Σ is partitioned according to the characteristic features, which are usually the equivalence relations, into a collection of nonempty subsets $S_1, S_2, \ldots, S_K$ such that $\cup_{i=1}^{K} S_i = \Sigma$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. Any partition of Σ into segments $S_1, S_2, \ldots, S_K$ can be represented by K–valued function having i's at the points of the subset S_i, $1 \leq i \leq K$. For example, each S_i is the set of pixels of Σ having gray level i, when Σ is partitioned into sets of constant gray level. For a picture of $L \times M$ pixels this representation requires $(L \times M) \log_2 K$ bits. The storage requirements of this basic representation are the same for all partitions of Σ into a given number of sets. An important special case occurs when $K = 2$, *i.e.*, Σ consists of a set S and its complement $\bar{S}$. This is the case of the binary representation of a digital picture Σ, having 1's at the points of S and 0's elsewhere (or ± 1, respectively).

Finding and studying representations which are more efficient for some special partitions is a problem of great practical interest. Referring to [2] for a more detailed description of representations, we only list here such of them as runs, binary trees, maximal blocks and quad trees which are based on maximal runs or maximal blocks of constant value. These approaches represent each S_i as the union of maximal runs or blocks that are contained in it. Another class of approaches to representation makes use of the fact that the sets S_i are determined by specifying their border sequences. The feature we would like to emphasize here is that these representations are derived from the primary matrix representation and they are one to one mappings.

It is well recognized [1–6] that the problem of finding an adequate and simple enough representation for digitized patterns is very much significant for image processing and computer vision due to the large amount of information to be stored and/or processed, providing efficient noise protection of information. A certain motivation for searching new representations can be found in the problem of encoding digital data including error-control techniques for digital communication [7, 12, 13]. One of the principal conclusions of Ref. [7] is that encoding binary data by noise-like images is preferable at low signal to noise ratios, and it provides extremely low probabilities of error for information storage and transmission. That is why it is very useful and convenient to have such a representation that combines deterministic and random features of image signal analysis, especially when one needs to analyze textured and/or speckle patterns [10, 11]. In addition, this representation could be useful for purposes of hierarchical organization and processing of information, specially for problems of image-signal classification.

We propose one of the possible approaches for solving the problem indicated above. The approach to the picture description, which we develop here, stems out of the coordinated cluster representation (CCR) for noncrystalline solids [8, 9]. Our goal here is to introduce the idea and discuss the fundamentals of the new representation for digital data processing.

In this work we consider binary digital images, that is, finite arrays of two-valued functions. For example, in the two dimensional case these images are naturally represented by (rectangular) matrices $S^{\alpha} = \{s_{lm}^{\alpha}\}$ or vectors. The last representation is readily obtained from the matrix one. Each element s_{lm}^{α} $(l = 1, \ldots, L;\ m = 1, \ldots, M)$ of the matrix can take on one of the two values $s_{lm}^{\alpha} = \pm 1$ (black or white colored pixel for $+1$ and -1 value, respectively). Note that, when it does not cause a confusion, we will sometimes use the values "1" and "0" for the elements s_{lm}^{α}. The Greek superscript α labels here different

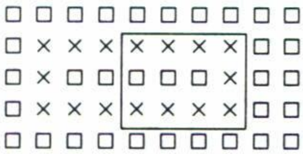

FIGURE 1. Local Scanning Basis. Given is the pattern of the size $L \times M = 5 \times 10$ pixels with the rectangle of the size 3×7 pixels as the object. LSB of the size 3×4 pixels is placed in the position with the left upper corner at the point $(2, 5)$.

images and takes on values in the range $\alpha = 1, \ldots, 2^N$, since configurational space of N numerated binary pixels consists of 2^N distinguishable patterns.

2. BASIC CONCEPTS OF THE REPRESENTATION

Now we proceed to the formal description of the transform, which we call the coordinated cluster representation (CCR). This transform is developed here only for binary signals.

Let us suppose that a binary image is given as a 2-D array of $L \times M = N$ pixels. This image can be represented by a $(L \times M)$ matrix of the binary image intensities $S^\alpha = \{s^\alpha_{lm}\}$. Let us introduce into consideration an auxiliary system of pixels, called here local scanning basis (LSB). LSB is a smaller size $I \times J = N_1$ matrix $B = \{b_{ij}\}$ superimposed for scanning onto the primary matrix S^α ($i = 1, \ldots, I \le L$ and $j = 1, \ldots, J \le M$). At each given position of the matrix B onto the primary pattern, while scanning consecutively with one position step (or with a few steps, if it is desired) over all the rows and the columns of the primary image, the elements b_{ij} take on the values of the associated s^α_{lm} they coincide with (see Fig. 1). In some sense the LSB can be considered as a rectangular scanning "window" with the linear sizes of $I \times J$ pixels.

The configuration space of LSB consists of all possible combinations (states, configurations) of the set of N_1 numerated binary pixels. There are 2^{N_1} such configurations. While scanning with the LSB, as it has been just described, one meets different states of the matrix B and keeps in mind the frequencies of occurrence of each state. In this way we can introduce into consideration a so called distribution function, which is defined on the configuration space of the LSB. This function gives the frequencies of occurrence of each possible configuration of the LSB while scanning over all the primary image. Thus we get the transform of a pattern $S^\alpha = \{s^\alpha_{lm}\}$ into the space of distribution functions (DF) defined on the configuration space of the system $B = \{b_{ij}\}$. In other words we associate any given picture $S^\alpha = \{s^\alpha_{lm}\}$ with its corresponding DF $F^\alpha_{(I,J)}(b)$ of the discrete argument b which, in fact, numbers the states of the system B, *i.e.*, b runs through the range of $b = 1, 2, \ldots, 2^{N_1}$. The subscript of DF explicitly indicates the size of LSB taken for scanning of an image. The transform, which has been just defined, is the core of the new description (representation) of a digital image-signal.

Configuration spaces of digitized patterns, which we used to work with, are countable (even finite) ones and, therefore, the states of any given space can be easily numbered. A simple way to number the states is, first, to convert a 2-D image into the vector form by column (or row) stacking operation of an image matrix $S^\alpha = \{s^\alpha_{lm}\}$ and stringing the elements together in a long vector (1-D object). Then, a vector with binary components is naturally considered as a number on the base 2, which can be readily converted into a decimal number. Therefore, hereafter for the configuration space of any given LSB we use numbers of states as representation of corresponding states.

It should be noted that the scanning procedure, which has been described above, can also be considered as a template matching, where the templates are sequentially taken out of the configuration space of LSB.

The considered map is related to the problem of covering a pattern with overlapping covering elements, which have been called coordinated clusters in Refs. [8, 9]. In the case of digital patterns the coordinated clusters are simply configurations of LSB of the size $I \times J = N_1$. For the case of one step scanning the following relation,

$$\sum_b F^\alpha_{(I,J)}(b) = (L - I + 1) \times (M - J + 1) \equiv A_1, \tag{1}$$

is valid for any pattern S^α. Equation (1) gives the sum of DF over the configuration space of the scanning system $B = \{b_{ij}\}$ via the number of positions which the "window" B occupies while scanning a pattern S^α. On can easily see that the distribution function $F^\alpha_{(I,J)}(b)$ is, in fact, a histogram on the configuration space of LSB, and, hence, the sum in Eq. (1) is the area of this histogram.

The distribution function $F^\alpha_{(I,J)}(b)$ is the only representation of a pattern S^α, if the LSB is chosen. So it is natural to discriminate pictures and find a resemblance between them via their DFs. If patterns S^α and S^β are represented by matrices $S^\alpha = \{s^\alpha_{lm}\}$ and $S^\beta = \{s^\beta_{lm}\}$, the similarity between them is usually defined by the Hamming normalized distance

$$d_H(S^\alpha, S^\beta) = (2N)^{-1} \sum_{l,m} |s^\alpha_{lm} - s^\beta_{lm}|. \tag{2}$$

In the case of CCR we define, correspondingly, un- and normalized distances between pictures as follows:

$$\rho(S^\alpha, S^\beta) = \sum_b |F^\alpha_{(I,J)}(b) - F^\beta_{(I,J)}(b)|, \tag{3}$$

$$\hat{\rho}(S^\alpha, S^\beta) = (2A_1)^{-1} \sum_b |F^\alpha_{(I,J)}(b) - F^\beta_{(I,J)}(b)|, \tag{4}$$

where the normalization factor A_1 is given by Eq. (1).

The unnormalized measure $\rho(S^\alpha, S^\beta)$ can be used not only for the patterns of equal size but even for those of unequal size. Due to this particularity the measure is not restricted, *i.e.*, the distance between unequal size pictures can be arbitrary large. The normalized measure $\hat{\rho}(S^\alpha, S^\beta)$ is applied only to the equal size pictures and it is restricted by the value $\hat{\rho}(S^\alpha, S^\beta) \leq 1$.

After the new description of digital images has been formally defined, we may now proceed to the consideration of fundamental properties of the representation and its possible applications.

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FIGURE 2. Distribution functions of the given pattern for four different LSB of the sizes $\{I \times J\} = \{1 \times 1; 1 \times 2; 2 \times 1; 3 \times 3\}$.
1. $F_{(1,1)}(0) = 25$, $F_{(1,1)}(1) = 10$. 2. $F_{(1,2)}(0) = 16$, $F_{(1,2)}(1) = 4$, $F_{(1,2)}(2) = 4$, $F_{(1,2)}(3) = 6$. 3. $F_{(2,1)}(0) = 12$, $F_{(2,1)}(1) = 6$, $F_{(2,1)}(2) = 6$, $F_{(2,1)}(3) = 4$. 4. $F_{(3,3)}(26) = 1$, $F_{(3,3)}(36) = 1$, $F_{(3,3)}(50) = 1$, $F_{(3,3)}(57) = 1$, $F_{(3,3)}(60) = 1$, $F_{(3,3)}(120) = 1$, $F_{(3,3)}(152) = 1$, $F_{(3,3)}(176) = 1$, $F_{(3,3)}(211) = 1$, $F_{(3,3)}(288) = 1$, $F_{(3,3)}(292) = 1$, $F_{(3,3)}(312) = 1$, $F_{(3,3)}(406) = 1$, $F_{(3,3)}(463) = 1$, $F_{(3,3)}(487) = 1$.

3. ILLUSTRATION AND DISCUSSION OF BASIC APPLICATIONS BY SIMULATION

3.1. Uniqueness and different LSB

The transformation considered above is not, in principle, unambiguous. But the connectivity of the representation plays a significant role for the uniqueness of the considered mapping and, hence, for converting the CCR into another representation. One can suppose that the greater the overlapping of LSB in adjacent sites, the more the uniqueness of the "representation".

It is easy to see that the smallest possible size of the scanning basis is equal to $I \times J = 1 \times 1 = 1$. In this case there is no overlapping between adjacent covering elements and all the patterns with equal number of black and white pixels are not distinguishable, since they have the same distribution function. So, all patterns are separated into the equivalent classes —each class contains all images having a common DF, which simply gives us the numbers of black and white pixels. This limit case of the representation contains little information about pictures.

However, if we increase the size of LSB and, consequently, the connectivity of covering, then in the opposite extreme case, when $I = L$ and $J = M$, we certainly have one to one map. In an intermediate case, if the size of LSB is reasonably chosen, one can keep almost all the information about any taken picture via its distribution function. More will be said in our consideration of translationally invariant (periodic) signals. The detailed analysis of the uniqueness problem and that of the questions related with it needs to be done, and it is now under intensive study.

To demonstrate this transform in the process and to emphasize its main features we give below this pattern representation in application to pictures with rather simple synthesized geometrical objects.

Figure 2 shows picture of the field size $L \times M = 5 \times 7$ pixels with the rectangle of the size $Yr \times Xr = 3 \times 4$, taken as the object. The left upper corner of the rectangle is at the point $(Yr_0, Xr_0) = (2, 3)$. The pattern in Fig. 2 is followed by the set of DFs, each being the distribution function of the considered picture but for different LSB. We take here four different LSB of the size $\{I \times J\} = \{1 \times 1; 1 \times 2; 2 \times 1; 3 \times 3\}$ pixels, correspondingly.

The configuration space of the smallest size LSB with $I = J = 1$ consists of two states —one is black and the other is white pixel. The DF corresponding to this situation gives us simply the number of black and white pixels in the picture. Other distribution functions corresponding to increasing sizes of LSBs give us examples of different possible representations of the pattern. The choice of a LSB's dimension for the representation depends on the spatial scale of an image texture or a temporal scale of a signal to investigate, and degree of uniqueness needed. If there is no a priori knowledge of the scale, in order to find out the latter, one can choose sequentially different sizes of LSB. Moreover, one can take a combination of distribution functions corresponding to different LSBs for simultaneous pattern representation —this increases uniqueness. A detailed discussion of this question is out of scope of this work. The set of LSB can be easily extended till the largest size $I = L = 5$ and $J = M = 7$, when one has as simple distribution function as

$$F^{\alpha}_{(5,7)}(b) = \begin{cases} 0, & \text{if } b \neq b^{\alpha}, \\ 1, & \text{if } b = b^{\alpha}, \end{cases} \tag{5}$$

where b^{α} is a number in the range $b = 1, \ldots, 2^{35}$ matching the configuration S^{α}. This limit case of LSB gives us an example of one to one map as it has been mentioned above.

3.2. *Texture classification and discrimination*

A typical scene usually contains various objects whose appearance displays a microstructure or a texture. That is one reason why the synthesis and classification of textures is a fundamental task in image analysis applications. Without going to depth, we give here a simple example of artificial structure and discuss capabilities of the CCR for problems of texture analysis and classification.

Let us consider a 2-D chessboard texture on the rectangular lattice $S = \{s_{lm}: 1 \leq l \leq L,\, 1 \leq m \leq M\}$ of $L \times M = N$ sites (pixels). In matrix representation this pattern is given with the matrix $S = \{s_{lm}: s_{lm} = ((-1)^{l+m} - 1)/2\}$, where zero and unit value matrix elements can be considered as white and black pixels, correspondingly.

The DFs for chessboard image, *i.e.* $L = M = 8$, for different LSB of the size $\{I \times J\} = \{1 \times 3; 3 \times 1; 3 \times 3; 3 \times 4\}$ and the only configurations of LSB which contribute to the corresponding DF are as follows:

1. $F_{(1,3)}(2) = 24$, $F_{(1,3)}(5) = 24$, (2) $\square\times\square$, (5) $\times\square\times$.

2. $F_{(3,1)}(2) = 24$, $F_{(3,1)}(5) = 24$, (2) $\begin{matrix}\square\\ \times\\ \square\end{matrix}$, (5) $\begin{matrix}\times\\ \square\\ \times\end{matrix}$.

3. $F_{(3,3)}(170) = 18$, $F_{(3,3)}(341) = 18$ (170) $\begin{matrix}\square\times\square\\ \times\square\times\\ \square\times\square\end{matrix}$, (341) $\begin{matrix}\times\square\times\\ \square\times\square\\ \times\square\times\end{matrix}$.

4. $F_{(3,4)}(1445) = 15$, $F_{(3,4)}(2650) = 15$, (1445) $\begin{matrix}\times\square\times\square\\ \square\times\square\times\\ \times\square\times\square\end{matrix}$, (2650) $\begin{matrix}\square\times\square\times\\ \times\square\times\square\\ \square\times\square\times\end{matrix}$.

The DFs for the chessboard-like image with $L = 11, M = 20$ for different LSB of the size $\{I \times J\} = \{1 \times 3; 3 \times 1; 3 \times 3; 3 \times 4; 4 \times 3\}$ are very much similar to that given above for chessboard image. In fact we have the following data:

1. $F_{(1,3)}(2) = 99$, $F_{(1,3)}(5) = 99$, (2) $\square\times\square$, (5) $\times\square\times$.

2. $F_{(3,1)}(2) = 90$, $F_{(3,1)}(5) = 90$, (2) $\begin{array}{c}\square\\ \times\\ \square\end{array}$, (5) $\begin{array}{c}\times\\ \square\\ \times\end{array}$.

3. $F_{(3,3)}(170) = 81$, $F_{(3,3)}(341) = 81$. (170) $\begin{array}{c}\square\times\square\\ \times\square\times\\ \square\times\square\end{array}$, (341) $\begin{array}{c}\times\square\times\\ \square\times\square\\ \times\square\times\end{array}$.

4. $F_{(3,4)}(1445) = 77$, $F_{(3,4)}(2650) = 76$, (1445) $\begin{array}{c}\times\square\times\square\\ \square\times\square\times\\ \times\square\times\square\end{array}$, (2650) $\begin{array}{c}\square\times\square\times\\ \times\square\times\square\\ \square\times\square\times\end{array}$.

5. $F_{(4,3)}(1365) = 72$, $F_{(4,3)}(2730) = 72$, (1365) $\begin{array}{c}\times\square\times\\ \square\times\square\\ \times\square\times\\ \square\times\square\end{array}$, (2730) $\begin{array}{c}\square\times\square\\ \times\square\times\\ \square\times\square\\ \times\square\times\end{array}$.

One can easily see that the contributing configurations of LSB are the same ones, and the only difference, caused by different dimensions of the patterns, appears in the values of DF on the corresponding configurations of LSB (the height of spikes of corresponding histograms).

Certain observations, based on analysis of CCR for chessboard and other textures, can be made.

As compared with the DFs of the pattern in Fig. 2, the DFs of chessboard structure is found to be more regular and simpler ones. This is obviously related to global and local symmetry of the chessboard texture. Thus, we can state the following proposition.

PROPOSITION 1: A textured pattern with local or global symmetry posses a simpler DF structure, which displays only few sharp spikes in the corresponding histogram.

This general property of the CCR, together with measures given by Eqs. (3) and (4), can be readily used for synthetic and natural texture classification and discrimination.

It somehow reminds the situation when one needs to process only sinusoidal signals. The latter can be readily described with only three parameters: amplitude, frequency and phase shift of a signal. Moreover, to some extent, one can think of the CCR-transform as a tool for a kind of "spectral analysis" in the configuration space of LSB. It means that different configurations of LSB are considered as counterparts of harmonic (sine- or cosine-like) functions, and the values of the distribution function on different configurations of LSB can be considered as analogs of Fourier components. But, at the moment, this analogy does not go further, because of the lack of the inverse transform.

Local symmetry in a pattern is also manifested by an invariant part of the DF under the local symmetry operations. In other words, let a texture posses symmetrical elements, *i.e.*, there is a certain space symmetry group for these elements. When extended from a texture element to the entire pattern, any operation of this symmetry group is called here local symmetry operation. Then, we can state the following proposition.

PROPOSITION 2: If the size of a LSB is smaller than or equal to that of the symmetrical elements, then all the spikes of the DF, which arise from the scanning of the area, occupied by symmetrical elements, will remain unchanged under the local symmetry operations. It is obviously true for the case of patterns with a global symmetry and an arbitrary size of a LSB.

Also, it should be noted that for the chessboard texture case, one can unambiguously invert the CCR into the matrix representation, which is related with the "global" symmetry of the pattern. Later on, we give more proofs of the regularities noted above.

3.3. I-D textured and random signals-processes

To investigate some peculiarities of CCR concerned with correlation analysis of patterns we consider here a 1-D periodic signal. The primitive cell of the carrying periodic signal consists of five binary pixels which take on the values in the sequence $PC_1 = 1$, $PC_2 = -1$, $PC_3 = 1$, $PC_4 = 1$, $PC_5 = -1$, *i.e.*, in black and white pixel representation the primitive cell looks like the sequence, × □ × × □. The total periodic signal is obtained by 1000–fold replication of the primitive cell.

Distribution functions of the considered periodic signal for six LSBs of the size $J = 2, 3, 4, 5, 6, 7$ are as follows (here we give only non-zero values of DFs and corresponding configurations of LSB, the values of the DF on the rest of configurations of the corresponding LSB are all equal to zero):

1. $F_{(2)}(1) = 2000$, $F_{(2)}(2) = 1999$, $F_{(2)}(3) = 1000$, (1) ×□, (2) □×, (3) ××.
2. $F_{(3)}(2) = 999$, $F_{(3)}(3) = 1000$, $F_{(3)}(5) = 1999$, $F_{(3)}(6) = 1000$, (2) □×□, (3) ××□, (5) ×□×, (6) □××.
3. $F_{(4)}(5) = 999$, $F_{(4)}(6) = 1000$, $F_{(4)}(10) = 999$, $F_{(4)}(11) = 999$, $F_{(4)}(13) = 1000$, (5) ×□×□, (6) □××□, (10) □×□×, (11) ××□×, (13) ×□××.
4. $F_{(5)}(11) = 999$, $F_{(5)}(13) = 1000$, $F_{(5)}(21) = 999$, $F_{(5)}(22) = 999$, $F_{(5)}(26) = 999$, (11) ××□×□, (13) ×□××□, (21) ×□×□×, (22) □××□×, (26) □×□××.
5. $F_{(6)}(22) = 999$, $F_{(6)}(26) = 999$, $F_{(6)}(43) = 999$, $F_{(6)}(45) = 999$, $F_{(6)}(53) = 999$, (22) □××□×□, (26) □×□××□, (43) ××□×□×, (45) ×□××□×, (53) ×□×□××.
6. $F_{(7)}(45) = 999$, $F_{(7)}(53) = 999$, $F_{(7)}(86) = 999$, $F_{(7)}(90) = 998$, $F_{(7)}(107) = 999$, (45) ×□××□×□, (53) ×□×□××□, (86) □××□×□×, (90) □×□××□×, (107) ××□×□××.

One can easily observe the regularity such that any of the DFs of the considered periodic signal has no more than five non-zero spikes. This fact is naturally related to the size of the primitive cell, and we can generalize it to the following proposition.

PROPOSITION 3. Let a binary periodic signal have a primitive cell of size T pixels. Then, any DF in the coordinated cluster representation takes no more than T non-zero values on the corresponding configuration space of LSB.

The proof to the last proposition is quite obvious. We should only mention that any LSB, having been displaced from any initial position to the new one distant for the period T (primitive cell length) from the primary position, encounters the same configuration, because of the periodicity of the signal. From this observation the proof follows.

We can quite straightforwardly extend Proposition 3 to the following theorem.

THEOREM 1: Given is a translationally invariant N–dimensional binary image-signal, which is presented as a matrix $S^{\alpha}(\vec{m})$, where $\vec{m} = (m_1, \dots, m_N)$ is an N–dimensional integer valued index vector. Let a primitive cell be constructed on the primitive lattice-matrix translations τ_n along the corresponding axes ($n = 1, 2, \dots, N$), their lengths $T_n = |\tau_n|$ being measured in "pixels". Then, any distribution function $F^{\alpha}_{(I_1,\dots,I_N)}(b)$ in the coordinated cluster representation takes no more than $T = \prod_{n=1}^{N} T_n$ non-zero values on a configuration space of the corresponding N-dimensional LSB.

Proof: Consider any N–dimensional LSB located in an arbitrary initial position on an image matrix $S^{\alpha}(\vec{m})$. Note that the LSB, having been displaced from the initial position to the new one, which is at the distance of the primitive translation τ_n ($n = 1, \dots, N$) from the primary position, encounters the same configuration, because of the translational invariance of the image-signal. Hence, the LSB, while scanning from the initial position, encounters the new configurations, if and only if translations (scanning shifts) of the LSB do not go out of the volume of the primitive cell. But, the latter is equal to $T = \prod_{n=1}^{N} T_n$. QED.

We should mention that periodicity and translational invariance are considered here up to boundary effects. Turning back to the chessboard texture we find out that it is composed of the primitive cell with 2×2 pixels, but the latter is symmetric with respect to rotation by angle π and to reflection. That is why the DF of chessboard texture has only two (less than four) non-zero spikes, which is in accordance with Theorem 1.

Theorem 1 and Proposition 3 can be used for recognition of periodical and/or translational invariant images-signals. Later on we shall indicate how it can be used to find out quasi-periodical and translationally quasi-invariant signals, *i.e.*, translationally invariant signals subject to noise.

Noise affects an image signal and will certainly change its DF in CCR. One of the traditional ways to study fuzzy or random images is to make correlation analysis. The autocorrelation function of an 1-D signal-process is given by the expression

$$G(k) = \langle S^{\alpha}(m+k) S^{\alpha}(m) \rangle = \lim_{M \to \infty} M^{-1} \sum_{m=1}^{M} S^{\alpha}(m+k)\, S^{\alpha}(m), \tag{6}$$

where pixels $S^{\alpha}(n)$ are supposed to take on values ± 1, m is a positive integer position of a pixel, and k is a positive integer distance-separation between pixels. It can be readily seen, that the only configurations of LSB of the length $(k+1)$, on which DF is not equal to zero and which have value "± 1" for the first and the same value for the last pixels of LSB and arbitrary values for the pixels between them, contribute positively to the autocorrelation function $G(k)$ of the signal. But those, on which DF is not equal to zero and which have opposite values for the first and for the $(k+1)$–th pixels, contribute negatively to the auto correlation function $G(k)$. The numbers of the $\mathrm{LSB}_{(k+1)}$ configurations, which contribute positively and negatively to the autocorrelation function $G(k)$, are readily calculated from the corresponding DF. Thus, one can readily calculate the autocorrelation function $G(k)$ of any signal represented by its DF in CCR.

A similar consideration for multi-dimensional images signals leads us to the following theorem.

THEOREM 2: Let $S^{\alpha}(\vec{m})$ be a matrix of N–dimensional binary digital image-signal, where $\vec{m} = (m_1, \dots, m_N)$ is an integer valued position vector of a pixel and the size of the image-signal is defined with the positive integer valued vector $\vec{M} = (M_1, \dots, M_N)$. Let $F^{\alpha}(K_1, \dots, K_N)$ be the distribution function corresponding to the image signal $S^{\alpha}(\vec{m})$ in the Coordinated Cluster Representation with Local Scanning Basis of the size $\vec{K} = (K_1, \dots, K_N)$.

Then, for any integer valued separation vector $\vec{k} = (k_1, \dots, k_N)$ between "pixels" such that $|k_1| < K_1$, $|k_2| < K_2, \dots, |k_N| < K_N$, the autocorrelation function

$$G(\vec{k}) = \langle S^{\alpha}(\vec{m} + \vec{k}) S^{\alpha}(\vec{m}) \rangle = \lim_{M \to \infty} \left(\prod_{n=1}^{N} M_n \right)^{(-1)} \sum_{\vec{m}}^{\vec{M}} S^{\alpha}(\vec{m} + \vec{k})\, S^{\alpha}(\vec{m}) \tag{7}$$

can be uniquely reconstructed from $F^{\alpha}_{(K_1, \dots, K_N)}$. Here K_n and $m_n \le M_n$ are positive integers, k_n are integers $(n = 1, \dots, N)$.

Proof: Note that we can set the one to one correspondence between each term $S^{\alpha}(\vec{m} + \vec{k}) S^{\alpha}(\vec{m})$ in Eq. (7) and each LSB position on the matrix $S^{\alpha}(\vec{m})$, while scanning the image-signal. For example, for each element of the sum we can prescribe the LSB position with the left upper pixel in the point $\vec{m}$.

It can be readily seen that for any given separation $\vec{k} = (k_1, ..k_N)$ between "pixels" such that $|k_1| < K_1$, $|k_2| < K_2, \dots, |k_N| < K_N$, the only configurations of LSB, which contribute positively by a unit value to the autocorrelation function $G(\vec{k})$, are those, on which the distribution function $F^{\alpha}_{(K_1, \dots, K_N)}$ is not equal to zero and which have the same value "± 1" on both, the "first" and the $\vec{k}$–th ($\vec{k}$–separated) pixels of LSB, and arbitrary values on other pixels of LSB. But those configurations, on which the distribution function is not equal to zero and which have the opposite values for both the "first" and the $\vec{k}$–separated pixels, contribute negatively by a value "-1" to the function $G(\vec{k})$.

The type (positive or negative) of contribution for each LSB configuration is easily defined. So, in order to calculate $G(\vec{k})$ one needs to find out the contribution type of each $\mathrm{LSB}_{(\vec{K})}$ configuration on which the distribution function $F^{\alpha}_{(K_1, \dots, K_N)}$ has a nonzero value and, then, sum the values of the DF multiplied by the corresponding contribution sign. That proofs Theorem 2 in a constructive manner.

The most important consequence of the Theorem 2 is that a relation between DF and corresponding autocorrelation function does exist. The latter proves the intuitive assumption that for an image-signal $S^{\alpha}(\vec{m})$ its DF based on the LSB of the size $\vec{K} = (K_1, \dots, K_N)$ contains all the information based on and related to correlations between pixels (samples) separated by a vector $\vec{k} = (k_1, \ldots, k_N)$ which can be enclosed in the volume of LSB.

To illustrate the relationship between CCR and correlations of an image we proceed to investigate the periodic signal given above, but the latter is subject to the random change of pixel values with the probability $q = 0.05$, that is, in average, 95% of pixels keep their values unchanged.

For the pure periodic signal we give here the correlation function $G(k)$ calculated according to Eq. (6): $G(0) = 1$; $G(1) = -.59992$; $G(2) = .19968$; $G(3) = .20032$; $G(4) = -.60008$; $G(5) = 1$; $G(6) = -.59992$; $G(7) = .19968$; $G(8) = .200032$; $G(9) = -.60008$; $G(10) = 1$.

The correlation function $GF(k)$, calculated by means of the distribution function $F_{(7)}$ with LSB of 7 pixels in length, takes on the following values: $GF(0) = 1$; $GF(1) = -.59992$; $GF(2) = .19984$; $GF(3) = .20024$; $GF(4) = -.59992$; $GF(5) = 1$; $GF(6) = -.59992$.

The functions $G(k)$ and $GF(k)$ naturally demonstrate periodicity with period $T = 5$ pixels, and perfectly coincide with each other up to negligible boundary effect of the order $|L - 1 - k|/(M - k) \ll 1$, where k is a separation between pixels, M is the duration of a signal and L is the length of a LSB.

The periodic signal affected by noise, as described above, has correlation function $G(k)$ as follows: $G(0) = 1$; $G(1) = -.48709$; $G(2) = .15886$; $G(3) = .16109$; $G(4) = -.48559$; $G(5) = .81822$; $G(6) = -.48778$; $G(7) = .16122$; $G(8) = .16827$; $G(9) = -.48587$; $G(10) = .82284$.

The correlation function GF(k) for the same "fuzzy" signal has the following values: $GF(0) = 1$; $GF(1) = -.48698$; $GF(2) = .15899$; $GF(3) = .16099$; $GF(4) = -.48538$; $GF(5) = .81818$; $GF(6) = -.48778$.

According to Theorem 2, the correlation functions $G(k)$ and $GF(k)$ calculated by the two different ways demonstrate excellent coincidence up to the boundary effect due to the finite duration of the signal. The quasi periodicity of the signal is clearly seen from the correlation function.

We should note here that in matrix (vector) representation of a signal, one has to perform correlation analysis to observe quasi periodicity. But in coordinated cluster representation one can find out the quasi periodicity directly from DF, because in the DF of a fuzzy signal the fundamental spikes of the DF of the primary periodic signal are still kept prominent, yet reduced in values, on the background of other possible spikes which have been caused by noise. The configurations of LSB which give a prominent contribution into the DF of a fuzzy signal can be readily used to reconstruct the carrying periodic signal. For that purpose one can use, for example, a combinatorial technique.

3.4. Specific features of an object

In practice, we are interested in particular objects present in a pattern. Hence, it is important to see how individual an object is in the representation discussed here. In other words, are there any characteristic spikes inherent to the object in the histogram of the distribution function and, if it is so, how stable are they with respect to the boundary conditions? The following patterns illustrate the answer.

Figure 3 represents digitized binary images with the field size of $L \times M = 5 \times 7$ pixels and with the objects as following: 1) circle centered at the point $(Y_0, X_0) = (3, 4)$ with

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1	2	3

FIGURE 3. Characteristic peaks of DF. Boundary and noise influence.
1. $F^1_{(3,3)}(1) = 1$, $F^1_{(3,3)}(4) = 1$, $F^1_{(3,3)}(8) = 1$, $F^1_{(3,3)}(10) = 1$, $F^1_{(3,3)}(21) = 1$, $F^1_{(3,3)}(32) = 1$, $F^1_{(3,3)}(34) = 1$, $F^1_{(3,3)}(64) = 1$, $F^1_{(3,3)}(81) = 1$, $F^1_{(3,3)}(136) = 1$, $F^1_{(3,3)}(160) = 1$, $F^1_{(3,3)}(170) = 1$, $F^1_{(3,3)}(256) = 1$, $F^1_{(3,3)}(276) = 1$, $F^1_{(3,3)}(336) = 1$. 2. $F^2_{(3,3)}(0) = 9$, $F^2_{(3,3)}(73) = 3$, $F^2_{(3,3)}(146) = 3$. 3. $F^3_{(3,3)}(1) = 1$, $F^3_{(3,3)}(8) = 1$, $F^3_{(3,3)}(10) = 1$, $F^3_{(3,3)}(21) = 1$, $F^3_{(3,3)}(64) = 1$, $F^3_{(3,3)}(81) = 1$, $F^3_{(3,3)}(107) = 1$, $F^3_{(3,3)}(136) = 1$, $F^3_{(3,3)}(150) = 1$, $F^3_{(3,3)}(170) = 1$, $F^3_{(3,3)}(178) = 1$, $F^3_{(3,3)}(233) = 1$, $F^3_{(3,3)}(336) = 1$, $F^3_{(3,3)}(349) = 1$, $F^3_{(3,3)}(402) = 1$.

the radius $Yb = Xa = 1$; 2) rectangle (straight line) of the size $Yr = 5$, $Xr = 1$ with the left upper corner at the point $(Yr_0, Xr_0) = (1, 2)$; 3) superposition of the objects given above. The system of $I \times J = 3 \times 3$ pixels is taken here to be LSB. The configuration space of LSB consists of $2^{N1} = 2^9 = 512$ states. DFs are defined on the discrete interval $b = 0, 1, \dots, 511$ and they discriminate the patterns given above. One can observe that there are some peaks in the DF for the pattern of Fig. 3 (picture 1) (for example, numbered with $1, 8, 10, 21, 64, \dots$) which retain in the DF for the superimposed pattern of Fig. 3 (picture 3), but there are no peaks of the distribution function for the pattern of Fig. 3 (picture 2). To clarify the meaning of this question one can make the following simulation with one of the objects.

Let us increase the size $L \times M$ of a pattern keeping an object unchanged. As soon as the object, "dressed" into the covering coordinated clusters (that is the set of configurations of LSB which have, at least, one pixel in common with the object), stops being in touch with the border then the characteristic part of the distribution function remains constant, even if the pattern is increased. Hence, if an object is present in the pattern, one can get pure characteristic peaks in the DF only in the case when "dressed" object does not touch the border of the pattern or any other object. A similar conclusion is obtained in the case of pattern recognition affected by noise.

Particularly, the previous results lead us to the conclusion that for the purpose of tracking an object one has to choose as large a LSB as the "dressed" object stays in touch with one of the edges of the pattern field or with the set of other simpler objects used as a reference system.

4. CONCLUDING REMARKS

In this paper we presented the fundamentals of a new quasi statistical description, called here the Coordinated Cluster Representation, of binary digital images. Two theorems have been proved, one of which gives the structure of the distribution function in the CCR for translationally invariant images, and the other one establishes the relationship between CCR and second order statistics of an image. The theorems provide mathematical basis for CCR's applications to image feature extraction and texture discrimination, specially when one needs to analyze the elements of local symmetry and/or statistical properties of an image-signal. Moreover, the CCR can be efficiently used for the problem of image classification, since it provides an important characteristic feature due to Theorem 2. The other promising application of CCR is in digital data encoding for economical and noise tolerant information storage and transmission, but, in the general case, it still needs to have the inverse problem to be solved.

The representation has been explicitly formulated here only for digital binary signals. General concepts, such as the measure of pattern resemblance, the problem of uniqueness of the mapping, effects of noise, which are necessary components for applications, have been discussed and illustrated.

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