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Low frequency approximation of a vertically averaged ocean model with thermodynamics

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ABSTRACT. A low frequency approximation of a primitive equations ocean model with a horizontally inhomogeneous layer is developed and shown to have a singular Hamiltonian structure and the same set of integrals of motion as the original system. Disturbances to a reference state without currents are Rossby waves and rearrangements of the buoyancy and depth fields that leave the velocity field unaltered. Pseudoenergy and pseudomomentum integrals of motion are constructed and their relationship with the instability of a basic state with currents is discussed.

RESUMEN. Se desarrolla una aproximación de bajas frecuencias a un modelo oceánico de ecuaciones primitivas con una capa horizontalmente inhomogénea. El nuevo modelo tiene una estructura Hamiltoniana singular y las mismas integrales de movimiento que el sistema original. Las perturbaciones a un estado de referencia sin corrientes son ondas de Rossby y redistribuciones de los campos de flotabilidad y profundidad, que dejan al campo de velocidad inalterado. Se construyen las integrales de movimiento de pseudoenergía y pseudomomentum y se discute su relación con el problema de inestabilidad de un estado básico con corrientes.

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1. INTRODUCTION

The shallow water equations, invented by the genius of Laplace for the study of tides (see for instance, Refs. [1, 2]), become the "primitive equations" (PE) once buoyancy effects are included, and constitute a powerful starting point for ocean modelling in time scales larger than a few hours (see Acronyms and Notation in Appendix C). A very popular vertical setup for the primitive equations model consists of a stack of homogeneous layers (HLPEM), with a depth-independent velocity field in each one. The simplest of these models has but one (active) layer, with evolution equations of the form

$$\text{HLPEM:} \begin{cases} \partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} + \vartheta \nabla h = 0, \end{cases} \tag{1}$$

where $f = f_0 + \beta y$ is the Coriolis parameter, and I am including neither forcing nor dissipation —here and thereof— for simplicity. The symbol ϑ represents the buoyancy of the top (active) ocean layer relative to the bottom (passive) layer, *i.e.*, $g(\rho - \rho_{\text{down}})/\bar{\rho}$. (In Laplace tidal equations ϑ is gravity; here it is taken as a constant some three orders of magnitude smaller than g.) A very important theorem of (1) is that of potential vorticity conservation,

$$(\partial_t + \mathbf{u}\cdot\nabla)\left(\frac{f+\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}}{h}\right) = 0. \tag{2}$$

The system (1) has both high and low frequency solutions, *e.g.* Poincaré and Rossby waves, when linearized from a state of rest. Many times the interest is on the nonlinear behavior at long time scales, and then these equations are approximated by systems like the "quasi-geostrophic" model (QGM) in which $\mathbf{u} \approx \hat{\mathbf{z}} \times \nabla\phi = \mathbf{u}_g$, where $h = H + f_0\phi/\vartheta$ (with H a uniform reference depth), and potential vorticity is defined as the expansion up to linear terms in $\mathbf{u}$, $h - H$ and y, namely

$$\frac{f+\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}}{h} \approx \frac{f+\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}_g - f_0(h-H)}{H} = q. \tag{3}$$

In terms of the streamfunction ϕ, $qH = f + \nabla^2\phi - R_d^{-2}\phi$, where $R_d = \sqrt{\vartheta H/f_0^2}$ is the deformation radius. The evolution equations are constancy of circulation in each disconnected part of the boundary and the QG version of potential vorticity conservation, namely

$$\text{HLQGM: } (\partial_t + \mathbf{u}_g\cdot\nabla)q = 0, \tag{4}$$

or $\partial_t q + [\phi, q] = 0$, where

$$[A, B] = \hat{\mathbf{z}}\cdot\nabla A \times \nabla B$$

is the horizontal Jacobian. (Figure 1 shows the relationship between the different types of models discussed in this Introduction.) Linearizing (4) from a state of rest, only Rossby waves are found, there are no Poincaré wave solutions in this system.

One disadvantage of the popular HLPEM —or its low frequency approximation HLQGM— is that they cannot incorporate thermodynamic effects since, by definition, density is constant in each layer. To remedy this limitation, the primitive equations models with *inhomogeneous* layers (ILPEM) were developed and used often in the last decade or so (see references in [3], where these models are generalized and their Hamiltonian structure and conservation laws are derived) . For the simplest case of only one layer, the system (1) is extended into

$$\text{ILPEM: } \begin{cases} \partial_t\vartheta + \mathbf{u}\cdot\nabla\vartheta = 0, \\ \partial_t h + \nabla\cdot(h\mathbf{u}) = 0, \\ \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + f\hat{\mathbf{z}}\times\mathbf{u} + \vartheta\nabla h + \tfrac{1}{2}h\nabla\vartheta = 0, \end{cases} \tag{5}$$

where now ϑ is a (horizontal) position and time dependent field. In a recent paper [4] the normal modes of this system, linearized with respect to a state of rest $(\vartheta,\, h,\, \mathbf{u}) =$

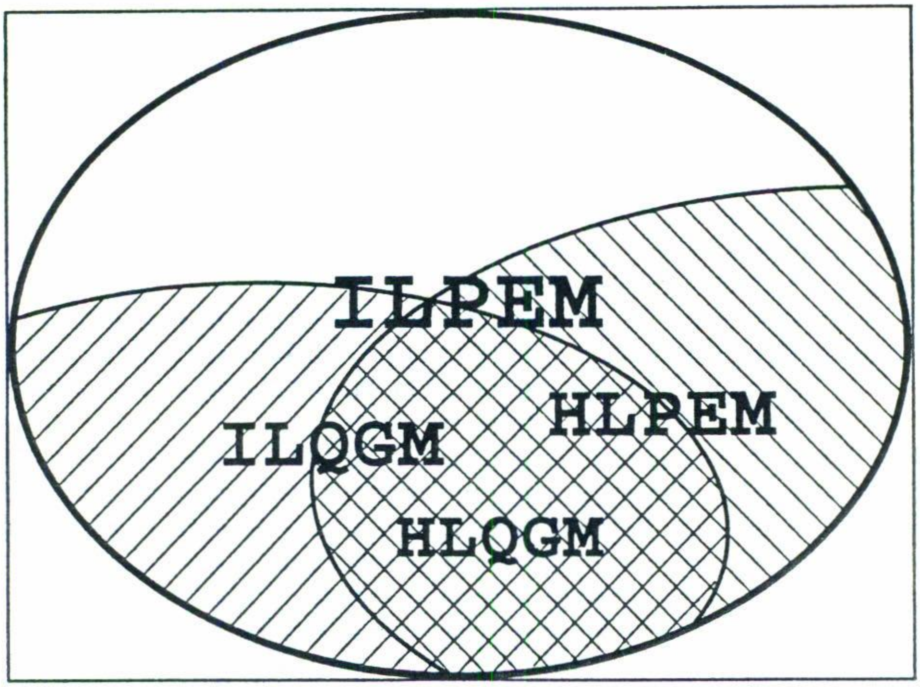

FIGURE 1. Homogeneous layer (HL) models are invariant submanifolds of the corresponding inhomogeneous layer (IL) ones. Quasi-geostrophic (QG) models are an approximation to the primitive equations (PE) ones.

$(\Theta,\, H,\, \mathbf{0})$, were shown to be Poincaré and Rossby waves and a "force compensating mode", for which $\Theta\delta h + \frac{1}{2}H\delta\vartheta = \delta\mathbf{u} = 0$. The potential vorticity equation derived from (5) is

$$(\partial_t + \mathbf{u}\cdot\nabla)\left(\frac{f + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}}{h}\right) = \tfrac{1}{2}h^{-1}[h, \vartheta], \tag{6}$$

instead of (2). It can be shown [5] that the right hand side of this equation is not a deficiency of the model but, rather, the vertical average of the baroclinic torque. This potential vorticity q is not conserved if density gradients are allowed within the layer. However, this equation plays an important role in low frequency dynamics [4]. The purpose of this paper is to develop a low frequency approximation of (5), in the same sense that (4) is an approximation of (1). In other words, the goal is to develop the inhomogeneous layers quasi-geostrophic model (ILQGM). In particular, the Hamiltonian structure of the new model is discussed, because it provides a framework useful for dealing with conservation laws and with the stability/instability problem. Four different types of models are compared in this paper: their relationship is depicted in Fig. 1.[1]

[1] The ILPEM and ILQGM are but an approximation of more exact dynamics, obtained through a vertical average of the dynamical fields. In Ref. [4] this approximation is improved by allowing an explicit vertical shear and variable density stratification. The models discussed here are denoted by IL^0PEM and IL^0QGM in [4], where the superscript 0 indicates the lack of vertical variation.

The HLPEM have, in the absence of forcing and dissipation, an interesting Hamiltonian structure [6–8]. The instantaneous state of the system can be seen as a point z on a singular manifold M_{HLPEM} (see, for instance Ref. [9]) whose evolution $z(t)$ is controlled by a Hamiltonian $\mathcal{H}[z]$, defined modulo the Casimir integrals of motion $\mathcal{C}[z]$ (the generators of null transformations). $\mathcal{H}$ can be shown to be a "free energy", positive definite in the deviation from a suitably chosen motionless reference state. The definiteness of $\mathcal{H}$ implies that the free evolution of the system is bounded, *i.e.*, solutions of the fully nonlinear equations cannot "explode" from a state of rest [10]. Moreover, in the study of disturbances $\delta z(t)$ to a steady basic state Z, which might have sheared currents, an integral of motion $\mathcal{H}+\mathcal{C}$ is usually found (the so-called pseudoenergy), whose first variation at Z vanishes, $\delta(\mathcal{H}+\mathcal{C}) = 0$; for some Z, the second variation can be shown to be positive definite, $\delta^2(\mathcal{H}+\mathcal{C}) > 0$. Therefore those basic states are stable.

The HLQGM is a low frequency approximation that can be seen as a metric projection into a subspace $M_{\mathrm{HLQGM}} \subset M_{\mathrm{HLPEM}}$ [11], in which the Casimirs restrict the evolution of the system more than in the HLPEM system. As a consequence, those models have more powerful stability theorems, since there are states that can be proved stable because the second variation of the pseudoenergy is *negative* definite, $\delta^2(\mathcal{H}+\mathcal{C}) < 0$; this is the so-called Arnol'd's second theorem. Moreover, there are also cases in which the total variation $\Delta(\mathcal{H}+\mathcal{C})$ is definite, and therefore the finite amplitude growth of a disturbance δz can be bounded [7, 12–14, 3].

In Ref. [3] it is shown that the ILPEM have also a singular Hamiltonian structure, and that M_{HLPEM} is an invariant submanifold of the larger state space M_{ILPEM} (see Fig. 1). There is a loss of Casimirs —or a "change in the rank" of the Poisson tensor [15]— when going from M_{ILPEM} to M_{HLPEM} (something that is not experienced when going from M_{HLPEM} to M_{HLQGM}). From a practical point of view, this is related to the non-existence of sufficient stability conditions for the ILPEM.

The ILQGM (a low frequency approximation of the ILPEM) is developed here and its Hamiltonian structure and conservation laws are also discussed. It is shown that, just like the ILPEM, there are not sufficient stability conditions. However the integrals of motion can nevertheless be used to analyze the instability problem. These results are illustrated with the problem of uniform flow instability, which is peculiar of the inhomogeneous layers models. A comparison of this ILQGM and that of Young [16], in terms of their Hamiltonian structure and conservation laws is also presented.

2. THE ILQG MODEL

In Ref. [4] it was shown to be convenient to change variables from the layer thickness h and buoyancy ϑ fields to $\varphi := \sqrt{\vartheta}h$ and $\gamma := \sqrt{\vartheta}$. The model evolution equations (5) using these variables take the form

$$\begin{gathered}\partial_t\gamma + \mathbf{u}\cdot\nabla\gamma = 0,\\ \partial_t\varphi + \nabla\cdot(\varphi\mathbf{u}) = 0,\\ \partial_t\mathbf{u} + \tfrac{1}{2}\nabla\mathbf{u}^2 + (f + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u})\hat{\mathbf{z}}\times\mathbf{u} + \gamma\nabla\varphi = 0,\end{gathered} \tag{7}$$

where I have used $\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla\mathbf{u}^2 + (\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u})\hat{\mathbf{z}}\times\mathbf{u}$. The reference state used in [4] in order to study the waves was one without currents, which corresponds to $\varphi(\mathbf{x},t) = \Phi$ and $\gamma(\mathbf{x},t) = \Gamma(\mathbf{x})$, where Φ is constant but $\Gamma(\mathbf{x})$ is an arbitrary field. The buoyancy and layer thickness fields in this reference state are then given by $\Theta(\mathbf{x}) = \Gamma^2(\mathbf{x})$ and $H(\mathbf{x}) = \Phi/\Gamma(\mathbf{x})$, respectively. The same reference state will be used here. Defining

$$\varphi(\mathbf{x},t) = \Phi + m\,\psi(\mathbf{x},t) \qquad (m = f_0/\Phi),$$

it is assumed that the velocity field is mainly in geostrophic balance, namely

$$\mathbf{u} \approx H^{-1}\,\hat{\mathbf{z}}\times\nabla\psi = \mathbf{u}_g. \tag{8}$$

The deviation fields ψ and $(\gamma-\Gamma)$ are considered to be small, but not necessarily infinitesimal, unlike in the study of linear waves done in Ref. [4]. The potential vorticity of the primitive equations system (7) is replaced here, like in quasi-geostrophic theory (3), by an expansion up to linear terms in the deviation fields $\delta\psi$ and $\delta\gamma$, *viz,*

$$\frac{f + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}_g - f_0(h-H)/H}{H} \approx \frac{\beta y + \nabla\cdot(H^{-1}\nabla\psi) - m^2\psi}{H} + m\,\gamma =: q, \tag{9}$$

where I have used the linear relation $\delta\varphi/\Phi \approx \delta h/H + \delta\gamma/\Gamma$. (Notice that the relative vorticity is mainly given by $\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}_g = \nabla\cdot(H^{-1}\nabla\psi)$.)

Even though the main contribution to the velocity field is the geostrophic one $\mathbf{u}_g$, higher order terms are needed for the evaluation of $\nabla\cdot(\varphi\mathbf{u})$, which is the driving term in (7b). Thus, Eq. (7c) is approximated in the following way:

$$\begin{aligned} f_0\mathbf{u} &\equiv \gamma\hat{\mathbf{z}}\times\nabla\varphi - (\beta y + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u})\mathbf{u} + \hat{\mathbf{z}}\times\partial_t\mathbf{u} + \tfrac{1}{2}\hat{\mathbf{z}}\times\nabla\mathbf{u}^2 \\ &\approx \gamma\hat{\mathbf{z}}\times\nabla\varphi - (\beta y + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}_g)\mathbf{u}_g + \hat{\mathbf{z}}\times\partial_t\mathbf{u}_g + \tfrac{1}{2}\hat{\mathbf{z}}\times\nabla\mathbf{u}_g^2 \\ &= (2m\gamma - q - H^{-1}m^2\psi)\,\hat{\mathbf{z}}\times\nabla\psi - H^{-1}\partial_t\nabla\psi + \tfrac{1}{2}\hat{\mathbf{z}}\times\nabla\mathbf{u}_g^2. \end{aligned} \tag{10}$$

This approximation consists of replacing $\mathbf{u}$ by the geostrophic approximation (8), unless it is multiplied by f_0. Using this approximation of $\mathbf{u}$ in the evolution Eq. (7b) of ψ, *viz.* $\partial_t\varphi + \nabla\cdot(\varphi\mathbf{u}) = m\partial_t\psi + \nabla\cdot((\Phi + m\psi)\mathbf{u}) = 0$, gives $\partial_t(q - m\gamma) = \mathbf{u}_g\cdot\nabla A + B$, where $A = (1 + m\psi/\Phi)(2m\gamma - q - H^{-1}m^2\psi) \equiv 2m\gamma - q + (2m\gamma - q - m\Gamma)\psi/\Phi$ and $B = mH^{-1}\nabla\cdot(\psi H^{-1}\nabla\partial_t\psi) + m\frac{1}{2}\mathbf{u}_g\cdot\nabla\mathbf{u}_g^2/\Phi$. Finally, A is replaced by $2m\gamma - q$, *i.e.*, the second term is neglected, because it is $O((\gamma-\Gamma)\psi^2,\ \psi^3,\ \beta y\psi^2)$, and B is similarly neglected. This provides one of the equations for the new model; the other one is that of γ, (7a), advected by the geostrophic velocity (8). Therefore, ILQGM dynamics is set up by the following two equations:

$$\text{ILQGM:} \begin{cases} \partial_t\gamma + \mathbf{u}_g\cdot\nabla\gamma = 0, \\ \partial_t q + \mathbf{u}_g\cdot\nabla q = mH^{-1}[\psi,\gamma]; \end{cases} \tag{11}$$

recall that $\mathbf{u}_g \cdot \nabla(\cdots) = H(\mathbf{x})^{-1}[\psi, \cdots]$. These equations hold on a certain horizontal domain D; appropriate boundary conditions are vanishing normal flux and constant circulations in each disconnected part ∂D_i of the boundary, namely

$$\hat{\mathbf{n}} \times \nabla\psi = 0 \quad (\mathbf{x} \in \partial D), \qquad \oint_{\partial D_i} H^{-1} \nabla\psi \cdot \hat{\mathbf{n}}\, dl = \tau_i = \text{const.}, \tag{12}$$

where $\hat{\mathbf{n}}$ is the outward normal unit vector. This problem is well posed because the field q and the conditions (12) uniquely determine the transport function ψ (see Appendix A) necessary to "advance" Eqs. (11). The evolution equations are then the set (11) plus $d\tau_i/dt = 0$ from (12b).

The physical nature of the approximation proposed here is the following. In the ILPEM, potential vorticity $(f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u})/h$ is not conserved; its rate of change (6), when written in terms of the variables of the low frequency approximation, is given precisely by (11b), namely $\frac{1}{2}h^{-1}[h, \vartheta] \equiv h^{-1}[\varphi, \gamma] \approx H^{-1}[\varphi, \gamma]$. Consequently, the new model is controlled by the evolution equations of the buoyancy and potential vorticity fields, under the assumptions that these fields do not depart much from their reference values and that the velocity field $\mathbf{u}$ is mainly geostrophic; evaluation of q requires the leading non-geostrophic contribution to $\mathbf{u}$.

I have developed the ILQGM using φ and γ as variables because they give a particularly simple representation of the "force compensating mode" and the free energy integral. However, this is not the only possible choice. One could work with the original variables ϑ and h, redefining

$$\mathbf{u}_g = f_0^{-1} \hat{\mathbf{z}} \times \left(\Theta \nabla h + \tfrac{1}{2} H \nabla \vartheta\right),$$

and

$$q = \frac{f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}_g - f_0(h - H)/H}{H},$$

instead of (8) and (9), respectively. (Recall that the reference state must be such that $\Theta \nabla H + \frac{1}{2} H \nabla \Theta = 0$). The whole formalism carries over with these variables, as long as deviations from the reference state are corresponded in a linear fashion, *e.g.* $\delta\vartheta \approx 2\Gamma\delta\gamma$.

I will show next that the system proposed here has the low frequency solutions of the original primitive Eqs. (7) or (5), is Hamiltonian, and has the right integrals of motion.

2.1. Linear waves

An important comparison between the original and the new system is in the dispersion relation of the linearized perturbations. Making

$$\begin{pmatrix} \varphi \\ \gamma \end{pmatrix} = \begin{pmatrix} \Phi \\ \Gamma \end{pmatrix} + \varepsilon \begin{pmatrix} m\hat{\psi} \\ \hat{\gamma} \end{pmatrix} \operatorname{cis}\left(\textstyle\int \mathbf{k} \cdot d\mathbf{x} - \omega t\right) + O(\varepsilon^2) \tag{13}$$

($\operatorname{cis} x = \cos x + i \sin x$) yields two eigensolutions,

$$\text{force compensating mode} \quad \begin{cases} \omega = 0, \\ \hat{\gamma} \neq 0, \ \hat{\psi} = 0, \end{cases} \tag{14}$$

or

$$\text{Rossby wave} \quad \begin{cases} \omega = \dfrac{\boldsymbol{\beta} \cdot \mathbf{k}}{\mathbf{k}^2 + R_d^{-2}}, \\ \hat{\gamma} = \hat{\psi} H^{-1} \hat{\mathbf{z}} \cdot \nabla \Gamma \times \frac{\mathbf{k}}{\omega} \end{cases} \tag{15}$$

where $R_d^2(\mathbf{x}) = \Gamma\Phi/f_0^2$ is the local deformation radius and $\boldsymbol{\beta}(\mathbf{x}) = -H\hat{\mathbf{z}} \times \nabla[(f_0 - \beta y)/H]$. The eigensolutions of the primitive equations system are these two modes and the high-frequency Poincaré waves, which obviously are not present here (see Ref. [4]); the only difference between (13) and the results in Ref. [4] being that, in the latter, the squared deformation radius and the beta vector field are given by $R_d^2 = \Gamma\Phi_0/f^2$ and $\boldsymbol{\beta} = H^{-1}\hat{\mathbf{z}} \times \nabla(fH)$. This shows that the model in this paper, like quasi-geostrophic theory, requires $\beta y \ll f_0$ and does not exhibit refraction of Rossby waves [17], because the variation of the deformation radius with the latitude is not modeled.

3. HAMILTONIAN STRUCTURE

The system (11) can be derived from the Hamiltonian functional

$$\mathcal{H}[\gamma, q, \tau] := \tfrac{1}{2} \iint_D H\mathbf{u}_g^2 + m^2\psi^2 \tag{16}$$

and the Lie-Poisson bracket

$$\begin{aligned} \{\mathcal{A}, \mathcal{B}\} := \iint_D q \left[\frac{1}{H}\frac{\delta\mathcal{A}}{\delta q}, \frac{1}{H}\frac{\delta\mathcal{B}}{\delta q}\right] \\ + \gamma \left[\frac{1}{H}\frac{\delta\mathcal{A}}{\delta\gamma}, \frac{1}{H}\frac{\delta\mathcal{B}}{\delta q}\right] + \gamma \left[\frac{1}{H}\frac{\delta\mathcal{A}}{\delta q}, \frac{1}{H}\frac{\delta\mathcal{B}}{\delta\gamma}\right] \end{aligned} \tag{17}$$

(see Appendix B.1). Recall that $\mathbf{u}_g$ is the geostrophic velocity defined in (8) and ψ is a functional of $[\gamma, q, \tau]$, as explained in Appendix A. Thus, for any functional of state $\mathcal{A}[\gamma, q, \tau, t]$

$$\frac{d\mathcal{A}}{dt} = \frac{\partial\mathcal{A}}{\partial t} + \{\mathcal{A}, \mathcal{H}\} \equiv \frac{\partial\mathcal{A}}{\partial t} + \iint_D \left(\frac{\delta\mathcal{A}}{\delta\gamma}\frac{\partial\gamma}{\partial t} + \frac{\delta\mathcal{A}}{\delta q}\frac{\partial q}{\partial t}\right). \tag{18}$$

In particular, using $\mathcal{A} = \mathcal{H}$ in (18) implies that the value of the Hamiltonian is an integral of motion, because $\partial\mathcal{H}/\partial t = 0$ and $\{\mathcal{H}, \mathcal{H}\} = 0$ by the antisymmetry of the Lie-Poisson bracket.

The bracket in Eq. (17) is singular, *i.e.*, there are non-trivial solutions of $\{\mathcal{A},\mathcal{C}\} = 0\ \forall\mathcal{A}$; these Casimirs $\mathcal{C}$ are of the form

$$\mathcal{C} = \iint_D (qHA(\gamma) + HB(\gamma)) + \sum a_i\tau_i, \tag{19}$$

where $A(.)$ and $B(.)$ are arbitrary differentiable functions and a_i are arbitrary constants (see Appendix B.2). Using $\mathcal{A} = \mathcal{C}$ in (18), it follows that the value of a Casimir is also an integral of motion because $\partial\mathcal{C}/\partial t = 0$ and $\{\mathcal{C},\mathcal{H}\} = -\{\mathcal{H},\mathcal{C}\} = 0$ by construction.

If both the domain D and the field $\Gamma(\mathbf{x})$ are x–independent, then there must exist a momentum $\mathcal{M}$ such that

$$\{\mathcal{M},\mathcal{B}\} = \iint_D \left(\frac{\delta\mathcal{B}}{\delta\gamma}\frac{\partial\gamma}{\partial x} + \frac{\delta\mathcal{B}}{\delta q}\frac{\partial q}{\partial x}\right), \tag{20}$$

$\forall\,\mathcal{B}[\gamma,\,q,\tau]$. This momentum is given by

$$\mathcal{M} = \iint_D q(\mathbf{x},t)H(y)\,\Xi(y), \quad \Xi'(y) = H(y), \tag{21}$$

modulo Casimirs (see Appendix B.3). Using $\mathcal{B} = \mathcal{H}$ in Eq. (20) yields $\{\mathcal{M},\mathcal{H}\} = 0$ because the coordinate x does not appear explicitly in definition (16) of $\mathcal{H}$. Therefore, using $\mathcal{A} = \mathcal{M}$ in (18) and the fact that $\partial\mathcal{M}/\partial t = 0$ it is found that the value of the momentum is also an integral of motion.

Finally, notice that if the original buoyancy field is uniform, say $\gamma = \Gamma =$ const., then the system (11) reduces to the classical quasi-geostrophic model (4), which has new Casimirs (namely the integral of an arbitrary function of q). In the language of Hamiltonian theory, it is said that there is change in the rank of the Poisson tensor [15] when going from the whole state space ILQGM to this invariant submanifold HLQGM (see Fig. 1). From a practical point of view, these additional Casimirs allow for the existence of stability theorems in the homogeneous layer case, something which will be shown below not to be possible when $\nabla\gamma \not\equiv 0$.

3.1. Conservation laws

The integrals of motion of the original system (7) or (5) are the circulations τ_i^{PE}, the energy $\mathcal{E}^{\mathrm{PE}}$, the Casimirs $\mathcal{C}^{\mathrm{PE}}$ and, if the reference state and domain are symmetric, the momentum $\mathcal{M}^{\mathrm{PE}}$, given by (see Ref. [3])

$$\begin{aligned}
\tau_i^{\mathrm{PE}} &= \oint\limits_{\partial D_i} \mathbf{u}\cdot d\mathbf{x}, \\
\mathcal{E}^{\mathrm{PE}} &= \frac{1}{2}\iint\limits_D (h\mathbf{u}^2 + \vartheta h) \equiv \frac{1}{2}\iint\limits_D (\varphi\gamma^{-1}\mathbf{u}^2 + \varphi^2), \\
\mathcal{C}^{\mathrm{PE}} &= \iint\limits_D \left(q^{\mathrm{PE}} h A(\vartheta) + hB(\vartheta)\right) \equiv \frac{1}{2}\iint\limits_D \left(q^{\mathrm{PE}} h\tilde{A}(\gamma) + \varphi\tilde{B}(\gamma)\right) \\
\mathcal{M}^{\mathrm{PE}} &= \iint\limits_D h(u - f_0 y - \tfrac{1}{2}\beta y^2) \equiv \iint\limits_D \varphi\gamma^{-1}(u - f_0 y - \tfrac{1}{2}\beta y^2),
\end{aligned} \tag{22}$$

where $q^{\mathrm{PE}} = (f + \hat{\mathbf{z}}\cdot\nabla\times\mathbf{u})/h$. The integrals of motion of the new system (11) are the circulations τ_i (12b), the Hamiltonian (16), the Casimirs (19), and, if the reference state and domain are symmetric, the momentum (21). Since the original system ILPEM and the low frequency approximation ILQGM have the same type of integrals of motion, it is interesting to compare the corresponding expressions. One difference between the ILPEM and the ILQGM is that in the former it is used: *i*) the complete velocity field $\mathbf{u}$, not just the geostrophic velocity $\mathbf{u}_g$, *ii*) the actual layer thickness h $(= \varphi/\gamma)$, not just the one in the reference state H $(= \Phi/\Gamma)$, and *iii*) the complete potential vorticity q^{PE}, not just the approximation (9). In addition to this:

- The constancy of the circulations is a theorem for ILPEM, whereas for ILQGM it is one of the equations of motion.
- Total energy (22b) seems to differ from the value of the Hamiltonian (16) in more than the replacement of $(h, \mathbf{u})$ for $(H, \mathbf{u}_g)$ because $\varphi = \Phi + m\psi$. However, the difference between the term $\iint \varphi^2/2$ in (22b) and the term $\iint m^2\psi^2/2$ in (16) is equal to $\iint(m\Phi\psi + \frac{1}{2}\Phi^2)$, which is a Casimir. Consequently, $\mathcal{H}$ is a "free energy", *i.e.*, the energy minus a trivial constant of motion chosen so that the potential energy part is quadratic (to lowest order) in the deviation from the reference state (this is usually called "available potential energy").
- For the Casimirs the only difference is the use of H instead of h, and q instead of q^{PE}.
- Finally, the momentum is an integral of motion which seems to be different in the primitive equations case (22d) or in the low frequency model proposed here (21). However, the latter corresponds to the approximation $\mathcal{M} \approx \iint q^{\mathrm{PE}} h\,\Xi = \iint[\partial_x v - \partial_y(u - f_0 y - \beta y^2/2)]\Xi$. On one hand, the term $\partial_x v\,\Xi(y)$ integrates to zero because the domain is assumed to be x–independent in order for $\mathcal{M}$ to be well defined. On the other hand $-\iint \partial_y(u - f_0 y - \beta y^2/2)\Xi(y)$ can be integrated by parts to give $\iint H(y)(u - f_0 y - \beta y^2/2) \approx \iint h(u - f_0 y - \beta y^2/2)$, which is the expression for the momentum of (5), plus a linear combination of the circulations. Therefore, both expressions of the momentum are equivalent.

4. Pseudo energy-momentum and the instability problem

Consider a steady basic state with currents, $\mathbf{U} = H^{-1}\hat{\mathbf{z}} \times \nabla\Psi$, and a finite amplitude deviation from it, *i.e.*,

$$\psi = \Psi(\mathbf{x}) + \delta\psi(\mathbf{x}, t), \qquad \gamma = \Gamma(\mathbf{x}) + \delta\gamma(\mathbf{x}, t).$$

In order to satisfy Eq. (11), the basic state must be such that $[\Psi, Q - m\Gamma] = 0$ and $[\Psi, \Gamma] = 0$, where $Q = [\beta y + \nabla \cdot (H^{-1}\nabla\Psi) - m^2\Psi]/H + m\Gamma \equiv [f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{U} - m^2\Psi]/H$. (Notice that the layer thickness in the basic state is not H but $H + m\Psi/\Gamma$.) Consequently

$$\Psi = \Psi(\Gamma) \quad \text{and} \quad Q = Q(\Gamma).$$

The pseudoenergy is defined as $\mathcal{H} + \mathcal{C}_E$, where $\mathcal{C}_E$ is a Casimir chosen so that $\mathcal{H} + \mathcal{C}_E$ has a vanishing first variation on the basic state, $\delta(\mathcal{H} + \mathcal{C}_E) = 0$. That is, the pseudoenergy is an integral of motion whose lowest order contribution in the perturbation $(\delta\gamma, \delta q, \delta\tau)$ is quadratic, $\frac{1}{2}\delta^2(\mathcal{H} + \mathcal{C}_E)$. In Appendix B.4 it is shown that for the low frequency dynamics (11) represented by ILQGM it is

$$\delta^2(\mathcal{H} + \mathcal{C}_E) = \iint \left(H\delta\mathbf{u}_g^2 + m^2\delta\psi^2 - mH\frac{d\Psi}{d\Gamma}\delta\gamma^2 + H\frac{d\Psi}{dQ}\left(\delta q^2 - \delta\tilde{q}^2\right)\right), \tag{23}$$

where

$$\delta\tilde{q} := \delta q - \frac{dQ}{d\Gamma}\delta\gamma.$$

The variable $\delta\tilde{q}/Q_y$ is equal to the variation of the meridional distance between the isolines of q and γ (see Ref. [4]). Since γ is a Lagrangian constant, that distance would vanish if q were also conserved. Consequently, $\delta\tilde{q}$ is a measure of the non- conservation of potential vorticity q.

If the basic state is not only steady but also x–symmetric, $\Gamma = \Gamma(y)$ and $\Psi = \Psi(y)$, a conserved pseudomomentum $\mathcal{M} + \mathcal{C}_M$ can be similarly constructed choosing $\mathcal{C}_M$ so that $\delta\mathcal{C}_M = -\delta\mathcal{M}$. The lowest order (quadratic) term in the perturbation is given by

$$\delta^2(\mathcal{M} + \mathcal{C}_M) = -\iint \frac{H^2}{Q_{,y}}(\delta q^2 - \delta\tilde{q}^2). \tag{24}$$

Finally, the most general integral of motion in the symmetric case, quadratic to lowest order in the perturbation, is an arbitrary combination of the pseudoenergy and the pseudomomentum, say $\mathcal{H} - \alpha\mathcal{M} + \mathcal{C}$, with $\mathcal{C} = \mathcal{C}_E - \alpha\mathcal{C}_M$, whose second variation is given by (see Appendix B.4)

$$\delta^2(\mathcal{H} - \alpha\mathcal{M} + \mathcal{C}) = \iint \left(H\delta\mathbf{u}_g^2 + m^2\delta\psi^2 - mH\frac{\Psi_{,y}}{\Gamma_{,y}}\delta\gamma^2 - H^2\frac{U - \alpha}{Q_{,y}}(\delta q^2 - \delta\tilde{q}^2)\right). \tag{25}$$

This is an integral of motion for linearized dynamics.

In the case with homogeneous layers, there are two types of sufficient stability conditions in QG models, namely

$$\frac{d\Psi}{dQ} > 0 \quad \text{or} \quad \frac{d\Psi}{dQ} < -\lambda^2 H \tag{26}$$

everywhere for a general steady basic state, and

$$\frac{U-\alpha}{Q_{,y}} < 0 \quad \text{or} \quad \frac{U-\alpha}{Q_{,y}} > \lambda^2 \tag{27}$$

for all y and some α, in the steady and symmetric case[2] [13, 14, 18]. The first or second condition, known as Arnol'd's first or second theorem, guarantees positive or negative definiteness of the corresponding integral of motion: the pseudoenergy $\delta^2(\mathcal{H}+\mathcal{C}_E)$ in the case of conditions (26), or an arbitrary combination of pseudoenergy and pseudomomentum $\delta^2(\mathcal{H}-\alpha\mathcal{M}+\mathcal{C})$ for conditions (27).

With inhomogeneous layers, on the other hand, the integrals of motion (23) and (25) cannot be sign definite, *i.e.*, there is no formal stability theorem for the model of this paper, except for the case of a uniform flow. In this case, using $\alpha = U$ (= const.) in (25) it is clearly seen that if $-md\Psi/d\Gamma > 0$ then the flow is stable. [From (8) and (15) it follows that in the f-plane, $\nabla f = 0$, $-md\Psi/d\Gamma$ equals the ratio of U to the long Rossby waves phase speed.]

Even though there are no stability theorems, except for the case of uniform U, the integrals of motion (23) and (25) can be used for the instability problem as in Ref. [19]. The idea is the following: if the basic state is unstable, then there must be growing perturbations for which $\delta^2(\mathcal{H}+\mathcal{C}_E) = 0$; also, in the symmetric case, $\delta^2(\mathcal{H}-\alpha\mathcal{M}+\mathcal{C}) = 0$ for all α. These conditions represent a balance between the positive and negative contributions of the integrands in Eqs. (23) or (25).

For instance, if $Q_{,y}$ has only one sign, then the pseudomomentum integral (24) shows that for a growing perturbation δq and $\delta\tilde{q}$ must be equally important, *i.e.*, $\|\delta q\| = \|\delta\tilde{q}\|$ in an L_2 metric weighted by $|H^2/Q_{,y}|$. On the other hand, consider Eq. (25): If there is a value of α such that $(U-\alpha)/Q_{,y} < 0$ and $md\Psi/d\Gamma < 0$ for all y, then for a growing perturbation $\delta\tilde{q}$ is more important than δq, *i.e.* non-conservation of potential vorticity will be an important ingredient of that type for instability. However, if $(U-\alpha)/Q_{,y} < 0$ but $md\Psi/d\Gamma > 0$ then a perturbation could grow because $\delta\gamma$ is important, whereas $\delta\tilde{q}$ and δq are not. In sum, even though there are no stability theorems, Eqs. (23), (24) and (25) can be used as a means to classify the different types of instabilities. For instance, in a calculation of the normal modes of an unstable basic state, the classification can be based on the relative size of the different sign-definite contributions to those integrals of motion.

[2] The positive constant λ^2 depends on the geometry of the domain. For instance, for an infinite channel of width L_y, it is $\lambda^{-2} = (\pi/L_y)^2 + R_d^{-2}$, whereas for a periodic channel of length L_x it is $\lambda^{-2} = (\pi/L_y)^2 + (2\pi/L_x)^2 + R_d^{-2}$ (see, for instance Ref. [14]).

5. Normal modes

Consider a basic state with a parallel flow and an infinitesimal disturbance on top, namely

$$\begin{aligned}\gamma &= \Gamma(y) + \varepsilon\,\Gamma(y)\,G(y)\,\mathrm{cis}[k(x-ct)] + O(\varepsilon^2),\\ \psi &= \Psi(y) + \varepsilon\,\Phi\,F(y)\,\mathrm{cis}[k(x-ct)] + O(\varepsilon^2),\end{aligned} \tag{28}$$

where $F(y)$ and $G(y)$ are two structure functions to be determined. The current in the basic state is given by $U(y) = -H(y)^{-1}\Psi'(y)$, where the prime indicates a derivative with respect to y. Upon substitution in Eq. (11) it is found that $F(y)$ and $G(y)$ must satisfy

$$\begin{gathered}(U-c)G + \Gamma' F = 0,\\ (U-c)\left[(H^{-1}F')' - H^{-1}k^2F - m^2F\right] + (Q - 2m\Gamma)'\,F - mU\,G = 0.\end{gathered} \tag{29}$$

If these equations are solved in the channel $y_1 \le y \le y_2$, then appropriate boundary conditions are $kF = kG = 0$ at $y = y_{1,2}$. This eigenvalue problem determines the values of $c(k)$; complex eigenvalues $\mathrm{Im}\,(kc) \neq 0$ correspond to growing or decaying perturbations to a normal-mode unstable basic flow.

If $\Gamma' = 0$, the "equivalent barotropic" instability problem is recovered (see for instance Ref. [14]), which requires that $\beta - U'' + R_d^{-2}U$ to change sign if the basic flow is unstable. With $\Gamma' \neq 0$, there is a new type of instability, which result from the nonlinear coupling of a Rossby wave and "force compensating mode" [4]; this instability mechanism does not require $U' \neq 0$. Fukamachi *et al.* [20] studied the instability of the $U =$ const. flow in the f-plane ($\beta = 0$), within the context of the ILPEM (5) and found very good agreement with a similar calculation in a continuously stratified model, which is more precise. A uniform flow can be obtained from

$$\left.\begin{aligned}\Gamma &= \Gamma_0 \exp(\mu y)\\ m\Psi &= \nu\,\Phi[\exp(-\mu y) - 1]\end{aligned}\right\}\ |y| \le \frac{L}{2}.$$

which corresponds to $U = \mu\nu c^2/f$, where $c^2 = \Gamma_0\Phi$. (The case studied by Fukamachi *et al.* roughly corresponds to $\nu = -1$). Assuming $\mu L \ll 1$ for simplicity, the eigensolutions coincide with those found in Ref. [4] for the ILPEM: F and G are approximately sinusoids of the form $\sin[l(y + \frac{1}{2}L)]$, where $l = n\pi/L$ with integer n. The eigenvalues are given by

$$c = \frac{1}{2}c_1 + \frac{1+2\kappa^2}{2(1+\kappa^2)}U \pm \sqrt{\frac{1}{4}\left(c_1 - \frac{U}{1+\kappa^2}\right)^2 + Uc_1},$$

where $\kappa^2 = (k^2 + l^2)c^2/f^2$ and $c_1 = (\Phi\Gamma'_{,y}/f)/(1+\kappa^2)$, which is the phase speed of Rossby waves, as can be seen from (15) with $\nabla f = 0$, *i.e.*, $\beta = -f\,\hat{\mathbf{z}} \times \nabla \ln(\Gamma)$. Instability, $\mathrm{Im}(c) \neq 0$, requires $c_1U < 0$, which implies $\Psi'\Gamma' > 0$, because $U = -\Gamma\Psi'/f$. This is a prediction of the conservation law for a combination of pseudoenergy and pseudomomentum. Choosing

$\alpha = U$ in Eq. (25) it follows that if the basic state is unstable there must be growing perturbations for which $\delta^2(\mathcal{H} - U\mathcal{M} + \mathcal{C}) = 0$,[3] which means

$$\iint \left((\delta \mathbf{u}_g)^2 + m^2(\delta\psi)^2\right) = \iint (\Psi'/\Gamma')(\delta\gamma)^2;$$

in order for both sides to be able to grow, the coefficient (Ψ'/Γ') cannot be negative. More information on this type of instability can be found in Refs. [20] and [4].

It may seem strange that a uniform flow can be unstable, since instabilities are usually associated to a shear of the basic current. However, one can interpret this instability as a manifestation of *baroclinic instability* due to the *implicit* vertical shear given by the "thermal wind balance" [21,5]

$$f_0 U_{,z} = -\Theta_{,y} = -2\Gamma\,\Gamma_{,y}. \tag{30}$$

6. CONCLUSIONS

Four types of models of ocean dynamics are considered in this paper, classified according to whether they have homogeneous or inhomogeneous layers (HL or IL), and whether they are of the "primitive equations" (PE) or "quasi-geostrophic" (QG) kind. Three of them are extensively used in the literature, whereas the fourth one, ILQGM, is the one proposed here. Even though all the examples in this paper correspond to only one active layer of fluid (on top of a motionless passive layer) the generalization to a finite number of layers is straightforward.

HL models have a uniform buoyancy ϑ, whereas in the IL models treated here, ϑ may vary with horizontal position and time.[4] Since ϑ is conserved following fluid particles, formally speaking, an HL model is a particular case of an IL one. On the other hand, PE models have both high and low frequency fluctuations (*e.g.* Poincaré and Rossby waves), whereas the QG are low frequency approximations of the former.

The four types of model are compared in Table I in terms of *i*) their independent and dependent fields and *ii*) the form of the free energy $\mathcal{E}$ and Casimir $\mathcal{C}$ integrals of motion. The "free energy" is the sum of $\mathcal{E}$ and one of the $\mathcal{C}$, chosen so that its first variation from a state of rest ($\vartheta = \Theta$, $h = H$, $\mathbf{u} = \mathbf{0}$) vanishes.[5] Notice that the free energy is positive definite in HL models, but only non-negative definite in IL models, owing to the existence of the "force-compensating mode".

[3] Just one normal mode (28) with $\mathrm{Im}(kc) \neq 0$ being an example.

[4] The replacement of the ϑ evolution equation by separate equations for the temperature and salinity fields is straightforward.

[5] The IL models can be posed using ϑ and h, or in terms of $\vartheta^{\frac{1}{2}}$ $(=\gamma)$ and $\vartheta^{\frac{1}{2}}h$ $(=\varphi)$, as done here. In either case, the potential vorticity q is defined up to linear terms in the perturbation fields, *i.e.*, in the first case it is used $\vartheta = \Theta + \delta\vartheta \approx \Gamma^2 + 2\Gamma\,\delta\gamma$ and $h = H + \delta h \approx \Phi/\Gamma + \delta\varphi/\Gamma - \delta\gamma\,\Phi/\Gamma^2$. The free energy is exactly quadratic in the corresponding fields, *i.e.*, the available energy density is given by $\frac{1}{2}\Theta(\delta h + \frac{1}{2}\delta\vartheta\, H/\Theta)^2$ or $\frac{1}{2}(\delta\varphi)^2$, respectively.

TABLE I.

Model	Fields		Integral of Motion Density	
	Ind.	Dep.	2 × Free Energy	Casimir
HLPEM	h, $\mathbf{u}$	q, τ	$h\,\mathbf{u}^2+\vartheta(h-H)^2$	$h\,F(q)$
HLQGM	q, τ	h, $\mathbf{u}$	$H\,\mathbf{u}^2+\vartheta(h-H)^2$	$H\,F(q)$
ILPEM	ϑ, h, $\mathbf{u}$	q, τ	$h\,\mathbf{u}^2+\left(\sqrt{\vartheta}\,h-\sqrt{\Theta}H\right)^2$	$hq\,A(\vartheta)+h\,B(\vartheta)$
ILQGM	ϑ, q, τ	h, $\mathbf{u}$	$H\,\mathbf{u}^2+\left(\sqrt{\vartheta}\,h-\sqrt{\Theta}H\right)^2$	$Hq\,A(\vartheta)+H\,B(\vartheta)$
N.B. $F(\cdots)$, $A(\cdots)$ and $B(\cdots)$ arbitrary				

The independent variables of the new model (11) are (in addition to the buoyancy) those of the classical QGM (4), potential vorticity q and circulations τ. This is not surprising since, in fact, the HLQGM belongs to an invariant submanifold of the ILQGM. The main difference with the classical HLQGM is that, with buoyancy gradients, q is not a Lagrangian constant $Dq/Dt \neq 0$. However, the evolution equation for q is one of the fundamental ones for low frequency dynamics. On the other hand, the new model has the same integrals of motion (energy, momentum, and Casimirs) and low frequency solutions (force-compensating modes and Rossby waves) than the original one, the ILPEM (5).

Young [16] developed a low frequency inhomogeneous layer model in the f–plane, through a method quite different from the one employed here. A very interesting aspect of Young's model is that he includes mixing of density and momentum and furthermore he implicitly takes into account the effect of the vertical shear of the currents (through the thermal wind balance). In the limit in which the inertial period is much larger than the interval between momentum mixing events, Young's equations can be shown to be the $\beta=0$ case of the set (11). In the opposite limit (inertial period is much smaller than the interval between momentum mixing events) Young's equations can be written, in the notation of this paper, as

$$\begin{aligned}\partial_t\gamma+\mathbf{u}_g\cdot\nabla\gamma &= 0,\\ \partial_t q+\mathbf{u}_g\cdot\nabla q &= mH^{-1}[\psi,\gamma]-\tfrac{1}{3}m^{-2}H^{-1}[\gamma,\nabla^2\gamma],\end{aligned}\tag{31}$$

i.e., compared with the set (11) there is an extra term in the q equation, which models the effect of the velocity vertical shear [5]. An interesting property of these equations is that instead of satisfying $d\mathcal{H}/dt=0$ with $\mathcal{H}$ from (16), they fulfill

$$\mathcal{H}_Y=\tfrac{1}{2}\iint\left(H\mathbf{u}_g^2+m^2\psi^2-\tfrac{1}{3}Hm^{-2}(\nabla\gamma)^2\right)=\text{const.}\tag{32}$$

It is easy to see that $\mathcal{H}_Y$ is a Hamiltonian for set (31), with *the same Lie-Poisson bracket* (17). Consequently, I have shown that Young's model is Hamiltonian, in the absence of forcing and dissipation, and that the Casimirs and momentum are those of (19) and (21), but the pseudoenergy (23) has to be changed in order to accommodate the "extra" term. Since the last term in (32) has a sign opposite to the other two, this energy integral does not prevent the possibility of an "explosion" of the system from initial conditions arbitrarily close to a resting state [10].

It is not difficult to evaluate the effect of the current vertical shear in the total energy if one assumes, following [16], that the depth dependence of the buoyancy can be neglected. In the "reduced gravity" case the dynamic pressure p vanishes at the base of the mixed layer ($z = -h$), since its z derivative equals γ^2, the vertical average of the buoyancy within the layer ($-h < z < 0$). Therefore, $p = (z+h)\gamma^2 = z\gamma^2 + \gamma\varphi$. Consequently, it is $\nabla p = \gamma\nabla\varphi + (2z+h)\gamma\nabla\gamma$, the expression whose vertical average is $\gamma\nabla\varphi$, which is the forcing term employed in (7b). However, if one keeps the vertical shear when calculating the geostrophic velocity, the result is $\mathbf{u} = H^{-1}\hat{\mathbf{z}} \times \nabla\psi + (2z/H+1)m^{-1}\hat{\mathbf{z}} \times \nabla\gamma$ and thus

$$\int_{-H}^{0} \mathbf{u}^2\, dz = H^{-1}(\nabla\psi)^2 + \tfrac{1}{3}Hm^{-2}(\nabla\gamma)^2 = H\mathbf{u}_g^2 + \tfrac{1}{3}Hm^{-2}(\nabla\gamma)^2.$$

Consequently, the total energy is given by

$$\mathcal{E} = \tfrac{1}{2}\iint \left(H\mathbf{u}_g^2 + m^2\psi^2 + \tfrac{1}{3}Hm^{-2}(\nabla\gamma)^2\right),$$

instead of the value of $\mathcal{H}$ in (16). Notice that, surprisingly enough, the "extra" term is exactly that of the Hamiltonian $\mathcal{H}_Y$ for Young's model [16], defined in Eq. (32), but with the opposite sign. Consequently, $\mathcal{H}_Y$ is not the free energy; its sign indefiniteness may be related to the implicit mixing of density and momentum (which correspond to an increase and decrease of the energy of the system, respectively).

It is desirable to have a model of low frequency dynamics in an inhomogeneous layer model in which a conserved free energy is positive definite. This could arrest the exponential growth of finite amplitude disturbances, something that the law (32) for the system (31) of Young [16] cannot do. The ILPEM is improved in Ref. [5] by means of an explicit representation of the velocity shear and density stratification. A finite value of the vertical gradient of density allows for the construction of a positive definite free energy; this may be the key to develop a similarly well-behaved low frequency approximation, as an improvement of the one developed here.

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A. UNIQUENESS

Let ψ_1 and ψ_2 be two solutions of $\nabla\cdot(H^{-1}\nabla\psi)-m^2\psi=(q-m\gamma)H-\beta y$ ($\mathbf{x}\in D$), and $\hat{\mathbf{n}}\times\nabla\psi=0$ ($\mathbf{x}\in\partial D$), $\int H^{-1}\nabla\psi\cdot\hat{\mathbf{n}}\,dl=\tau_i$ ($\mathbf{x}\in\partial D_i$). Defining $\tilde{\psi}:=\psi_1-\psi_2$, it is easy to see that $\tilde{\psi}$ satisfies

$$\begin{aligned}\nabla\cdot(H^{-1}\nabla\tilde{\psi})-m^2\tilde{\psi}&=0\quad(\mathbf{x}\in D),\\ \hat{\mathbf{n}}\times\nabla\tilde{\psi}&=0\quad(\mathbf{x}\in\partial D),\\ \oint_{\partial D_i} H^{-1}\nabla\tilde{\psi}\cdot\hat{\mathbf{n}}\,dl&=0\quad(\mathbf{x}\in\partial D_i).\end{aligned}$$

Multiplying the first equation by $\tilde{\psi}$ and integrating gives

$$\iint H^{-1}(\nabla\tilde{\psi})^2+m^2(\tilde{\psi})^2=0,$$

and therefore $\tilde{\psi}\equiv 0$, *i.e.*, the solution is unique, because Hm^2 is positive. □

B. GEOMETRICAL STRUCTURE

B.1. Hamiltonian

The functional $\mathcal{H}$ in (16) can be rewritten as

$$\mathcal{H}=\tfrac{1}{2}\iint H^{-1}(\nabla\psi)^2+m^2\psi^2, \tag{33}$$

and then its first variation is found to be

$$\begin{aligned}\delta\mathcal{H}&=\iint H^{-1}\nabla\psi\cdot\nabla\delta\psi+m^2\psi\delta\psi\\ &=\sum\oint_{\partial D_i}H^{-1}\psi\nabla\delta\psi\cdot\hat{\mathbf{n}}\,dl-\iint\psi\left(\nabla\cdot(H^{-1}\nabla\delta\psi)-m^2\delta\psi\right)\\ &=\sum\psi_i\delta\tau_i-\iint\psi H\delta(q-m\gamma),\end{aligned}$$

where ψ_i is equal to the value of ψ at ∂D_i, which is a constant in virtue of the first boundary condition in (12). Consequently, the functional derivatives of the Hamiltonian are

$$\frac{\delta\mathcal{H}}{\delta q}=-H\psi,\quad\frac{\delta\mathcal{H}}{\delta\gamma}=mH\psi,\quad\frac{\partial\mathcal{H}}{\partial\tau_i}=\psi_i.$$

Using these and the Lie-Poisson bracket defined in Eq. (17) yields the time derivative (18) of a general functional of state.

B.2. Casimirs

These integrals of motion are the solution of $\{\mathcal{A},\mathcal{C}\} = 0\ \forall\mathcal{A}$, which either are $\mathcal{C} = \tau_i$ (because the circulations do not appear in the Lie-Poisson bracket) or must satisfy

$$\frac{\delta\mathcal{A}}{\delta q}\left(\left[q, H^{-1}\frac{\delta\mathcal{C}}{\delta q}\right] + \left[\gamma, H^{-1}\frac{\delta\mathcal{C}}{\delta\gamma}\right]\right) + \frac{\delta\mathcal{A}}{\delta\gamma}\left[\gamma, H^{-1}\frac{\delta\mathcal{C}}{\delta q}\right] = 0.$$

Since $\mathcal{A}$ is arbitrary, the expressions multiplying its functional derivatives must both vanish. The second term gives $[\gamma, H^{-1}(\delta\mathcal{C}/\delta q)] = 0$, which implies that $\mathcal{C}$ is the integral of

$$HqC_1(\gamma) + C_2(\gamma, \mathbf{x}),$$

with C_1 and C_2 arbitrary. The first term then gives

$$[q, C_1] + \left[\gamma, q\frac{dC_1}{d\gamma} + H^{-1}\frac{\partial C_2}{\partial\gamma}\right] = \left[\gamma, H^{-1}\frac{\partial C_2}{\partial\gamma}\right] = 0,$$

i.e., $C_2 = HC_3(\gamma)$. Consequently, $\mathcal{C}$ is of the form (19).

B.3. Momentum

In order to find the functional $\mathcal{M}$, the bracket (17) is substituted on the left hand side of Eq. (20) and the coefficients of the functional derivatives of $\mathcal{A}$ in both sides are equated. Definition (20) then requires

$$\left[q, H^{-1}\frac{\delta\mathcal{M}}{\delta q}\right] + \left[\gamma, H^{-1}\frac{\delta\mathcal{M}}{\delta\gamma}\right] = H\frac{\partial q}{\partial x},$$

and

$$\left[\gamma, H^{-1}\frac{\delta\mathcal{M}}{\delta q}\right] = H\frac{\partial\gamma}{\partial x}.$$

The second equation implies $\partial_x[H^{-1}(\delta\mathcal{M}/\delta q)] = 0$ and $\partial_y[H^{-1}(\delta\mathcal{M}/\delta q)] = H(y)$. Using this result in the first equation it is found that $[\gamma, H^{-1}(\delta\mathcal{M}/\delta\gamma)] = 0$. Consequently, (20) implies $\delta\mathcal{M}/\delta\gamma = 0$ and $\delta\mathcal{M}/\delta q = \Xi(y)H(y)$, *i.e.*, the momentum functional is given by (21).

B.4. Pseudoenergy

In order to find this integral of motion, a Casimir of the form (19) is chosen so that $\delta\mathcal{C}_E = -\delta\mathcal{H}$. Using $\delta\mathcal{H}[q,\gamma,\tau]$ from Appendix B.1, it easy to see that $\mathcal{C}_E$ is determined by

$$a_i = -\psi_i,$$
$$A(\Gamma) = \Psi(\Gamma)/m,$$
$$Q(\Gamma)F'(\Gamma) + G'(\Gamma) = -\Psi(\Gamma).$$

Therefore, $Q(\Gamma)F''(\Gamma)+G''(\Gamma) = -\Psi'(\Gamma)-Q'(\Gamma)\Psi'(\Gamma)/m$. (The primes indicate derivatives with respect to the argument.) The second variations are then

$$\delta^2\mathcal{H} = \iint H\delta\mathbf{u}_g^2 + m^2\delta\psi^2,$$
$$\delta^2\mathcal{C}_E = \iint H\Psi'\left(2\delta q\delta\gamma - (Q'+m)\delta\gamma^2\right);$$

adding these two equations, expression (23) is found.

In the case of a symmetric basic state, the pseudomomentum is similarly found choosing $\mathcal{C}_M$ so that $\delta\mathcal{C}_M = -\delta\mathcal{M}$. Using Eq. (21) in Eq. (19), implies

$$a_i = 0,$$
$$A(\Gamma) = -\Xi(y),$$
$$Q(\Gamma)F'(\Gamma) + G'(\Gamma) = 0.$$

Consequently $Q(\Gamma)F''(\Gamma) + G''(\Gamma) = -Q'(\Gamma)F'(\Gamma)$ where $F'(\Gamma) = -H/\Gamma_y$. The second variations are $\delta^2\mathcal{M} = 0$ and

$$\delta^2\mathcal{C}_M = -\iint (H^2/\Gamma_y)\left(2\delta q\,\delta\gamma - Q'\,\delta\gamma^2\right) = -\iint (H^2/Q_y)\left(\delta q^2 - \delta\tilde{q}^2\right)$$

Finally, using $\Psi'(\Gamma) = -HU/Q_y$ in the last expression for $\delta^2(\mathcal{H}+\mathcal{C}_E)$, expression (25) follows immediately.

C. Notation and acronyms

-

$\hat{\mathbf{z}}$	vertical unit vector,
$\mathbf{x}$	horizontal position,
t	time,
∇	horizontal nabla operator,
$\vartheta(\mathbf{x},t)$	buoyancy,
$h(\mathbf{x},t)$	layer thickness,
$\mathbf{u}(\mathbf{x},t)$	horizontal velocity,
$\boldsymbol{\Omega}$	Earth's angular velocity,
$f\ (=2\boldsymbol{\Omega}\cdot\hat{\mathbf{z}})$	Coriolis parameter,
β	northward gradient of f,
y	northward coordinate.

- **PEM:** Primitive equations model. These are Euler equations with the Boussinesq approximation (density variations are neglected, $\rho \approx \bar{\rho}$, except in buoyancy), the "traditional" approximation (only the Coriolis force due to the vertical component of $\boldsymbol{\Omega}$ is included) and the hydrostatic approximation (vertical pressure gradient in balance with buoyancy force).

- **QGM:** Quasi-geostrophic model. A low frequency approximation of PEM, in which the horizontal velocity is diagnosed from the geostrophic balance (between the horizontal pressure gradient and the Coriolis force).
- **HL:** Homogeneous layer(s) [model]. Density ρ is constant in each active layer (where velocity is assumed depth independent).
- **IL:** Inhomogeneous layer(s) [model]. Density may vary horizontally and with time in each active layer, but not with depth (just like for the horizontal velocity field).
- **Main balances:** *Hydrostatic:* between the vertical pressure gradient and buoyancy. *Geostrophic:* between the horizontal pressure gradient and the Coriolis force (proportional to the horizontal velocity). *Thermal wind*: between horizontal density gradient and the vertical shear of horizontal velocity (this balance is but a consequence of the hydrostatic and geostrophic ones).
- **Poincaré wave:** High frequency oscillation (gravity wave affected by the Coriolis effect). The period may not be larger than the inertial one, $2\pi/|f|$.
- **Rossby wave:** Low frequency oscillation. The period is much larger than the inertial one.

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