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On the evaluation of the magnetostatic field due to stationary currents in toroidal solenoids

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ABSTRACT. The study of the magnetostatic field due to a stationary current in a coil wound around a toroid is usually limited to the determination of the toroidal magnetic induction due to the poloidal component of the current using Ampère's circuital law. This paper emphasizes that poloidal (toroidal) currents produce toroidal (poloidal) magnetic induction fields, presenting the explicit integration of Poisson's equation for the toroidal magnetic induction based on the use of the harmonic expansion of the inverse of the source point-field point distance; the presence and importance of the circling field due to the azimuthal currents associated with the pitch in the coil is also discussed.

RESUMEN. El estudio del campo magnetostático debido a una corriente estacionaria en un embobinado alrededor de un toroide usualmente se limita a la determinación de la inducción magnética toroidal debida a la componente poloidal de la corriente usando la ley circuital de Ampère. Este artículo destaca que las corrientes poloidales (toroidales) producen campos de inducción magnética toroidales (poloidales), presentando la integración explícita de la ecuación de Poisson para la inducción magnética toroidal basada en el desarrollo armónico del inverso de la distancia del punto fuente y el punto campo; también se discute la presencia y la importancia del campo circulante debido a las corrientes azimutales asociadas al paso del embobinado.

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1. INTRODUCTION

Toroidal solenoids are interesting from the point of view of practical applications, such as the confinement of charged particles [1] and the storage of energy [2]; as well as in fundamental physics studies, such as the Aharonov-Bohm effect [3], Zeldovich's parity-violating anapole [4], and the recognition of the existence of toroidal moments [5]. From a didactical point of view and when the pitch is ignored [6, 7], the magnetostatic fields of infinitely straight and toroidal solenoids have the following common features: *i*) the magnetic induction is restricted to the interior of the solenoids vanishing outside, *ii*) the evaluation of the magnetic induction follows from a straightforward application of Ampère's circuital law and the symmetries of the respective solenoids, and *iii*) these properties are valid for any cross section of the solenoids. Features *i*) and *ii*) are found in many textbooks [8–16], of which only few show that both features cease to be valid when the pitch is taken into account [13–16]. Feature *iii*) has been analyzed in detail for the case of straight solenoids [17–24], but the counterpart for toroidal solenoids is practically absent in textbooks and the didactic literature.

The typical half-a-page treatment of Refs. [8–12] has been practically the only one used in the classroom for teaching generations of scientists and engineers about toroidal solenoids; few of them have learned about the effect of the pitch [13–16] as witnessed by the need of reminders [6,7]. The new generations facing the currents problems [1–5] need a better basis to understand the latter qualitatively and quantitatively, before solving them. This paper presents an alternative for the study of toroidal solenoids at the junior-senior level. Section 2 contains the general discussion of the correspondence between the poloidal and toroidal components of the current and the toroidal and poloidal components of the magnetic induction, respectively. Section 3 presents the explicit integration of Poisson's equation for the magnetic induction due to the poloidal current in a toroidal solenoid with a rectangular cross section, using the harmonic expansion of the Green function in circular cylindrical coordinates. Then the case of a general cross section is also analyzed by using the superposition principle. Section 4 consists of a discussion of the results and their connections and analogies with other situations of magnetostatics and electrostatics; the effect of the pitch is discussed specifically, and here we can point out that the circling magnetic induction arising from the azimuthal currents is not explicitly evaluated, since it may be illustrated through the familiar situation of a circular loop. Formulas for the harmonic functions and the expansions in terms of them in circular cylindrical coordinates are included in the Appendix.

2. POLOIDAL (TOROIDAL) CURRENTS PRODUCE TOROIDAL (POLOIDAL) MAGNETIC INDUCTION FIELDS

The description of the currents in solenoids in textbooks is usually limited to a description with words and some drawing illustration [8–16]. Such descriptions in the case of toroidal solenoids imply the invariance of the system under any rotation around the axis of the solenoid, even though the drawings often contradict such an implication. This situation indicates the need of a quantitative description of the currents involved.

In any magnetostatic situation, the current density and the magnetic induction have the common property of being divergenceless or transverse vector fields. Additionally, they are transverse to each other being related through Ampère's law. Any vector field in three dimensional space has in general three components; however, the transversality condition is equivalent to a vanishing longitudinal component, thus reducing the number of independent components to two. In the remainder of this section we describe the corresponding components of the current in toroidal solenoids and identify the respective components of the magnetic induction produced by them.

The filamentary current elements in a toroidal solenoid with a total of N turns and a current I can be written in circular cylindrical coordinates [25] in the form

$$I_{\mathrm{f}}\, d\vec{\ell} = \frac{NI}{2\pi}\left(\hat{k}\, dz + \hat{R}\, dR + \hat{\phi} R\, d\phi\right). \tag{1}$$

The axial and radial components contained in meridian planes follow the periphery of the corresponding cross sections of the toroid constituting the so-called poloidal current. This poloidal component of the current distribution is invariant under rotations around the axis of the toroid, and is responsible for the magnetic induction field to be in the azimuthal direction, inversely proportional to the distance from the axis and restricted to the interior of the solenoid. On the other hand, the azimuthal component of the current distribution, *i.e.*, the third term in Eq. (1), is associated with the pitch in the winding of the coil; it breaks the rotational invariance and invalidates features *i*) and *ii*) of the magnetostatic field described in the Introduction [6,7,13–16].

It is instructive to compare the complementary characteristics of the appropriate components of currents and magnetostatic induction fields in toroidal solenoids and circular loops. The poloidal currents in the toroidal solenoids produce a toroidal magnetic induction. The current in a circular loop is toroidal, being in the azimuthal direction and of the same intensity at a fixed radial distance, and produces a poloidal magnetic induction characterized by closed lines in meridian planes. This complementarity is a consequence of Ampère's law and the axial rotational symmetry of both systems.

3. Integration of Poisson's equation for the magnetic induction

From Ampere's and Gauss' laws it follows that the magnetic induction field satisfies Poisson's equation

$$\nabla^2 \vec{B} = -4\pi K \nabla \times \vec{J}, \tag{2}$$

where the curl of the current density plays the role of the source. The magnetic induction as a solution of this equation can be written as

$$\vec{B}(\vec{r}) = K \int \frac{d^3r' \, \nabla' \times \vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}. \tag{3}$$

In order to evaluate the magnetic induction of a toroidal solenoid we consider first the case of a toroid with a rectangular cross section and a poloidal current (Fig. 1a). Such a toroid is defined in circular cylindrical coordinates by its edge parallel circles: inner lower ($R = a, \phi, z = z_1$), inner upper ($R = a, \phi, z = z_2$), outer upper ($R = b, \phi, z = z_2$) and outer lower ($R = b, \phi, z = z_1$) with $0 < a < b$ and $z_1 < z_2$. The poloidal current density can then be written as

$$\begin{aligned} \vec{J}(\vec{r}) = \frac{NI}{2\pi R} \Big\{ &\hat{k} \left[\delta(R-a) - \delta(R-b)\right] \left[\Theta(z-z_1) - \Theta(z-z_2)\right] \\ &+ \hat{R} \left[\delta(z-z_2) - \delta(z-z_1)\right] \left[\Theta(R-a) - \Theta(R-b)\right] \Big\}, \end{aligned} \tag{4}$$

in terms of the Dirac delta functions, which define coil elements along which the current flows, and of the Heaviside step functions, which define the extent of those elements.

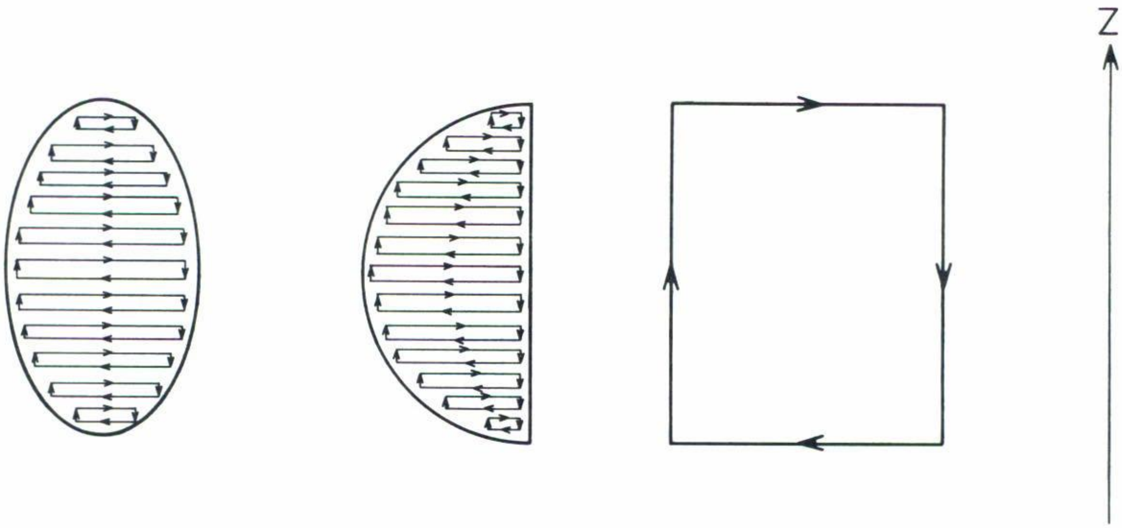

FIGURE 1. Any cross section of a toroid can be decomposed into rectangular cross sections, such that the current around the perimeter of the general cross section is equivalent to the superposition of the currents around the respective rectangular loops. The total magnetic induction field of the general toroidal solenoid is also the superposition of the fields produced by the individual toroidal solenoids with rectangular cross sections.

The integral of Eq. (3) requires the curl of the current density

$$\nabla' \times \vec{J}(\vec{r}\,') = \frac{NI}{2\pi}\hat{\phi}'\left\{\frac{\Theta(R'-a)-\Theta(R'-b)}{R'}\frac{\partial}{\partial z'}\left[\delta(z'-z_2)-\delta(z'-z_1)\right]\right.$$
$$\left. - \frac{\partial}{\partial R'}\left[\frac{\delta(R'-a)-\delta(R'-b)}{R'}\right]\left[\Theta(z'-z_1)-\Theta(z'-z_2)\right]\right\}, \qquad (5)$$

which is recognized to be of toroidal character.

The integral in Eq. (3) also requires the expression of the inverse of the source point field point distance in circular cylindrical coordinates, for which we use the harmonic expansion of Eq. (A1).

With these ingredients the integral for the magnetic induction becomes

$$\vec{B}(\vec{r}) = \frac{KNI}{2\pi}\int_0^{2\pi}\hat{\phi}'\,d\phi'\int_0^{\infty}R'\,dR'\int_{-\infty}^{\infty}dz'$$
$$\left\{\frac{\Theta(R'-a)-\Theta(R'-b)}{R'}\frac{\partial}{\partial z'}\left[\delta(z'-z_2)-\delta(z'-z_1)\right]\right.$$
$$\left. - \frac{\partial}{\partial R'}\left[\frac{\delta(R'-a)-\delta(R'-b)}{R'}\right]\left[\Theta(z'-z_1)-\Theta(z'-z_2)\right]\right\}$$
$$\frac{2}{\pi}\sum_{m=0}^{\infty}\int_0^{\infty}dk\, I_m(kR_<)\,K_m(kR_>)\cos k(z'-z)\epsilon_m\cos m(\phi'-\phi). \qquad (6)$$

The integration over the azimuthal angle,

$$\int_0^{2\pi} \hat{\phi}'\, d\phi'\, \epsilon_m \cos m(\phi' - \phi) = 2\pi\delta_{m1}\hat{\phi}, \tag{7}$$

selects the $m = 1$ term in the sum of Eq. (6) and determines the direction of the magnetic induction field. The factors involving the differences of the Heaviside step functions restrict the intervals of integration to the extent of the respective coil elements. The integrations over the axial coordinate can be done by parts:

$$\int_{-\infty}^{\infty} dz' \frac{\partial}{\partial z'} \left[\delta(z' - z_2) - \delta(z' - z_1)\right] \cos k(z' - z) = k \left[\sin k(z - z_1) - \sin k(z - z_2)\right], \tag{8}$$

and directly

$$\int_{z_1}^{z_2} dz' \cos k(z' - z) = \frac{1}{k} \left[\sin k(z - z_1) - \sin k(z - z_2)\right]. \tag{9}$$

Thus Eq. (6) is reduced to

$$\begin{aligned}\vec{B}(\vec{r}) = \frac{2KNI}{\pi}\hat{\phi} \int_0^{\infty} & dk \left[\sin k(z - z_1) - \sin k(z - z_2)\right] \\ & \left\{ k \int_a^b dR' I_1(kR_<) K_1(kR_>) \right. \\ & \left. - \frac{1}{k} \int_0^{\infty} dR'\, R' I_1(kR_<)\, K_1(kR_>) \frac{\partial}{\partial R'} \left[\frac{\delta(R' - a) - \delta(R' - b)}{R'}\right] \right\}. \end{aligned} \tag{10}$$

Before we can go any further it is necessary to distinguish between the different locations of the field point $R < a$, $a < R < b$, or $b < R$.

The field inside the inner cylinder

$$\begin{aligned} B(R < a, \phi, z) &= \frac{2KNI}{\pi}\hat{\phi} \int_0^{\infty} dk \left[\sin k(z - z_1) - \sin k(z - z_2)\right] I_1(kR) \\ & \qquad \left\{ k \int_a^b dR'\, K_1(kR') - \frac{1}{kR'} \frac{\partial}{\partial R'} \left[R' K_1(kR')\right] \Big|_{R'=a}^{R'=b} \right\} \\ &= 0 \end{aligned} \tag{11}$$

vanishes because the two radial integrals inside the curly brackets cancell each other, according to Eqs. (A2-A3).

Similarly the field outside the outer cylinder also vanishes:

$$\begin{aligned} B(R > a, \phi, z) &= \frac{2KNI}{\pi}\hat{\phi} \int_0^\infty dk\, [\sin k(z-z_1) - \sin k(z-z_2)]\, K_1(kR) \\ &\quad \left\{ k\int_a^b dR' I_1(kR') - \frac{1}{kR'}\frac{\partial}{\partial R'}\left[R' I_1(kR')\right]\Big|_{R'=a}^{R'=b} \right\} \\ &= 0. \end{aligned} \tag{12}$$

As for the field between the inner and outer cylinders

$$\begin{aligned} B(a < R < b, \phi, z) &= \frac{2KNI}{\pi}\hat{\phi} \int_0^\infty dk\, [\sin k(z-z_1) - \sin k(z-z_2)] \\ &\quad \left\{ k\left[K_1(kR)\int_a^R dR'\, I_1(kR') + I_1(kR)\int_R^b dR'\, K_1(kR')\right] \right. \\ &\quad -\frac{1}{kR'}\left[K_1(kR)\frac{\partial}{\partial R'}\left[R' I_1(kR')\right]\Big|_{R'=a}^{R'=R} \right. \\ &\quad \left.\left. +\; I_1(kR)\frac{\partial}{\partial R'}\left[R' K_1(kR')\right]\Big|_{R'=R}^{R'=b}\right]\right\} \\ &= \frac{2KNI}{\pi}\hat{\phi}\int_0^\infty dk\, [\sin k(z-z_1) - \sin k(z-z_2)] \\ &\quad \times \left[K_1(kR)I_0(kR) + I_1(kR)K_0(kR)\right]. \end{aligned} \tag{13}$$

Here we find the same cancellations that occurred in Eqs. (11) and (12), and the remaining two terms are identified with the Wronskian of the modified Bessel functions of order zero, Eq. (A4). The remaining integral

$$\begin{aligned} \vec{B}(a < R < b, \phi, z) &= \frac{2KNI}{R\pi}\hat{\phi}\int_0^\infty \frac{dk}{k}\, [\sin k(z-z_1) - \sin k(z-z_2)] \\ &= \frac{2KNI}{R}\hat{\phi}\,[\Theta(z-z_1) - \Theta(z-z_2)] \end{aligned} \tag{14}$$

can be identified with the difference of the step functions in the axial direction which ensure that the field vanishes outside the solenoid, Eqs. (A5-A6). Moreover the magnetic induction field is toroidal and varies inversely with the radial distance from the axis.

The case of a toroidal solenoid with a general cross section can be analyzed via the superposition principle after the case of the rectangular cross section has been established. In fact, any cross section can be approximated with a mesh of rectangles, of infinitesimal size if necessary, as illustrated in Fig. 1. By considering poloidal currents circulating in each small loop of the mesh, the net current distribution is along the periphery of the general cross section of the toroid because the currents in the inner components of the smaller loops cancel by pairs among neighboring loops in the mesh. Thus, any toroid with a general cross section is analyzed as decomposed into a collection of coaxial toroids with the chosen rectangular cross sections. According to the superposition principle the magnetic induction field of the toroidal solenoid is the superposition of the contributions of each and every one of the component toroidal solenoids. As it was established previously such contributions are zero outside each toroid and azimuthal and inversely proportional to the distance from the axis; the same properties are translated for the total field. In conclusion, the total magnetic induction for a toroid with a meridian cross section defined by the alternative equations $R = R_C(z)$ or $z = z_C(R)$ can be written in the form,

$$\begin{aligned}\vec{B}(R,\phi,z) = \frac{2KNI}{R}\hat{\phi}\sum_{i=1}^{H}\left[\Theta(R - R^{\mathrm{I}}_{C_i}(z)) - \Theta(R - R^{\mathrm{O}}_{C_i}(z))\right]\\ \sum_{j=1}^{V}\left[\Theta(z - z^{\mathrm{L}}_{C_j}(R)) - \Theta(z - z^{\mathrm{U}}_{C_j}(R))\right]\end{aligned} \tag{15}$$

where i describes each of the H successive horizontal segments at the chosen axial coordinates inside the toroidal cross section and defined by the radial coordinates $R^{\mathrm{I}}_{C_i}$ and $R^{\mathrm{O}}_{C_i}$ of their inner and outer end points, respectively; and similarly, j describes each one of the V successive vertical segments at the chosen radial coordinate inside the toroidal cross section and defined by the axial coordinates $z^{\mathrm{L}}_{C_j}$ and $z^{\mathrm{U}}_{C_j}$ of their lower and upper end points, respectively.

The differential form of Ampère's law can be used with Eq. (15) to obtain the current distribution in the toroidal solenoid:

$$\begin{aligned}\vec{J} &= \frac{1}{4\pi K}\nabla\times\vec{B}\\ &= \frac{NI}{2\pi R}\sum_{i=1}^{H}\sum_{j=1}^{V}\Bigg\{\left[\Theta(R - R^{\mathrm{I}}_{C_i}(z)) - \Theta(R - R^{\mathrm{O}}_{C_i}(z))\right]\\ &\quad\left[\delta(z - z^{\mathrm{U}}_{C_j}(R))\left(\hat{R} + \hat{k}\frac{dz^{\mathrm{U}}_{C_j}}{dR}\right) - \delta(z - z^{\mathrm{L}}_{C_j}(R))\left(\hat{R} + \hat{k}\frac{dz^{\mathrm{L}}_{C_j}}{dR}\right)\right]\\ &\quad+\left[\Theta(z - z^{\mathrm{L}}_{C_j}(R)) - \Theta(z - z^{\mathrm{U}}_{C_j}(R))\right]\\ &\quad\times\left[\delta(R - R^{\mathrm{I}}_{C_i}(z))\left(\hat{R}\frac{dR^{\mathrm{I}}_{C_i}}{dz} + \hat{k}\right) - \delta(R - R^{\mathrm{O}}_{C_i}(z))\left(\hat{R}\frac{dR^{\mathrm{O}}_{C_i}}{dz} + \hat{k}\right)\right]\Bigg\}.\end{aligned} \tag{16}$$

For given values of R and z only one term in each sum contributes, and only one of the four combinations (I,L), (I,U), (O,U) or (O,L) survives.

Here it should be noted that the presence of the Dirac delta functions reflect that the current is restricted to the periphery of the cross section, and that the direction of the current $\hat{R}\,dR_C + \hat{k}\,dz_C$ is tangential to that periphery. It is straightforward to check that Eq. (16) for $\vec{J}(\vec{r}\,')\,dv' \to I_{\mathrm{f}}\,d\vec{l}'$ reproduces the poloidal filamentary current of Eq. (1).

4. DISCUSSION

The study of the magnetostatic field of toroidal selenoids presented in this paper has emphasized the distinction of the poloidal and azimuthal components of the current and the characteristics of the fields associated with each one. The role of axial rotational symmetry in toroidal solenoids and circular loops was used in Sect. 2 to illustrate the complementary character of poloidal and toroidal currents and the corresponding toroidal and poloidal magnetic inductions for toroids with rectangular cross sections in circular cylindrical coordinates and gives the argumentation to show that the poloidal currents around a toroidal solenoid with any cross section produce a toroidal magnetic induction field that varies inversely proportional to the distance from the axis in the interior of the solenoid and vanishes outside. The space variation of the field is common with that of the field due to a straight line of current [8–16]. The connection can be traced by considering the latter as the limit situation of the toroids with rectangular cross sections for $(a \to 0, b \to \infty, z_1 \to -\infty, z_2 \to \infty)$. On the other hand, there is also the connection pointed out in textbooks [8–16] between the fields of toroidal and infinitely straight solenoids, corresponding to the limit situations of large distances from the axis where the field tends to be uniform.

The importance of the pitch in the winding of the coil around the torus has already been pointed out [6, 7, 13–16]. Going back to Eq. (1), it is necessary to take into account the azimuthal component of the current distribution, where $R(\phi, z)$ may break the axial rotational invariance. The contribution to the field of this azimuthal component may be evaluated by the same methods of Section 3. If the winding is such that the loops are kept in meridian planes and the connections among them take place at a chosen parallel circle, which could be the outer equatorial circle, then the circling field due to such a toroidal component of the current is poloidal and invalidates features *i*)–*iii*) of the Introduction.

The student of electrostatics is familiar with Gauss' law, the Coulomb field outside a uniformly charged spherical shell, and the vanishing of the field inside the sphere. The magnetostatic counterpart uses the name of Ampère instead of Gauss' and Coulomb's, toroids instead of sphere, and azimuthal instead of radial, and exchanges inside and outside. The analogies may be extended to non-spherical conductors and solenoids wound around closed curves, provided the pitch is ignored. Of course the effect of the latter can always be added, just as departures from being a perfect conductor.

A. APPENDIX

The harmonic function expansion of the inverse of the distance between the source point and the field point [26, 27]

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \frac{2}{\pi} \sum_{m=0}^{\infty} \int_0^{\infty} dk\, I_m(kR_<)\, K_m(kR_>) \cos k(z' - z) \epsilon_m \cos m(\phi' - \phi), \tag{A1}$$

is expressed in terms of modified Bessel functions in the radial coordinates and cosine functions in the axial and azimuthal coordinates; here $\epsilon_m = 1$ for $m = 0$ and $\epsilon_m = 2$

for $m = 1, 2, \ldots$. The derivatives of the modified Bessel functions of order zero give the corresponding functions of order one

$$\frac{dI_0(x)}{dx} = I_1(x), \qquad \frac{dK_0(x)}{dx} = -K_1(x). \tag{A2}$$

Also

$$\frac{1}{x}\frac{d}{dx}\Big[xI_1(x)\Big] = I_0(x), \qquad \frac{1}{x}\frac{d}{dx}\Big[xK_1(x)\Big] = K_0(x). \tag{A3}$$

The radial factor in the integral of Eq. (13) is identified with the Wronskian of the modified Bessel functions of order zero,

$$I_0(x)K_1(x) + I_1(x)K_0(x) = -I_0(x)\,\frac{dK_0(x)}{dx} + \frac{dI_0(x)}{dx}K_0(x) = \frac{1}{x}. \tag{A4}$$

The Dirac delta function in the axial coordinate has the harmonic function representation

$$\delta(z - z_i) = \frac{1}{\pi}\int_0^\infty dk\, \cos k(z - z_i). \tag{A5}$$

Its integral leads to the corresponding representation for the Heaviside step function

$$\Theta(z - z_i) = \int_{-\infty}^{z} \delta(z' - z_i)\, dz' = \frac{1}{\pi}\int_0^\infty \frac{dk}{k}\sin k(z - z_i). \tag{A6}$$

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