Revista Mexicana de Física **42**, *No. 2 (1996) 161–171*

Mechanics of self-affine cracks: the concept of equivalent traction, path integrals and energy release rate

ALEXANDER S. BALANKIN
Instituto Tecnológico y de Estudios Superiores de Monterrey
Campus Estado de México, México 52629

Recibido el 9 de agosto de 1995; aceptado el 15 de noviembre de 1995

ABSTRACT. A novel concept of equivalent traction is suggested to account for the mechanics of self-affine cracks. This concept is used to construct path-independent integrals for some problems with self-affine crack, which in turn are used to establish the asymptotic stress field near the wedge-like notch with self-affine edges and the energy release rate associated with self-affine crack propagation. The unloading of self-affine boundaries of an elastic solid is predicted for the first time. Some new useful relations are also derived. The theoretical results are discussed with respect to recent experimental observations.

RESUMEN. Se sugiere un nuevo concepto de tracción equivalente para la mecánica de las grietas auto-afines. Se construyen, empleando este concepto, algunas integrales independientes de la trayectoria para algunos problemas con grietas auto-afines, las cuales a su vez son utilizadas para establecer el campo de tensión asintótica cerca del corte en cuña con bordes auto-afines y la rapidez de liberación de energía asociada con la propagación de la grieta auto-afín. Se predice la descarga de fronteras auto-afines de un sólido elástico. Se derivan también algunas relaciones nuevas que son de utilidad. Se discuten los resultados teóricos respecto a observaciones experimentales recientes.

PACS: 03.40.Dz; 46.30N; 62.20M

The propagation of cracks is a problem of both technological and scientific interest. This has motivated a large amount of research into how cracks form, and how, once formed, they propagate.

It is well known that real cracks in solid materials have little resemblence to ideal cracks with smooth edges, which are usually considered in fracture mechanics [1, 2]. For this reason, in recent years, the quantitative analysis of fractured surfaces has become an integral part of the study of deformation and rupture of materials [3]. Such surface analysis often provides information about surface morphology which is complementary to that obtained by other metallurgical methods. It is now clearly established that fracture surfaces can be considered as self-affine objects [1, 4].[1] Hence the usual treatments based on the continuum elasticity theory do not provide simple tools for discussing the essential

[1] The concept of self-affine fractals is being greatly useful in identifying a hidden symmetry in a wide variety of objects and phenomena in nature [5].

nonlinearities of the problem. At the same time, it has been emphasized that a merely fractal description of crack faces can hardly benefit our understanding of the mechanism underlying failure phenomena [1,2,6–12].

Fractal geometry, developed by Mandelbrot [13], allows the description of irregular forms which are more complex than Euclidean shapes. A feature having fractal property can be characterized by its fractal dimension D [5,13]. It should be emphasized that most experimental studies indicate that fracture surfaces and crack propagation paths are self-affine rather than self-similar [1,4]. In the application of fractal models to real crack faces the concepts of self-similar and self-affine fractals must be carefully distinguished [11,12, 14].

The fundamental difference between self-similar and self-affine fractals is the way scaling will produce statistical equivalence.[2] Self-affine crack faces are statistically invariant under an affine transformation [13]: $x' \to \lambda_x x$, $y' \to \lambda_y y$, $z' \to \lambda_z z$. Requiring that such transformations be combined, a group structure is implied. As a consequence λ_y and λ_z have to be homogeneous functions of, say, λ_x; both scale as $\lambda_y \propto \lambda_x^{\nu_y}$, $\lambda_z \propto \lambda_x^{\nu_z}$, but the exponents ν_y and ν_z are in general different.[3] If so, then $\lambda_z \propto \lambda_y^H$, where the roughness (or Hurst) exponent [13] is given by the relationship[4]

$$H = \frac{\nu_z}{\nu_y}.$$

In the special case of an isotropic surface with mean plane parallel to the coordinate plane (x, y), we have $\nu_y \equiv 1$, so that $H = \nu_z$. The last relation is also valid for any self-affine profile on a two-dimensional plane. As an example of a self-affine curve we can refer to the graph of the Mandelbrot-Weierstrass function

$$z(x) = \sum_{k=0}^{\infty} \lambda^{-Hk} \cos\left(\lambda^k x\right), \quad \lambda > 1,$$

which is nowhere differentiable and is often used as a model for the real crack faces [15,16]. The standard deviation of this function obeys scaling behavior[5]

$$\langle z \rangle \propto \langle x \rangle^H. \tag{1}$$

However, in contrast with self-similar fractals, the fractal dimension of self-affine patterns is not uniquely defined. First of all we must distinguish between the local ($L \ll \xi_C$) and

[2] Self-similar fractals may be scaled equally in the x- and y-directions to produce statistically equivalent profiles, whereas self-affine fractals must be scaled by different amounts in the x- and y-directions to produce statistical equivalence.

[3] The exponents ν_y and ν_z are analogous to those of anisotropic correlation lengths in critical phenomena such as in liquid crystals [5].

[4] Notice that in the case of any self-similar fractal $\nu_y = \nu_z = 1/D$, so that $H \equiv 1$. This means that the Hurst exponent is useless when discussing both self-affine and self-similar fractals in the same context.

[5] It should be emphasized that the relation (1) is valid only for the standard deviations of coordinates $\langle z \rangle$ and $\langle x \rangle$, but is not valid for the self-affine function $z(x)$ itself.

the global ($L \gg \xi_C$) fractal dimensions.[6] The latter is always equal to the topological dimension of the self-affine fractal, while in the local limit there are various definitions for different fractal dimensions, which are associated with different scaling properties of a self-affine fractal. The relationships between Hurst exponent and various fractal dimensions are given in Ref. [11]. Here we note only that the metric (box-counting) fractal dimension D_B is equal to $d-H$, where d is the topological dimension of the surrounding Euclidean space, while the latent (local divider) fractal dimension is $D_D = (d-1)/H$, if $H \geq (d-1)/d$, or $D_D = d$, if $H \leq (d-1)/d$ [5, 11].[7]

It was shown in Refs. [11, 12] that the acceptance of the self-affine geometry of crack faces leads to a change in the asymptotic stress field near the tip of a self-affine crack, and affects in this way the fracture toughness. Namely, stresses associated with a self-affine crack are less singular than the stress field in the vicinity of a linear cut. The explicit expressions for stress singularity exponent, which is associated with a self-affine crack, were derived in Ref. [11] within a framework of dimensional analysis on the basis of the energy balance arguments. At the same time, the authors of Ref. [17] have adduced arguments to show the incompatibility of the conventional fracture mechanics with the fractal geometry of fracture patterns. Because of this, the incorporation of fractal concepts to fracture mechanics is a problem of critical importance. One way of doing this is suggested below.

Before proceeding further, we note that any real crack path obeys self-affine properties only within a bounded interval of length scales

$$\ell_0 < L < \xi_C, \tag{2}$$

where $\ell_0 \sim 10^{-6}$ m is the microscopic cutoff (the dislocation free zone size [1–3, 18]) and $\xi_C \sim 10^{-3}$ m is the self-affine correlation length [1–3, 19]). Hence, a crack face can be treated as a nonstandard curve (surface), the standard part of which is a self-affine fractal.[8] In this way we can build curvilinear coordinates along any "self-affine" crack face; so that we can define the change in the normal vector to the crack face along its trajectory, and in turn analyze a problem with self-affine crack within a framework of a powerful tools of the continuum mechanics.

Now, let us consider an infinite linear elastic solid with traction-free self-affine crack with a mean plane perpendicular to the direction of the tensile stress σ_{ii} prescribed at infinity (see Fig. 1). This problem is related to the tensile mode (Mode I) of loading for the problem with a linear crack (cut). At the same time, we note that, while in the problem with linear crack (cut) the crack tip is the unique singular point for the elastic fields, in the case of a self-affine crack singular points of the elastic field occur not only at the crack tip but also on the rough crack faces, and within the interval (2) singularities exist at all scales. Moreover, in the case of self-affine crack, the crack singularity field produced by any linear part of self-affine crack[9] is a linear superposition of the three basic modes of

[6] L is the characteristic scale of measurements and ξ_C is the self-affine correlation length [5, 11].

[7] Notice that for self-similar fractals $D_D = D_B$ (see also footnote 4).

[8] The foundations of non-standard analysis and its application to some problems of fractal geometry is considered in [20].

[9] These parts have lengths of the order of ℓ_0.

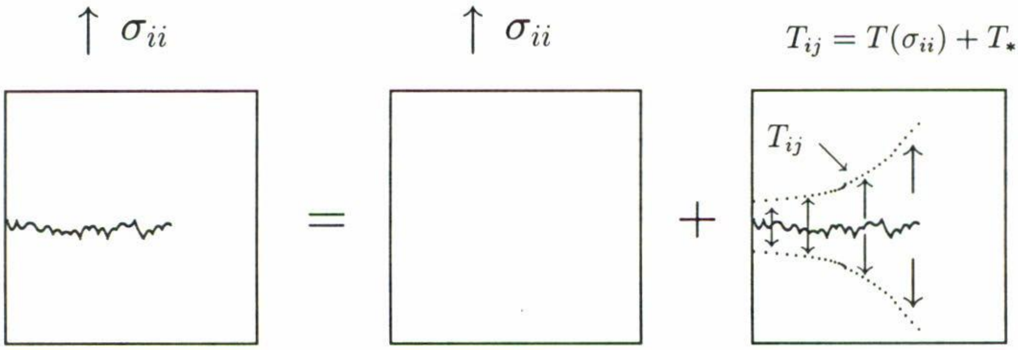

FIGURE 1. Reduction of the problem with self-affine crack and longitudinal tensile stress σ_{ij} to the problem with self-affine crack loaded by unknown traction T_{ij}.

cracking [21,22]. However, in any plane problem this field is a linear superposition of only two modes [22][10] and can be represented as [6]

$$
\begin{aligned}
\sigma_{xx} &= \frac{K_{\rm I}}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - k\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\right],\\
\sigma_{yy} &= \frac{K_{\rm I}}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + k\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right],\\
\sigma_{zz} &= \nu(\sigma_{xx}+\sigma_{yy}), \qquad k = \frac{K_{\rm II}}{K_{\rm I}} \propto \sin\left(\frac{\pi}{2}-\psi\right),
\end{aligned}
\tag{3}
$$

where ν is the Poisson's ratio, $K_{\rm I}$ and $K_{\rm II}$ are the stress intensity factors associated with modes I and II, respectively; ψ is the angle between the stress applied at infinity and the linear nanocrack face, and θ is the angle between the (*nano*)crack face and the direction of observation (see Fig. 2a).

Furthermore, if the principle of superposition [22] for contributions from all parts (nanocracks) is valid, the problem under consideration can be replaced by an equivalent one: *self-affine crack faces are loaded by a traction T_{ij} and stresses vanish at infinity*, as shown in Fig. 1.

Traction T_{ij} consists of the regular σ–induced traction $T(\sigma_{ii})$ and the unknown additional traction T^*_{ij} due to the contribution from all singular points along the crack faces. The last (singular) term of stress field near the self-affine can be represented as a sum of the contributions (3) from all linear parts (nanocracks) along the crack edges (within the interval of self-affinity (2)!). By virtue of the fact that $\ell_0 \ll \xi_{\rm C}$ this sum can be replaced by the integral

$$
\sigma_{ij} \propto \phi(\theta,\nu)\int_{-\xi_{\rm C}}^{X} \frac{n(-x)}{|x|\sqrt{|x|}}\,\Theta(-x)\,dx \propto K_{\rm f} X^{-\alpha}, \tag{4}
$$

[10] Namely, the tensile (Mode I) and in-plane shear (Mode II) modes.

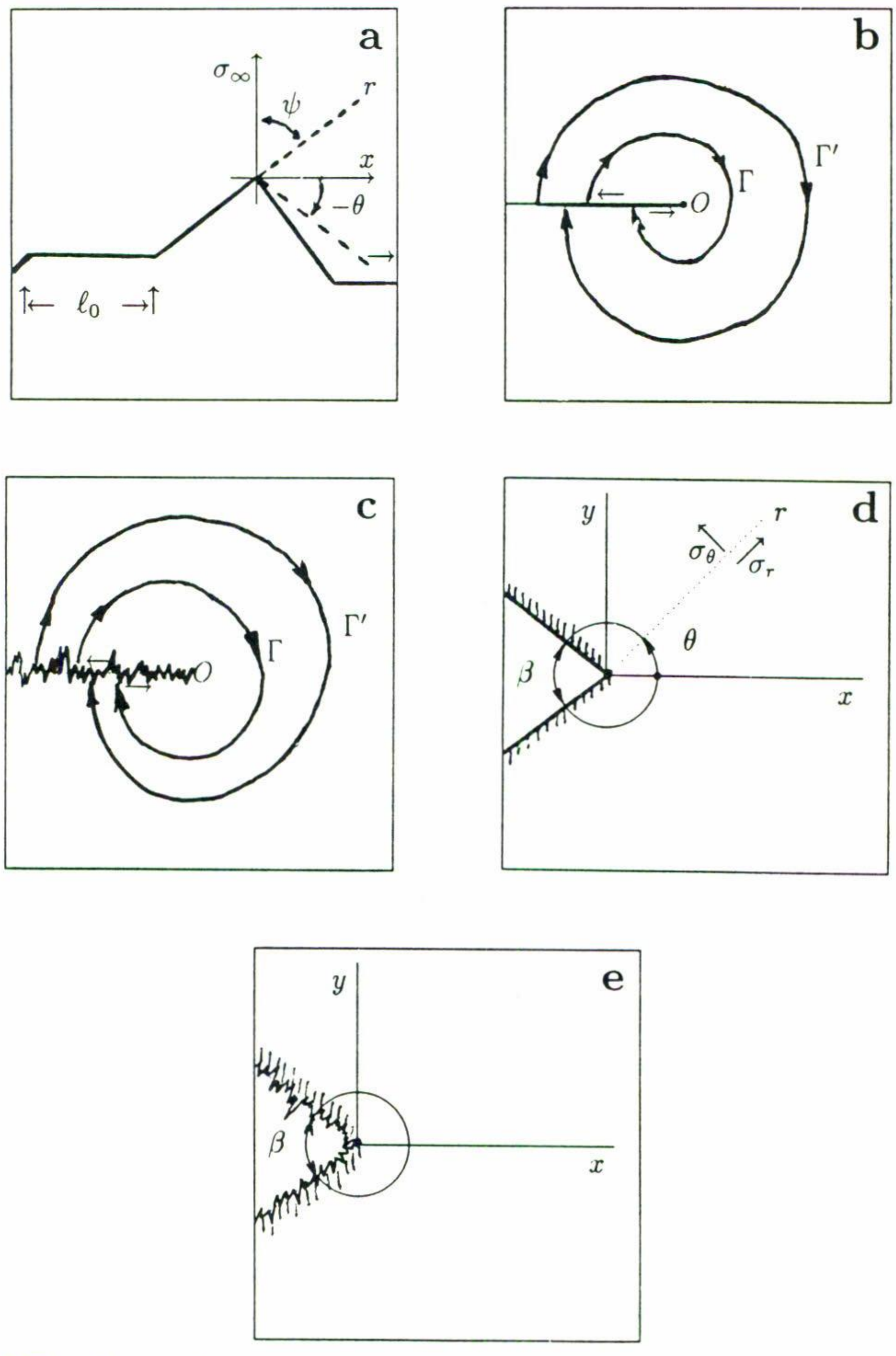

FIGURE 2. The polar coordinate system (a); paths of J integration for cracks with smooth edges (b): $J(\Gamma) = J(\Gamma') = G$; and with self-affine edges (c): $J(\Gamma) < J(\Gamma')$, but $J_f(\Gamma) = J_f(\Gamma')$; and polar stresses at a wedge-like notches with smooth (d) and self-affine (e) edges.

where $X = r/\ell_0$, r is the distance from the crack tip and x is a variable; $n(-x) \propto x^\eta$ is the number of singular points within the interval $(-x, 0)$, and $\Theta(\ldots)$ is the Heaviside unit function, $\Theta(-x) = 1$ for $x \leq 0$ and $\Theta = 0$ for $x > 0$; $\phi_{ij}(\theta, \nu)$ is the dimensionless function of θ and ν.[11] The coefficient $K_{\mathrm{f}}(\theta\nu)$ may be treated as the stress intensity factor for self-affine crack (instead of common stress intensity factors K_{I} and K_{II} in conventional fracture mechanics).

For the problem with a regular (smooth) crack (that is, the special case of the problem under consideration!) we simply have $H \equiv 1$ and $\eta = 0$, so that the singular term of an elastic field obeys the standard inverse square root asymptotic [21],

$$\sigma_{ij} = \frac{K_{\mathrm{I}}}{\sqrt{r}} \phi_s(\theta, \nu), \tag{5}$$

while for the graph of an independent random (Wiener) process (which was used in Ref. [21] as a model of crack trajectory) $H = 1/2$ and $\eta = 1/2$, so that there is no stress field (*power law*)singularity, because $\alpha = 0$ ($\sigma_{ij} \propto \ln X$).

Generally, we can expect that η is a monotone increasing function of the roughness (Hurst) exponent H. The last varies from 0 to 1 [5], so that always $\eta > 0$ and $0 \leq \alpha \leq 1/2$.[12] Hence in the problem with a self-affine crack the asymptotic stress field near the crack tip always should be less singular than the classical asymptotic (5) for regular cracks. At the same time at distances $r \ll \ell_0$ (*i.e.*, within the dislocation free zone) and $r \gg \xi_{\mathrm{C}}$ (*i.e.*, when self-affine crack can be treated as a smooth cut) the asymptotic stress field always obeys the classical behavior (5)[13], but with different stress intensity factors K_1 and K_{I} for $r \ll \ell_0$ and $r \gg \xi_{\mathrm{C}}$, respectively [11].

One may classify fracture of solids into two broad categories, namely the brittle fracture and the ductile fracture. The last is associated with a high roughness of crack faces which is characterized by the local fractal (box) dimension $D_{\mathrm{B}} = d - H$ more than a certain critical value $D_{\mathrm{B}}^* = d - H^*$; while a brittle fracture surface is characterized by the local fractal dimension $D_{\mathrm{B}} < D_{\mathrm{B}}^*$ [1,2,11,23]. It should be noted that the global metric dimension of any crack face is equal to its topological dimension $d - 1$.[14]

To gain greater insight into the nature of the difference between these two types of fracture let us consider a correlation function $C(r) = \langle -z(-r)z(r)\rangle / \langle z^2(r)\rangle$ [24], which for self-affine patterns is independent of r and possesses the remarkable equality

$$C(r) = \frac{\langle -z(-r)z(r)\rangle}{\langle z^2(r)\rangle} = 2\left(2^{dH-(d-1)} - 1\right), \tag{6}$$

[11] The explicit expression for $\phi(\theta, \nu)$ depends on the specific crack geometry and may be also derived by the integration of angular terms in (3) over the nanocrack orientation distribution (see also [22] and references therein).

[12] All $\alpha < 0$ must be excluded from the solution as physically unfeasible [21].

[13] Notice that here we consider only linear elastic solids; the stress field asymptotic in non-linear elastic solid with linear crack is characterized by power law exponent $\alpha > 0.5$, while the power law asymptotic stress field in elasto-plastic material with linear crack is characterized by $\alpha < 0.5$ [21].

[14] This is the reason for the classic behavior (5) at distances $r \gg \xi_{\mathrm{C}}$.

where $r \equiv x$ for a two dimensional ($d = 2$) problem, and $r = (x, y)$ for a three dimensional ($d = 3$) one [6]. It can be clearly seen, that if $H > H^*$, then $C(r) > 0$ and the crack trajectory (surface) displays persistence, *i.e.*, a trend at r (*e.g.*, a high or low value) is likely to be followed by a similar trend at $r + \Delta r$, whereas one has antipersistence ($C(r) < 0$) when $H < H^*$. When the roughness exponent assumes the critical value $H = H^*$ then one has a random pattern for which the correlations between increments vanishes for all r. The critical value of roughness (Hurst) exponent, which is associated with the brittle-to-ductile transition, may be defined from (6) as $H^* = (d-1)/d$,[15] where d is the dimension of the problem under consideration. If $H > H^*$, a variance of increments $\delta(\Delta L_x) = \langle |z(r+\Delta L_r) - z(r)| \rangle$ is a smooth (differentiable) function of the crack projection length increment ΔL_r, while for $H < H^*$ the graph of $\delta(\Delta L_r)$ is a nondifferentiable self-affine fractal, which obeys scaling behavior $\delta\,(\Delta L_r) \propto (\Delta L_r)^{2H}$ characterized by fractal dimension $d_\mathrm{B} = d - 2(1 - H)$[16] [6] within the interval of self-affinity of crack-faces $\ell_0 < \Delta L_r < \xi_\mathrm{C}$.

Now, it is easy to understand that there are two different types of stress behavior near the tip of a self-affine crack, which are associated with brittle and ductile fracture. Namely, if the roughness exponent $H < H^*$, *i.e.*, the local fractal dimension of crack face $D_\mathrm{B} > D_\mathrm{B}^*$[17] (very rough cracks!), $\alpha = 0$, so that stresses does not depend on r within the interval (2) and we have

$$\sigma_{ij}(\ell_0) = \sigma_{ij}(r) = \sigma_{ij}(\xi_\mathrm{C}), \tag{7}$$

while for a less rough brittle cracks, which are characterized by $H > H^*$ ($D_\mathrm{B} < D_\mathrm{B}^*$) the asymptotic of stress field obeys power law behavior (4), which is characterized by the scaling exponent

$$0 < \alpha = \tfrac{1}{2} - \eta < \tfrac{1}{2}. \tag{8}$$

Experimental studies have revealed that a crack propagates in a solid due to the initiation of new nanocracks at its tip [18]. This initiation results in a release of elastic energy ΔU_E which provides energy for further crack development. The well-known path-independent (invariant) J integral of fracture mechanics has been related to potential-energy-release rates associated with moving or extending cracks in linear elastic solids, as well as in non-linear elastic and elasto-plastic materials (see, for example, Ref. [21]).[18]

Considering a small circular contour Γ of radius ℓ, encompassing the crack tip (see Fig. 2b), we can write the J integral as

$$J = \int_\Gamma (W n_1 - \sigma_{ij} n_j u_{i1})\, ds, \tag{9}$$

[15] Notice that this relation was first obtained in [11] by other means.

[16] This formula is a generalization of the relation $d_\mathrm{B} = 2H$ which was derived in Ref. [25] for the Weierstrass-Mandelbrott function ($d = 2$).

[17] The critical value of the local fractal dimension D_B^* is equal to 1.5 for a two dimensional problem and to 2.33 for a three dimensional one.

[18] Various path-independent integrals considered in conventional fracture mechanics are associated with different conservation laws [21, 26].

where s is the arc length along Γ, σ_{ij} is the stress tensor, u_i is the displacement vector, $\{n_j\}$ is the normal vector,[19] and the strain-energy density W is a single-valued function of the strains u_{ij}.

It is well known that this integral is path-independent for any linear cut in an elastic solid [21, 26]. At the same time, it is easy to understand that in the case of a self-affine crack this integral is not invariant, because the number of singular points enclosed by different contours is different (see Fig. 2c).

Substituting (4) into (9) shows that within the interval (2) the J–integral scales as

$$J(\lambda\Gamma) = \lambda^{1-2\alpha} J(\Gamma), \tag{10}$$

where α is defined by Eq. (8). On the other hand, suppose that $\Delta_E = J\,\Delta S_L$ is the energy flux at the crack tip as the crack length (area) increases. In the case of a self-affine crack $\Delta S_L \propto (\Delta L_x)^{D_{\mathrm{D}}}$, where ΔL_x is the crack projection length and $D_{\mathrm{D}} = \min\{(d-1)/H, d\}$ is the latent dimension of the crack face [11]. Hence, if crack faces are self-affine the conventional J–integral (9) obeys scaling

$$J(\lambda\Gamma) = \lambda^{D_{\mathrm{D}}-(d-1)} J(\Gamma). \tag{11}$$

It immediately follows from Eqs. (10) and (11) that the stress singularity exponent α is equal to

$$\alpha = \frac{1-d(1-H)}{2H}, \text{ if } H > H^*; \text{ or } \alpha = 0, \text{ if } H \le H^*, \tag{12}$$

Notice that relation (12) was first derived in Ref. [11] by other means.

In a certain sense, a decrease of the stress singularity exponent α owing to increase in the crack roughness is similar to a decrease of the exponents of stress field singularity in the vicinity of a wedge-like notch on account of increase in its angle (see Fig. 2d). Therefore, the stress intensity factors and invariant J_{f} integral for self-affine crack can be also estimated using the weight function method [27].[20]

For a wedge-like notch with angle β (see Fig. 2d) the invariant J_β integral is defined by introducing the weight function F_β obeying scaling behavior $F_\beta = s^{-\varphi} f(\theta)$ [27] and reads

$$J_\beta = \int_\Gamma (W n_1 - \sigma_{ij} n_j u_{i1})\, F_\beta(s^{-\varphi}, \theta)\, ds, \tag{13}$$

where θ is the angular coordinate and $\varphi = \varphi(\beta)$.

[19] For the problem with self-affine crack, $\{n_j\}$ can be defined within a framework of non-standard analysis; notice that for the purposes of the present work we need to know only the change in $\{n_j\}$ along the crack trajectory.

[20] Weight functions were introduced in fracture mechanics by Bueckner [28]. They provide weights for the loads applied to the crack surfaces, such that their weighted integrals over the crack surfaces provide the stress intensity factors at a chosen point. Thus, they are related to Green's functions for the crack: they are, in fact, the stress intensity factors associated with concentrated point loading of the crack surfaces but they can also be constructed as solutions of the equations of equilibrium with zero tractions on the crack surfaces but with an unphysical singularity at the crack edge [28, 29].

It is important that in the case of self-affine crack the scaling behavior (10) occurs only within a bounded interval (2), so we can apply to the weight function $F_{\rm f}$ a scaling (incomplete self-similarity) representation (see Ref. [30])

$$F_{\rm f} = s^{\phi} f(\theta), \qquad \phi = 2\eta = 1 - 2\alpha, \tag{14}$$

where α is defined by Eq. (12). Now, it is easy to verify that the integral

$$J_{\rm f} = \int_{\Gamma} (W n_1 - \sigma_i j n_j u_{i1})\, F_{\rm f}\, ds, \tag{15}$$

with σ_{ij} and $F_{\rm f}$ given by Eqs. (4) and (14), respectively, does not depend on the chosen contour Γ.

The elastic energy release associated with the self-affine crack growth may be estimated as

$$\Delta U_{\rm E} \propto \int_{V_{\Delta L}} u_e(r)\, d^d r, \tag{16}$$

where $u_e(r) \simeq \langle \sigma^2(r) \rangle / 2E$ is the mean density of the elastic energy in the unloading zone $V_{\Delta L}$ near the crack tip, and E is the Young modulus. After averaging the stresses (4) along $\vec{r}$ in the unloading zone $V_{\Delta L}(\ell_0 < \Delta L_x < \xi_{\rm C}$, we find $\langle \sigma^2(r) \rangle \propto r^{-2\alpha}$, and after substituting $\langle \sigma^2 \rangle$ into (16) we obtain $\Delta U_{\rm E} \propto (\Delta L_x/\ell_0)^{d-2\alpha}$; hence, the energy release rate $G = \Delta U_{\rm E}/\Delta L_x$ scales with the increment of self-affine crack length in the direction of crack growth ΔL_x (within the interval $\ell_0 < \Delta L_x < \xi_{\rm C}$) as

$$G = G^* \left(\frac{\Delta L_x}{\ell_0} \right)^{\phi}, \qquad \phi = \frac{(d-1)(1-H)}{H}, \tag{17}$$

if fracture is brittle ($H > H^*$); or as

$$G = G^* \left(\frac{\Delta L_x}{\ell_0} \right), \tag{18}$$

if fracture is ductile ($H \le H^*$),[21] while at larger scales $G = J_{\rm C}$ is constant ($J_{\rm C}$ is the critical value of the conventional J-integral (9)).[22] The results obtained by Eqs. (17) and (18) are in good agreement with experimental data which were reported and analyzed in Refs. [23, 31, 32].

Now, let us consider a wedge-like notch with self-affine edges (see Fig. 2e). Under the assumption that the weight function can be represented as the product of the F_β and $F_{\rm f}$, we derive the asymptotic stress field which is produced by such wedge-like notch in the following form:

$$\sigma_{ij} \simeq K_{n{\rm f}}\, X^{-\alpha_n}\, \Phi_{ij}(\theta, \nu), \qquad \alpha_n = \lambda + \frac{d-1}{2H} - \frac{d+1}{2}, \qquad \ell_0 < r < \xi_{\rm C}, \tag{19}$$

[21] In the mechanics of straight crack, $G \equiv J_{\rm C} \equiv$ const.! [21].

[22] It is precisely this macroscopic value of energy release rate $J_{\rm C}$ the one estimated in standard experimental tests [21].

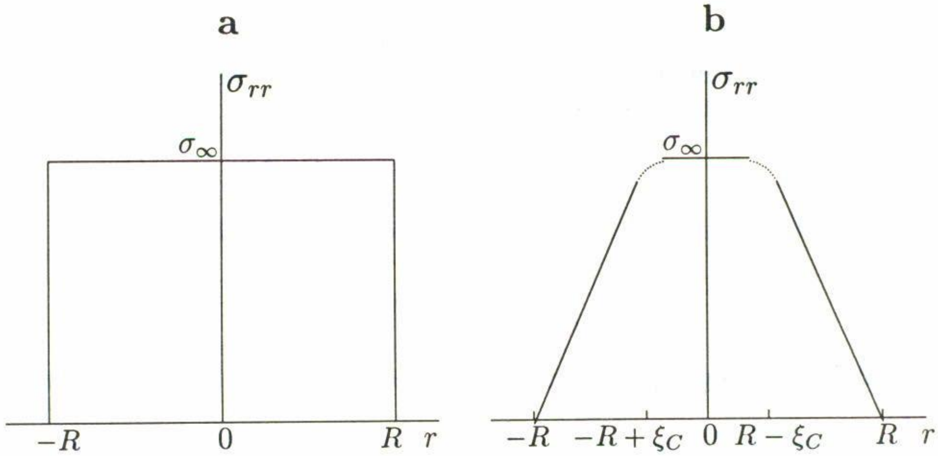

FIGURE 3. Stress distribution in the elastic cylinder of radius R under longitudinal tensile load σ_∞ in the case of smooth (a) and self-affine ($d = 3$, $H = 1/2$, $m = 1$) (b) boundary surface.

where λ is defined by the conventional equation [21]

$$\sin 2\lambda(\pi - \beta) = \pm\lambda \sin 2(\pi - \beta). \tag{20}$$

It is easy to see that stresses cease to be singular if $\beta \leq \beta^*$, where the critical value for the notch angle β^* is defined by the equation

$$\lambda(\beta^*) = \frac{(d+1)H - (d-1)}{2H}. \tag{21}$$

Moreover, from Eq. (14) it follows that a self-affine roughness leads to unloading of an elastic material near the free boundary ($\beta = \pi/2$), *i.e.*,

$$\sigma_{ij} \propto \left(\frac{r}{\ell_0}\right)^m, \qquad 0 < m = (d-1)\frac{1-H}{2H}, \qquad r < \xi_C, \tag{22}$$

so that the stress distribution in the specimen with a self-affine boundary surface has the form shown in Fig. 3b, instead of the stress distribution in the specimen with a smooth surface, which is shown in Fig. 3a.

ACKNOWLEDGMENTS

The author is grateful to G. Cherepanov, V. Ermishkin, V. Ivanova, D. Jeulin, M. Ostoja-Starzewski and P. Tamayo for useful discussions. The courtesy and support of the staff and faculty of the Materials Research Center at Monterrey University of Technology, and especially the Chairperson Professor Armando Bravo, are gratefully acknowledged.

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