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More about the $S = 0$ relativistic oscillator*

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ABSTRACT. I start from the Bargmann-Wiegner equations and introduce interaction in the form which is similar to a $S = 1/2$ case [M. Moshinsky & A. Szczepaniak, *J. Phys.* **A22** (1989) L817]. By means of an expansion of the wave function in the complete set of γ–matrices one can obtain the equations for a system which could be named as the $S = 0$ Kemmer-Dirac oscillator. Equations for the components ϕ_1 and ϕ_2 are different from the ones obtained by Y. Nedjadi & R. Barrett for the $S = 0$ Duffin-Kemmer-Petiau (DKP) oscillator [*J. Phys.* **A27** (1994) 4301]. As a result the energy spectrum of the $S = 0$ Kemmer-Dirac relativistic oscillator is dissimilar from the first versions of the DKP oscillator. Origins of this fact are given.

RESUMEN. A partir de la ecuación de Bargmann-Wigner, se introduce una interacción similar al caso $S = 1/2$ [M. Moshinsky y A. Szczepaniak, *J. Phys.* **A22** (1989) L817]. De la expansión en matrices γ de la función de onda, se puede obtener un sistema de ecuaciones que podría ser llamado el oscilador Kemmer-Dirac $S = 0$. Las ecuaciones para las componentes ϕ_1 y ϕ_2 son distintas a las obtenidas por Y. Nedjadi y R. Barrett para el oscilador (DKP) Duffijn-Kemmer-Petiau $S = 0$ [*J. Phys* **A27** (1994) 4301]; y el espectro de energías del oscilador relativista Kemmer-Dirac $S = 0$ es diferente al de la primera versión del oscilador DKP. Se presentan los motivos de esto.

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While the problem of interaction of a spinor particle with external fields is well understood, one cannot say that for the interactions of bosons and higher spin fermions [1]. In the present article I consider the oscillatorlike interaction of a $S = 0$ relativistic particle in the formalism first introduced by Kemmer [2–5]. The problem is shown to be exactly solvable.

A general system of relativistic wave equations for arbitrary spin was first written by Dirac [3] and Fierz [6]. In my presentation I use a reformulation of their formalism by Bargmann and Wigner [7]. For the cases of spin–0 and spin–1 the Bargmann-Wigner set[1]

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[1] This name usually refers to the case of a symmetric wave function, $S = 1$ and higher. However, it is easy to show that these equations (1) describe a $S = 0$ particle in the case of an antisymmetric wave function (see below).

reduces to two equations which can be written in the form (*e.g.*, Ref. [8])

$$\begin{cases} [i\gamma^\mu \partial_\mu - m]\,\Psi(x) = 0, \\ \Psi(x)\left[i(\gamma^\mu)^T \partial_\mu - m\right] = 0, \end{cases} \tag{1}$$

where the wave function is a 4×4 matrix (symmetric in the $S = 1$ case and antisymmetric in the $S = 0$ case) and the derivative acts to the left in the second equation.

Let us introduce the interaction as a $S = 1/2$ case, Ref. [9],

$$\left[i\gamma^\mu \partial_\mu - k\gamma^i \gamma^0 r^i - m\right]\Psi(x) = 0, \tag{2}$$

$$\Psi(x)\left[i(\gamma^\mu)^T \partial_\mu - k(\gamma^i \gamma^0)^T r^i - m\right] = 0. \tag{3}$$

Then, let us expand Ψ in terms of a set of sixteen γ–matrices. The wave function can be divided in two parts according to properties of simmetry. The set of matrices C, $\gamma^5 C$, $\gamma^5 \gamma^\mu C$, provides the antisymmetric part, and $\gamma^\mu C$, $\sigma^{\mu\nu} C$, the antisymmetric part. This form of interaction does not mix the $S = 0$ and $S = 1$ states. C is the matrix of charge conjugation.

Using the properties: $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$ and $C(\sigma^{\mu\nu})^T C^{-1} = -\sigma^{\mu\nu}$ in the case of the spin-0 wave function[2]

$$\Psi_{[\alpha\beta]} = C_{\alpha\beta}\varphi + \gamma^5_{\alpha\tau} C_{\tau\beta}\tilde{\varphi} + \gamma^5_{\alpha\delta}\gamma^\mu_{\delta\tau} C_{\tau\beta}\tilde{A}_\mu, \tag{4}$$

one can come to

$$\begin{cases} m\varphi = 0, \\ m\tilde{\varphi} = -i(\partial_\nu \tilde{A}^\nu), \\ m\tilde{A}_\mu = -i\partial_\mu \tilde{\varphi} + k\left[g^{0\nu} g_{\mu i} + g^{i\nu} g_{\mu 0}\right] r^i \tilde{A}_\nu. \end{cases} \tag{5}$$

Thus, the initial reducible representation is decomposed into the $(1/2, 1/2)$ vector representation, the $(0, 0)$ scalar representation and the trivial (pseudo)scalar representation, similar to the Duffin-Kemmer-Petiau algebra. Without interaction ($k = 0$) the above equations coincide (within the definition of κ, the constant which is proportional to mass) with Eqs. (26.12) in Ref. [4] and Eqs. $(247, 247')$ in Ref. [5a], which characterize the formalism of Kemmer:[3]

$$\begin{cases} m\tilde{\varphi} = -i\partial_\mu \tilde{A}^\mu, \\ m\tilde{A}_\mu = -i\partial_\mu \tilde{\varphi}. \end{cases} \tag{6}$$

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[2] The case of the spin $S = 1$ in the Bargmann-Wigner formalism will be reported elsewhere. Here we note that the higher spin relativistic oscillators have been considered earlier in the Duffin-Kemmer formalism, Ref. [10, 11], and in the Dowker fromalism [12, 13]. The latter [14] reveals surprising similarities with the $S = 1/2$ case, however, the problem of redundant components is not yet solved.

[3] This formulation is also contained in the more general formulation of Dirac [3] as mentioned in Ref. [4]. Therefore, I take a liberty to name the equations (17, 18) as the Kemmer-Dirac oscillator.

After substitution of the second equation into the first one, they yield the Klein-Gordon equation for a spinless $\tilde{\varphi}$ particle.

For stationary states $\tilde{\varphi}(x,t) = \tilde{\varphi}(x)\exp(-iEt)$, $\tilde{A}_\mu(x,t) = \tilde{A}_\mu(x)\exp(-iEt)$ the above set (5) is rewritten to[4]

$$\begin{cases} m\tilde{\varphi} = -E\tilde{A}_0 - i\nabla\vec{\tilde{A}}, \\ m\tilde{A}_0 = -E\tilde{\varphi} + k(\vec{r}\vec{\tilde{A}}), \\ m\vec{\tilde{A}} = i\nabla\tilde{\varphi} + k\vec{r}\tilde{A}_0. \end{cases} \tag{7}$$

I will show that the above equations describe an oscillatorlike system. $S = 0$ relativistic oscillators are also considered in Refs. [10–12, 15, 16].

After simple algebraic transformations one can come to the following set of equations:

$$\begin{cases} (E-m)\phi_1 = \vec{p}^{\,-}\vec{\tilde{A}}, \\ (E+m)\phi_2 = \vec{p}^{\,+}\vec{\tilde{A}}, \\ \vec{p}^{\,+}\phi_1 - \vec{p}^{\,-}\phi_2 = m\vec{\tilde{A}}, \end{cases} \tag{8}$$

where $\vec{p} = \frac{1}{i}\vec{\nabla}$, $\vec{p}^{\,\pm} = \frac{1}{\sqrt{2}}(\vec{p} \pm k\vec{r})$ and

$$\phi_1 = \frac{\tilde{A}_0 - \tilde{\varphi}}{\sqrt{2}}, \qquad \phi_2 = \frac{\tilde{A}_0 + \tilde{\varphi}}{\sqrt{2}}. \tag{9}$$

Now it is useful to compare our set of equations with the set of Ref. [11].[5] After application of the unitary transformation with the matrix

$$U = \begin{pmatrix} 0 & i & 0^T \\ i & 0 & 0^T \\ 0 & 0 & -\mathbb{1} \end{pmatrix} \tag{10}$$

to our 5-dimensional function $\psi = \mathrm{column}(\phi_1\ \phi_2\ \vec{\tilde{A}})$ one can recover the equation of the DKP approach with "tensor" interaction:

$$i\beta_0\frac{\partial\psi(x)}{\partial t} = \left[\vec{\beta}\cdot\vec{p} - k\beta_0(\vec{\beta}\cdot\vec{r}) + k(\vec{\beta}\cdot\vec{r})\beta_0 + m\right]\psi(x). \tag{11}$$

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[4] I chose a dependence of the wave function on time similar to Refs. [9, 11]. If we use $\tilde{\varphi}(x,t) = \tilde{\varphi}(x)\exp(iEt)$, $\tilde{A}_\mu(x,t) = \tilde{A}_\mu(x)\exp(iEt)$, the components ϕ_1 and ϕ_2 are only interchanged each other and $\omega \to -\omega$ in Eqs. (17, 18); surprisingly, this does not lead to any change in the energy spectrum.

[5] Eqs. (9) of the cited paper are the analogs of our equations (7), but with a different form of interaction. I would still like to point out that the method of solving the problem, used in the paper [11], does not take into account the degeneracy of the levels in the quantum number M.

Multiplying the first and the second equations, Eqs. (8), by m one finds

$$\begin{cases} m(E-m)\phi_1 = \vec{p}^{\,-}\vec{p}^{\,+}\phi_1 - \vec{p}^{\,-}\vec{p}^{\,-}\phi_2 \\ m(E+m)\phi_2 = \vec{p}^{\,+}\vec{p}^{\,+}\phi_1 - \vec{p}^{\,+}\vec{p}^{\,-}\phi_2, \end{cases} \tag{12}$$

and acting $m(E+m)$ on the first equation and $m(E-m)$ on the second one yields

$$\begin{cases} m^2(E^2-m^2)\phi_1 = m(E+m)\vec{p}^{\,-}\vec{p}^{\,+}\phi_1 - (\vec{p}^{\,-}\vec{p}^{\,-})(\vec{p}^{\,+}\vec{p}^{\,+})\phi_1 + (\vec{p}^{\,-}\vec{p}^{\,-})(\vec{p}^{\,+}\vec{p}^{\,-})\phi_2 \\ m^2(E^2-m^2)\phi_2 = -m(E-m)\vec{p}^{\,+}\vec{p}^{\,-}\phi_2 - (\vec{p}^{\,+}\vec{p}^{\,+})(\vec{p}^{\,-}\vec{p}^{\,-})\phi_2 + (\vec{p}^{\,+}\vec{p}^{\,+})(\vec{p}^{\,-}\vec{p}^{\,+})\phi_1 . \end{cases} \tag{13}$$

Finally, by means of the use of the following commutation relations:

$$\left[p_i^+ p_j^-\right]_- = ik\delta_{ij}, \qquad \left[p_i^\pm p_j^\pm\right]_- = 0, \tag{14}$$

$$\left\{p_i^- p_j^+ - p_i^+ p_j^-\right\} f(\vec{r}) = \left[-ik\delta_{ij} + k\epsilon_{jik}\hat{L}_k\right] f(\vec{r}), \tag{15}$$

$$\left\{\vec{p}^{\,-}\vec{p}^{\,+} + \vec{p}^{\,+}\vec{p}^{\,-}\right\} f(\vec{r}) = \left[\vec{p}^{\,2} - k^2\vec{r}^{\,2}\right] f(\vec{r}), \tag{16}$$

(with L_k being the operator of the angular momentum and $k = im\omega$) for the $S = 0$ case we obtain

$$(E^2 - m^2)\phi_1 = \left[\vec{p}^{\,2} + m^2\omega^2\vec{r}^{\,2} + (E+2m)\omega + \omega^2\hat{L}^2\right]\phi_1, \tag{17}$$

$$(E^2 - m^2)\phi_2 = \left[\vec{p}^{\,2} + m^2\omega^2\vec{r}^{\,2} + (E-2m)\omega + \omega^2\hat{L}^2\right]\phi_2. \tag{18}$$

In fact, one has the oscillator-behaved term $(m^2\omega^2\vec{r}^{\,2})$; however, there are additional terms comparing with Eq. (10) of the paper [11], the Duffin-Kemmer-Petiau oscillator. The operator of the angular momentum $\hat{L}^2$ is not present in the equations of Ref. [11] and there is no dependence of the "constant" term on the energy there. The presence of this term could lead to some speculations since one can show that a consequence of this fact is the "splitting" of energy levels in the both equations. Namely, one has two roots in each of equations. Moreover, if we pass to the nonrelativistic limit ($E = \epsilon + mc^2$, $\epsilon \ll mc^2$) one has the quantity $(2mc^2 - \hbar\omega)\epsilon$, which could be equal to zero or even negative. In the mean time, the sum of the remained terms on the *rhs* in the first equation (17) is positive. Does this fact signify that the oscillator system surveys not for all frequency values? More detailed analysis presented below permits us to answer these questions.

Now let us seek to solve Eq. (17). For identification purposes, in what follows it is $(E^2_{N,\ell} - m^2)/2m$ rather than $E_{N,\ell}$ which I seek since the first form reduces to the usual Schrödinger eigenvalue in the non-relativistic limit. If the basis functions of Ref. [9] are used, then $\hat{L}^2\phi_{1,2} = \ell_{1,2}(\ell_{1,2}+1)\phi_{1,2}$ and energy eigenvalues of the equation associated with Eq. (17) could be found from the algebraic equation

$$\frac{1}{2m}(E^2 - m^2) - (E+2m)\frac{\omega}{2m} - \ell_1(\ell_1+1)\frac{\omega^2}{2m} = \left(N_1 + \tfrac{3}{2}\right)\omega, \tag{19}$$

where the principal quantum number is a non-negative integer. This equation is quadratic in E and has therefore 2 roots. The solutions of Eq. (19) are

$$\frac{1}{2m}(E_{\pm}^2 - m^2) = \left(N_1 + \tfrac{5}{2}\right)\omega + \left(\ell_1(\ell_1+1) + \tfrac{1}{2}\right)\frac{\omega^2}{2m} \pm \Delta_1, \tag{20}$$

where

$$\Delta_1 = \frac{\omega}{2}\left(1 + (2N_1+5)\left(\frac{\omega}{m}\right) + \left(\ell_1 + \frac{1}{2}\right)^2\left(\frac{\omega}{m}\right)^2\right)^{\frac{1}{2}} \tag{21}$$

This formula has structural similarities with the eigenvalues found for the DKP oscillator, Ref. [11], *i.e.*, it involves the usual 3–dimensional harmonic oscillator energy, a term proportional to $\ell(\ell+1)$ which appears as some kind of rotational energy and the third contribution Δ to energy, which is a complicated function of the oscillator frequency, ℓ_1 and N_1.

In the limit where the oscillator frequencies are such that $\hbar\omega \ll mc^2$, keeping only the first-order term in ω in the equations leads to

$$\frac{1}{2m}(E_{+}^2 - m^2) \simeq \epsilon^{+} = (N_1+3)\omega, \tag{22}$$

$$\frac{1}{2m}(E_{-}^2 - m^2) \simeq \epsilon^{-} = (N_1+2)\omega. \tag{23}$$

I now seek to solve the second equation (18). Using the same procedure as above the two eigenvalues of the energies are

$$\frac{1}{2m}(E_{\pm}^2 - m^2) = \left(N_2 + \tfrac{1}{2}\right)\omega + \left(\ell_2(\ell_2+1) + \tfrac{1}{2}\right)\frac{\omega^2}{2m} \pm \Delta_2, \tag{24}$$

where

$$\Delta_2 = \frac{\omega}{2}\left(1 + (2N_2+1)\left(\frac{\omega}{m}\right) + \left(\ell_2 + \frac{1}{2}\right)^2\left(\frac{\omega}{m}\right)^2\right)^{\frac{1}{2}} \tag{25}$$

In the limit of low frequencies

$$\frac{1}{2m}(E_{+}^2 - m^2) \simeq \epsilon^{+} = (N_2+1)\omega, \tag{26}$$

$$\frac{1}{2m}(E_{-}^2 - m^2) \simeq \epsilon^{-} = N_2\,\omega. \tag{27}$$

The condition of compatibility of the set of equations (17, 18) ensures us that $N_2 = N_1 + 2$ and $\ell_2 = \ell_1$. Therefore, in the relativistic region we have two physical (positive and negative) values of the energy like to the other formulations of an interacting $S = 0$

relativistic particle. However, a remarkable feature of the presented formulation is the double degeneracy (in N) of the levels in the limit $\hbar\omega \ll mc^2$ except for the ground level. Let us note that such a phenomenon has been revealed in Ref. [15] (*cf.* $\epsilon^{\pm}$ with Eqs. (11a, 11b) of the cited work). However, reasons for the introduction of the matrix structure in the Klein-Gordon equation were not explained there. Next, I would like to note that the quantity $(E_{\pm}^2 - m^2)/2m$ is seen from Eqs. (20, 21) or (24, 25) to be non-negative even in the high-frequency limit.

Let me draw your kind attention to one more paradox,[6] which also can occur in the case of the Dirac oscillator of Moshinsky [9]. If we put one of the components (e.g., ϕ_2) equal to zero in the 5–component wave function $\psi = \text{column}\,(\phi_1\ \phi_2\ \vec{\vec{A}})$ at first sight one can obtain the different spectrum $E = m + (N + 3)\omega$. It follows from the resulting linear equation for the energy (as opposed to equations (17, 18), which are quadratic). However, let us not forget that, in fact, such a constraint $\phi_2 = 0$ leads to another constraint on ϕ_1, $\vec{p}^{+} \cdot \vec{p}^{+}\phi_1 = 0$, which substitutes the dynamical equation (8b). One can come to the same paradox in the set of equations (7) of Ref. [9] if we put one of the spinors (*e.g.*, ψ_2) to zero. Namely, the spectrum would be $E = m$ and the constraint $(\vec{\sigma} \cdot \vec{p}^{+})\psi_1 = 0$ would restrict the wave function there. The deeper consideration reveals the fact that in both cases we come over from a set of equations of the first order in energy (in fact, in time in the coordinate representation) to the one equation of the second order. In general, the mathematical validity of such a procedure is not clear, but we do not have any alternative way to solve them, *cf.* with the consideration of the Dirac particle in the uniform magnetic field in the well-known textbooks, *e.g.*, Ref. [17, p. 67]. Nevertheless, by using lengthy transformations of the operator $\hat{L}^2 = \epsilon_{ijk}r_j p_k \epsilon_{ilm} r_l p_m$ and of Eq. (17) as well as taking into account the mentioned constraints one can prove that the positive-energy part of the spectrum (19) reduces to the spectrum of the different physical system with $\phi_2 = 0$ (*i.e.*, it coincides with $E = m + (N + 3)\omega$). The constraint $\vec{p}^{+} \cdot \vec{p}^{+}\phi_1 = 0$ is equivalent to the connection between the principal quantum number and the orbital quantum number $\ell_1 = N_1 + 2 = N_2$. The problem of the negative-energy part of the spectrum of Eqs. (17, 18) deserves further elaboration. At the moment I would still like to mention the following. In the rest frame it is easy to see that $\phi_2 = 0$ (or $\tilde{A}_0 = -\tilde{\varphi}$) does eliminate negative energies under the used choice of the stationary states.

In conclusion, let me mention that a behavior of a scalar particle in external fields has been considered in many publications, see, *e.g.*, the bibliography in Refs. [18, 19]. Recent publications, Ref. [20], deal with a solution of the problem of finding the energy spectra of a scalar particle with polarizability in the constant magnetic, electric fields and in the field of the plane electromagnetic wave. However, as we learnt, the model of the $S = 0$ oscillator with the intrinsic spin structure has very specific peculiarities, which differ it from the other model used, *e.g.*, in descriptions of π- and K-mesons.

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